A Theoretical Study of ψ-waves

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"….To understand the properties of a quantum mechanical system, one need not know the nature of potential of the system. It is sufficient to describe the system using my equations knowing completely about system's ψ-function."

KEY WORDS: Quantum mechanical systems; linear operator analysis; generalized co-ordinates; many-particle systems; eigenoperator and eigenvalues; modern notations (Dirac notations); hermitian operators; hermitian conjugate; transposed operator; commutator.

Mathematics is clever enough to do everything
by itself without physicists' speculation.

(Werner Heisenberg).

1. Introduction

The description of a quantum mechanical system demands a fundamental modification of basic physical concepts and laws. Such a modification is presented by ψ-waves which are clever enough to describe the state of a quantum mechanical system.

The mathematical study of such a modified function provides us the new equations and results in the quantum theory.

I have performed such a theoretical study and I am presenting such new equations and results in the quantum theory.

2. Saurav equation

To describe the state of a quantum mechanical system without knowing the potential energy of the system, there plays a new equation which can describe the systems which are strange in the nature and one can not determine the potential nature of those systems.
Let

$$\psi(q_\alpha, t) = \exp\left(\frac{i}{\hbar} s(q_\alpha, t)\right) \tag{1}$$

and getting the exact differential of $\psi(q_\alpha, t)$ with respect to time, I get

$$\frac{d\psi(q_\alpha,t)}{dt} = \frac{i}{\hbar}\left\{\frac{\partial s}{\partial t} + \frac{\partial s}{\partial q_\alpha}\dot{q}_\alpha\right\}\psi(q_\alpha,t) \tag{2}$$

And taking the advantage of classical theoretical physics;

$$\frac{\partial s}{\partial t} = -H(q_\alpha, \frac{\partial s}{\partial q_\alpha}, t) \tag{3}$$

$$\frac{\partial s}{\partial q_\alpha} = p_\alpha \tag{4}$$

and

$$p_\alpha \dot{q}_\alpha = H(q_\alpha, \frac{\partial s}{\partial q_\alpha}, t) + L(q_\alpha, \dot{q}_\alpha, t) \tag{5}$$

and putting eq (3), eq (4) and eq (5) in eq (2), I get

$$L(q_\alpha, \dot{q}_\alpha, t)\psi(q_\alpha, t) = -i\hbar\frac{d\psi(q_\alpha,t)}{dt} \tag{6}$$

and taking the advantage of operator analysis, I get the quantum mechanical Lagrangian (Lagrangian operator),

$$\hat{L}(q_\alpha, \dot{q}_\alpha, t) = -i\hbar\frac{d}{dt} \tag{7}$$

and getting the partial differential of $\psi(q_\alpha, t)$ with respect to time in the consideration of eq (1) and eq (3), I get

$$H(q_\alpha, \frac{\partial s}{\partial q_\alpha}, t)\psi(q_\alpha, t) = i\hbar\frac{\partial\psi(q_\alpha,t)}{\partial t} \tag{8}$$

and taking the advantage of operator analysis, I get the quantum mechanical Hamiltonian (Hamiltonian operator),

$$\hat{H}(q_\alpha, \frac{\partial s}{\partial q_\alpha}, t) = i\hbar\frac{\partial}{\partial t} \tag{9}$$

Now getting the potential energy from the classical theoretical physics,

$$2V(q_\alpha,t) = H(q_\alpha,\frac{\partial s}{\partial q_\alpha},t) - L(q_\alpha,\dot{q}_\alpha,t) \tag{10}$$

and applying the linear operator analysis, I get the quantum mechanical potential energy (potential energy operator) in the consideration of eq (7), eq (9) and eq (10),

$$\hat{V}(q_\alpha,t) = \frac{i\hbar}{2}\left\{\frac{\partial}{\partial t} + \frac{d}{dt}\right\} \tag{11}$$

and getting the partial differential of $\psi(q_\alpha,t)$ with respect to q_α, in the consideration of eq (1) and eq (4), I get

$$P_\alpha \psi(q_\alpha,t) = -i\hbar\frac{\partial \psi(q_\alpha,t)}{\partial q_\alpha} \tag{12}$$

and taking the advantage of operator analysis, I get the quantum mechanical momentum (momentum operator),

$$\hat{P}_\alpha = -i\hbar\frac{\partial}{\partial q_\alpha} = -i\hbar\nabla_\alpha \tag{13}$$

And applying the Hamiltonian operator to $\psi(q_\alpha,t)$, I get

$$\hat{H}(q_\alpha,\frac{\partial s}{\partial q_\alpha},t)\psi(q_\alpha,t) = \frac{\hat{P}_\alpha^2}{2m}\psi(q_\alpha,t) + \hat{V}(q_\alpha,t)\psi(q_\alpha,t) \tag{14}$$

and in the consideration of eq (9), eq (11) and eq (13), I get

$$i\hbar\frac{\partial \psi(q_\alpha,t)}{\partial t} + \frac{\hbar^2}{m}\nabla_\alpha^2\psi(q_\alpha,t) - i\hbar\frac{d\psi(q_\alpha,t)}{dt} = 0 \tag{15}$$

or

$$\frac{d\psi(q_\alpha,t)}{dt} + \frac{i\hbar}{m}\nabla_\alpha^2\psi(q_\alpha,t) - \frac{\partial \psi(q_\alpha,t)}{\partial t} = 0 \tag{16}$$

eq (15) and eq (16) are Saurav equations discovered by me (27-8-2004).

and by transforming generalized co-ordinates to Cartesian one in eq (16), I get the Saurav equation in Cartesian co-ordinates,

$$\frac{d\psi(r,t)}{dt}+\frac{i\hbar}{m}\vec{\nabla}^2\psi(r,t)-\frac{\partial\psi(r,t)}{\partial t}=0 \tag{17}$$

where r is the Cartesian co-ordinate of particle and $\vec{\nabla}^2=\frac{\partial^2}{\partial r^2}=\left\{\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right\}$ is classical Laplacian operator.

3. Quantum mechanical Hamilton-Jacobi equation

The theoretical study of ψ-waves can derive the quantum mechanical Hamilton-Jacobi equation on its own and can introduce new equations in theoretical physics.
Let getting the first order partial differential of $\psi(q_\alpha,t)$ with respect to q_α in the consideration of eq (1), I get

$$\frac{\partial\psi(q_\alpha,t)}{\partial q_\alpha}=\frac{i}{\hbar}\frac{\partial s}{\partial q_\alpha}\psi(q_\alpha,t) \tag{18}$$

and getting the second order partial differential of $\psi(q_\alpha,t)$ with respect to q_α in the consideration of eq (1) and eq (18), I get

$$\frac{\partial^2\psi(q_\alpha,t)}{\partial q_\alpha^2}=\frac{i}{\hbar}\left\{\frac{\partial^2 s}{\partial q_\alpha^2}+\frac{i}{\hbar}\left(\frac{\partial s}{\partial q_\alpha}\right)^2\right\}\psi(q_\alpha,t) \tag{19}$$

and taking the advantage of operator analysis, I get the quantum mechanical Laplacian operator,

$$\hat{\nabla}_\alpha^2=\frac{i}{\hbar}\left\{[\nabla_\alpha^2 s]+\frac{i}{\hbar}[\nabla_\alpha s]^2\right\} \tag{20}$$

and putting the quantum mechanical Laplacian operator in the generalized Schrödinger equation,

$$i\hbar\frac{\partial\psi(q_\alpha,t)}{\partial t}+\frac{\hbar^2}{2m}\nabla_\alpha^2\psi(q_\alpha,t)-V(q_\alpha,t)\psi(q_\alpha,t)=0 \tag{21}$$

and in the consideration of eq (8) and eq (3), I get

$$\frac{\partial s}{\partial t}+\frac{[\nabla_{\alpha}s]^{2}}{2m}-\frac{i\hbar}{2m}[\nabla_{\alpha}^{2}s]+V(q_{\alpha},t)=0 \tag{22}$$

which is the quantum mechanical Hamilton-Jacobi equation.
and as a limiting case $(\hbar\to 0)$, it reduces to classical one,

$$\frac{\partial s}{\partial t}+\frac{[\nabla_{\alpha}s]^{2}}{2m}+V(q_{\alpha},t)=0 \tag{23}$$

which is the classical Hamilton-Jacobi equation.
and applying the quantum mechanical Laplacian operator (20) to Saurav equation (16), I get

$$\frac{d\psi(q_{\alpha},t)}{dt}-\frac{[\nabla_{\alpha}^{2}s]}{m}\psi(q_{\alpha},t)-\frac{i}{m\hbar}[\nabla_{\alpha}s]^{2}\psi(q_{\alpha},t)-\frac{\partial\psi(q_{\alpha},t)}{\partial t}=0 \tag{24}$$

and applying the quantum mechanical Laplacian operator (20) to Saurav equation (15) in the consideration of eq (8) and eq (6), I get

$$L(q_{\alpha},\dot{q}_{\alpha},t)\psi(q_{\alpha},t)+H(q_{\alpha},\frac{\partial s}{\partial q_{\alpha}},t)\psi(q_{\alpha},t)+\frac{i\hbar}{m}[\nabla_{\alpha}^{2}s]\psi(q_{\alpha},t)-\frac{[\nabla_{\alpha}s]^{2}}{m}\psi(q_{\alpha},t)=0 \tag{25}$$

and canceling $\psi(q_{\alpha},t)$ in eq (25), I get

$$L(q_{\alpha},\dot{q}_{\alpha},t)+H(q_{\alpha},\frac{\partial s}{\partial q_{\alpha}},t)+\frac{i\hbar}{m}[\nabla_{\alpha}^{2}s]-\frac{[\nabla_{\alpha}s]^{2}}{m}=0 \tag{26}$$

and as a limiting case $(\hbar\to 0)$, it reduces to classical one,

$$L(q_{\alpha},\dot{q}_{\alpha},t)+H(q_{\alpha},\frac{\partial s}{\partial q_{\alpha}},t)-\frac{[\nabla_{\alpha}s]^{2}}{m}=0 \tag{27}$$

which is a new equation in classical theoretical physics.
and putting eq (3) in eq (25), I get

$$\frac{\partial s}{\partial t}\psi(q_{\alpha},t)-L(q_{\alpha},\dot{q}_{\alpha},t)\psi(q_{\alpha},t)+\frac{[\nabla_{\alpha}s]^{2}}{m}\psi(q_{\alpha},t)-\frac{i\hbar}{m}[\nabla_{\alpha}^{2}s]\psi(q_{\alpha},t)=0 \tag{28}$$

and canceling $\psi(q_\alpha, t)$ in eq (28), I get

$$\frac{\partial s}{\partial t} + \frac{[\nabla_\alpha s]^2}{m} - \frac{i\hbar}{m}[\nabla_\alpha^2 s] - L(q_\alpha, \dot{q}_\alpha, t) = 0 \quad (29)$$

which is the quantum mechanical Saurav-Gaurav equation.
and as a limiting case $(\hbar \to 0)$, it reduces to classical one,

$$\frac{\partial s}{\partial t} + \frac{[\nabla_\alpha s]^2}{m} - L(q_\alpha, \dot{q}_\alpha, t) = 0 \quad (30)$$

which is the classical Saurav-Gaurav equation.

4. Saurav equation for many-particle systems

Saurav equation is clever enough to describe the many-particle system without knowing the nature of potential of the system.
The ψ for N-particle system is defined as,

$$\psi(q_1, q_2, ..., q_i, ...q_s, t) \qquad \forall \quad N \in s, (i = 1,2,3....s) \quad (31)$$

where q_i are the generalized co-ordinates of ith particle and s is the degrees of freedom of the system.
Now putting eq (31) in Saurav equation (16), I get

$$\frac{d\psi(q_1, q_2, ...q_s, t)}{dt} + i\hbar \sum_{i=1}^{N} \sum_{\alpha=1}^{\alpha} \frac{\nabla_{\alpha i}^2}{m_i} \psi(q_1, q_2, ...q_s, t) - \frac{\partial \psi(q_1, q_2, ...q_s, t)}{\partial t} = 0 \quad (32)$$

where α is the degrees of freedom of ith particle and m_i is the mass of ith particle of the system.
eq (32) is Saurav equation for many-particle systems which is proved by the additivity of kinetic energy.
Saurav equation for many-particle systems can also be written in Cartesian co-ordinates, as

$$\frac{d\psi(r_1, r_2, ...r_N, t)}{dt} + i\hbar \sum_{i=1}^{N} \frac{\vec{\nabla}_{r_i}^2}{m_i} \psi(r_1, r_2, ...r_N, t) - \frac{\partial \psi(r_1, r_2, ...r_N, t)}{\partial t} = 0 \quad (33)$$

where $\left\{ \vec{\nabla}_{r_i}^2 = \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} + \frac{\partial^2}{\partial z_i^2} \right\}$ is the Laplacian operator in Cartesian co-ordinates and $r_i (i = 1,2,....N)$ is the Cartesian co-ordinates of ith particle of the system.

eq (32) and eq (33) can describe the many-particle quantum mechanical systems without knowing the nature and behaviore of potential energy of the systems.

Remark

The second term of eq (32)

$$\sum_{i=1}^{N} \sum_{\alpha=1}^{\alpha} \frac{\nabla_{\alpha i}^2}{m_i} \psi(q_1, q_2,....q_s, t) \tag{34}$$

can be illustrated by transformation of generalized co-ordinates to Cartesian one. And eq (34) can be written for Cartesian, as

$$\sum_{i=1}^{N} \sum_{\alpha=1}^{3} \frac{\vec{\nabla}_{r_i}^2}{m_i} \psi(r_1, r_2,....r_N, t) \tag{35}$$

(The degrees of freedom of N-particle systems in Cartesian co-ordinate system is, s=3N and $\alpha = \frac{s}{N} = 3$ (See L. D. Landau and E. M. Lifshitz, Mechanics)).

And for $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, eq (35) can be written, as

$$\sum_{i=1}^{N} \frac{\vec{\nabla}_{x_i}^2}{m_i} \psi(r_1, r_2,....r_N, t) + \sum_{i=1}^{N} \frac{\vec{\nabla}_{y_i}^2}{m_i} \psi(r_1, r_2,....r_N, t) + \sum_{i=1}^{N} \frac{\vec{\nabla}_{z_i}^2}{m_i} \psi(r_1, r_2,....r_N, t) \tag{36}$$

where $\vec{\nabla}_{\beta_i}^2 (\beta = x, y, z)$ is illustrated as,

$$\vec{\nabla}_{\beta_i}^2 = \frac{\partial^2}{\partial \beta_i^2} \tag{37}$$

where β_i is the Cartesian co-ordinates of ith particle.

5. Saurav equation and energy eigenvalues

The energy eigenvalue of a quantum mechanical system can be determined by using Saurav equation without really knowing the potential energy of the system.
Let assuming the eigenvalue equation,

$$\hat{\mathcal{A}}\psi = \alpha\psi \tag{38}$$

where ψ is an eigenfunction, α its eigenvalue and $\hat{\mathcal{A}}$ is an operator.
and considering the eigenvalue equation (38) for Hamiltonian operator, I get the energy eigenvalue equation,

$$\hat{H}\psi = E\psi \tag{39}$$

where E is the energy eigenvalue of eigenfunction ψ.
and in the consideration of eq (39) and eq (9), I get

$$E\psi = i\hbar\frac{\partial\psi}{\partial t} \tag{40}$$

and putting eq (40) in Saurav equation (15), I get

$$E\psi = i\hbar\frac{d\psi}{dt} - \frac{\hbar^2}{m}\nabla^2\psi \tag{41}$$

which is Saurav equation for energy eigenvalues which is independent of the nature of potential of the system. Eq (41) can be used to determine the energy eigenvalues of those systems for which the determination of potential energy is impossible.
and in the consideration of eq (41) and eq (39), I get the new Hamiltonian operator,

$$\hat{H} = i\hbar\frac{d}{dt} - \frac{\hbar^2}{m}\nabla^2 = i\hbar\left\{\frac{d}{dt} + \frac{i\hbar}{m}\nabla^2\right\} \tag{42}$$

and in the consideration of eq (11), eq (13), eq (14) and eq (39), I get

$$E\psi = \frac{i\hbar}{2}\left\{\frac{d\psi}{dt} + \frac{i\hbar}{m}\nabla^2\psi + \frac{\partial\psi}{\partial t}\right\} \tag{43}$$

which is another Saurav equation for energy eigenvalues.
and in the consideration of eq (43) and eq (39), I get the another new Hamiltonian operator,

$$\hat{H} = \frac{i\hbar}{2}\left\{\frac{d}{dt} + \frac{i\hbar}{m}\nabla^2 + \frac{\partial}{\partial t}\right\} \tag{44}$$

which is the proved by Saurav equation (15)

6. Saurav equation and Lagrangian eigenvalues

Now considering the eigenvalue equation (38) for Lagrangian operator, I get the Lagrangian eigenvalue equation,

$$\hat{L}\psi = \lambda\psi \tag{45}$$

where λ is Lagrangian eigenvalue.
and in the consideration of eq (7) and eq (45), I get

$$\lambda\psi = -i\hbar\frac{d\psi}{dt} \tag{46}$$

and in the consideration of Saurav equation (15) and eq (46), I get

$$\lambda\psi = -i\hbar\frac{\partial\psi}{\partial t} - \frac{\hbar^2}{m}\nabla^2\psi \tag{47}$$

which is Saurav equation for Lagrangian eigenvalues.
and in consideration of eq (45) and eq (47), I get the new Lagrangian operator,

$$\hat{L} = -i\hbar\frac{\partial}{\partial t} - \frac{\hbar^2}{m}\nabla^2 = i\hbar\left\{\frac{i\hbar}{m}\nabla^2 - \frac{\partial}{\partial t}\right\} \tag{48}$$

which is proved by Saurav equation (15).
Now assuming

$$\lambda\psi = \hat{L}\psi = \frac{\hat{P}^2}{2m}\psi - \hat{V}\psi \tag{49}$$

and in the consideration of eq (11), eq (13) and eq (49), I get

$$\lambda\psi = \frac{i\hbar}{2}\left\{\frac{i\hbar}{m}\nabla^2\psi - \frac{d\psi}{dt} - \frac{\partial\psi}{\partial t}\right\} \quad (50)$$

and in the consideration of eq (45) and eq (50), I get the another new Lagrangian operator,

$$\hat{L} = \frac{i\hbar}{2}\left\{\frac{i\hbar}{m}\nabla^2 - \frac{\partial}{\partial t} - \frac{d}{dt}\right\} \quad (51)$$

7. Saurav equation and kinetic energy eigenvalues

Now considering the eigenvalue equation (38) for kinetic energy operator, I get the kinetic energy eigenvalue equation,

$$\hat{T}\psi = \tau\psi \quad (52)$$

where τ is the kinetic energy eigenvalue.
And according to classical theoretical physics;

$$T = \frac{1}{2}\{H + L\} \quad (53)$$

and in the consideration of linear operator analysis with the help of eq (7) and eq (42), I get the quantum mechanical kinetic energy (kinetic energy operator),

$$\hat{T} = -\frac{\hbar^2}{2m}\nabla^2 \quad (54)$$

and in the consideration of eq (48), eq (42) and eq (53) considering the linear operator analysis, I get the kinetic energy operator,

$$\hat{T} = \frac{i\hbar}{2}\left\{\frac{d}{dt} + 2\frac{i\hbar}{m}\nabla^2 - \frac{\partial}{\partial t}\right\} \quad (55)$$

and applying it to eigenfunction ψ in the consideration of kinetic energy eigenvalue, I get

$$\tau\psi = \frac{i\hbar}{2}\left\{\frac{d\psi}{dt} + 2\frac{i\hbar}{m}\nabla^2\psi - \frac{\partial\psi}{\partial t}\right\} \quad (56)$$

which is Saurav equation for kinetic energy eigenvalues.
and applying eq (54) to eigenfunction ψ in the consideration of (52), I get the kinetic energy eigenvalue equation,

$$\tau\psi = -\frac{\hbar^2}{2m}\nabla^2\psi \tag{57}$$

and in the consideration of eq (56) and eq (57), I get

$$\tau\psi = \frac{i\hbar}{2}\left\{\frac{\partial\psi}{\partial t} - \frac{d\psi}{dt}\right\} \tag{58}$$

which is another Saurav equation for kinetic energy eigenvalues.
and in the consideration of eq (58) and eq (52), I get the another new kinetic energy operator,

$$\hat{T} = \frac{i\hbar}{2}\left\{\frac{\partial}{\partial t} - \frac{d}{dt}\right\} \tag{59}$$

and in the consideration of eq (56) and eq (58), I get

$$i\hbar\left\{\frac{d\psi}{dt} + \frac{i\hbar}{m}\nabla^2\psi - \frac{\partial\psi}{\partial t}\right\} = 0 \tag{60}$$

which is Saurav equation (15 and 16).
and in the consideration of eq (57) and eq (58), I get the another Saurav equation for kinetic energy eigenvalues,

$$\tau\psi = i\hbar\left\{\frac{\partial\psi}{\partial t} - \frac{i\hbar}{2m}\nabla^2\psi - \frac{d\psi}{dt}\right\} \tag{61}$$

and in the consideration of eq (61) and eq (52), I get the another new kinetic energy operator,

$$\hat{T} = i\hbar\left\{\frac{\partial}{\partial t} - \frac{i\hbar}{2m}\nabla^2 - \frac{d}{dt}\right\} \tag{62}$$

and in the consideration of eq (56) and eq (61), I get the another new kinetic energy eigenvalue equation,

$$\tau\psi = 2i\hbar\left\{\frac{d\psi}{dt} + \frac{5}{4}\frac{i\hbar}{m}\nabla^2\psi - \frac{\partial\psi}{\partial t}\right\} \tag{63}$$

and in the consideration of eq (63) and eq (52), I get the another new kinetic energy operator,

$$\hat{T} = 2i\hbar\left\{\frac{d}{dt} + \frac{5}{4}\frac{i\hbar}{m}\nabla^2 - \frac{\partial}{\partial t}\right\} \tag{64}$$

And considering eq (53) in the consideration of linear operator analysis, I get

$$\hat{T} = \frac{1}{2}\left\{\hat{H} + \hat{L}\right\} \tag{65}$$

and in the consideration of eq (9), eq (51) and eq (65), I get

$$\hat{T} = \frac{i\hbar}{4}\left\{\frac{i\hbar}{m}\nabla^2 - \frac{d}{dt} + \frac{\partial}{\partial t}\right\} \tag{66}$$

and in the consideration of eq (66) and eq (52), I get

$$\tau\psi = \frac{i\hbar}{4}\left\{\frac{i\hbar}{m}\nabla^2\psi - \frac{d\psi}{dt} + \frac{\partial\psi}{\partial t}\right\} \tag{67}$$

8. Saurav equation and momentum eigenvalues

Now considering the eigenvalue equation (38) for momentum operator, I get the momentum eigenvalue equation,

$$\hat{P}\psi = \Pi\psi \tag{68}$$

where Π is the momentum eigenvalue.
and in the consideration of eq (68) and eq (13), I get the momentum eigenvalue equation,

$$\Pi\psi = -i\hbar\nabla\psi \tag{69}$$

Now considering the eigenvalue equation (38) for $\hat{P}^2$ operator, I get

$$\hat{P}^2\psi = \Pi^2\psi \tag{70}$$

And assuming;

$$\hat{P}^2 = 2m\hat{T} \tag{71}$$

and in the consideration of eq (71) and eq (59), I get

$$\hat{P}^2 = i\hbar m\left\{\frac{\partial}{\partial t} - \frac{d}{dt}\right\} \tag{72}$$

and in the consideration of eq (72) and eq (70), I get

$$\Pi^2\psi = i\hbar m\left\{\frac{\partial\psi}{\partial t} - \frac{d\psi}{dt}\right\} \tag{73}$$

and in the consideration of eq (55) and eq (71), I get

$$\hat{P}^2 = i\hbar m\left\{\frac{d}{dt} + 2\frac{i\hbar}{m}\nabla^2 - \frac{\partial}{\partial t}\right\} \tag{74}$$

and in the consideration of eq (74) and eq (70), I get

$$\Pi^2\psi = i\hbar m\left\{\frac{d\psi}{dt} + 2\frac{i\hbar}{m}\nabla^2\psi - \frac{\partial\psi}{\partial t}\right\} \tag{75}$$

and in the consideration of eq (62) and eq (71), I get

$$\hat{P}^2 = 2i\hbar m\left\{\frac{\partial}{\partial t} - \frac{i\hbar}{2m}\nabla^2 - \frac{d}{dt}\right\} \tag{76}$$

and in the consideration of eq (70) and eq (76), I get

$$\Pi^2\psi = 2i\hbar m\left\{\frac{\partial\psi}{\partial t} - \frac{i\hbar}{2m}\nabla^2\psi - \frac{d\psi}{dt}\right\} \tag{77}$$

and in the consideration of eq (71) and eq (64), I get

$$\hat{P}^2 = 4i\hbar m\left\{\frac{d}{dt} + \frac{5}{4}\frac{i\hbar}{m}\nabla^2 - \frac{\partial}{\partial t}\right\} \tag{78}$$

and in the consideration of eq (78) and eq (70), I get

$$\Pi^2\psi = 4i\hbar m\left\{\frac{d\psi}{dt} + \frac{5}{4}\frac{i\hbar}{m}\nabla^2\psi - \frac{\partial\psi}{\partial t}\right\} \tag{79}$$

Now considering eq (5) in the consideration of linear operator analysis, I get

$$\hat{P}^2 = m\{\hat{H} + \hat{L}\} \tag{80}$$

and in the consideration of eq (80), eq (9) and eq (51), I get

$$\hat{P}^2 = \frac{i\hbar m}{2}\left\{\frac{i\hbar}{m}\nabla^2 - \frac{d}{dt} + \frac{\partial}{\partial t}\right\} \tag{81}$$

and in the consideration of consideration of eq (81) and eq (70), I get

$$\Pi^2\psi = \frac{i\hbar m}{2}\left\{\frac{i\hbar}{m}\nabla^2\psi - \frac{d\psi}{dt} + \frac{\partial\psi}{\partial t}\right\} \tag{82}$$

9. Saurav equation and potential energy eigenvalues

Now considering the eigenvalue equation (38) for potential energy operator, I get the potential energy eigenvalue equation,

$$\hat{V}\psi = \text{Б}\,\psi \tag{83}$$

where Б is the potential energy eigenvalues.
and in the consideration of eq (11) and eq (83), I get

$$\text{Б}\,\psi = \frac{i\hbar}{2}\left\{\frac{d\psi}{dt} + \frac{\partial\psi}{\partial t}\right\} \tag{84}$$

which is Saurav equation for potential energy eigenvalues.
and in the consideration of Saurav equation (15) and eq (84), I get

$$\text{Б}\,\psi = i\hbar\frac{d\psi}{dt} - \frac{\hbar^2}{2m}\nabla^2\psi \tag{85}$$

which is another Saurav equation for potential energy eigenvalues.
and in the consideration of eq (85) and eq (83), I get the new potential energy operator,

$$\hat{V} = i\hbar\frac{d}{dt} - \frac{\hbar^2}{2m}\nabla^2 = i\hbar\left\{\frac{d}{dt} + \frac{i\hbar}{2m}\nabla^2\right\} \tag{86}$$

and in the consideration of Saurav equation (15) and eq (84), I get

$$Б\psi = i\hbar\frac{\partial\psi}{\partial t} + \frac{\hbar^2}{2m}\nabla^2\psi \tag{87}$$

and in the consideration of eq (87) and eq (83), I get the another new potential energy operator,

$$\hat{V} = i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\nabla^2 = i\hbar\left\{\frac{\partial}{\partial t} - \frac{i\hbar}{2m}\nabla^2\right\} \tag{88}$$

Now assuming

$$\hat{V} = \hat{H} - \hat{T} \tag{89}$$

and

$$\hat{V} = \hat{T} - \hat{L} \tag{90}$$

and in the consideration of eq (89), eq (42) and eq (59), I get

$$\hat{V} = \frac{i\hbar}{2}\left\{3\frac{d}{dt} + 2\frac{i\hbar}{m}\nabla^2 - \frac{\partial}{\partial t}\right\} \tag{91}$$

and applying it to eigenfunction ψ in the consideration of potential energy eigenvalue equation (83), I get

$$Б\psi = \frac{i\hbar}{2}\left\{3\frac{d\psi}{dt} + 2\frac{i\hbar}{m}\nabla^2\psi - \frac{\partial\psi}{\partial t}\right\} \tag{92}$$

and in the consideration of eq (89), eq (9) and eq (55), I get

$$\hat{V} = \frac{i\hbar}{2}\left\{3\frac{\partial}{\partial t} - 2\frac{i\hbar}{m}\nabla^2 - \frac{d}{dt}\right\} \tag{93}$$

and applying it to ψ in the consideration of eq (83), I get

$$Б\,\psi = \frac{i\hbar}{2}\left\{3\frac{\partial\psi}{\partial t} - 2\frac{i\hbar}{m}\nabla^2\psi - \frac{d\psi}{dt}\right\} \tag{94}$$

and in the consideration of eq (89), eq (42) and eq (62), I get

$$\hat{V} = i\hbar\left\{2\frac{d}{dt} + \frac{3}{2}\frac{i\hbar}{m}\nabla^2 - \frac{\partial}{\partial t}\right\} \tag{95}$$

and applying it to ψ in the consideration of eq (83), I get

$$Б\,\psi = i\hbar\left\{2\frac{d\psi}{dt} + \frac{3}{2}\frac{i\hbar}{m}\nabla^2\psi - \frac{\partial\psi}{\partial t}\right\} \tag{96}$$

and in the consideration of eq (89), eq (9) and eq (64), I get

$$\hat{V} = i\hbar\left\{3\frac{\partial}{\partial t} - \frac{5}{2}\frac{i\hbar}{m}\nabla^2 - 2\frac{d}{dt}\right\} \tag{97}$$

and applying it to ψ in the consideration of eq (83), I get

$$Б\,\psi = i\hbar\left\{3\frac{\partial\psi}{\partial t} - \frac{5}{2}\frac{i\hbar}{m}\nabla^2\psi - 2\frac{d\psi}{dt}\right\} \tag{98}$$

and in the consideration of eq (89), eq (64) and eq (42), I get

$$\hat{V} = i\hbar\left\{2\frac{\partial}{\partial t} - \frac{3}{2}\frac{i\hbar}{m}\nabla^2 - \frac{d}{dt}\right\} \tag{99}$$

and applying it to ψ in the consideration of eq (83), I get

$$Б\,\psi = i\hbar\left\{2\frac{\partial\psi}{\partial t} - \frac{3}{2}\frac{i\hbar}{m}\nabla^2\psi - \frac{d\psi}{dt}\right\} \tag{100}$$

and in the consideration of eq (89), eq (64) and eq (44), I get

$$\hat{V} = \frac{i\hbar}{2}\left\{5\frac{\partial}{\partial t} - 4\frac{i\hbar}{m}\nabla^2 - 3\frac{d}{dt}\right\} \tag{101}$$

and applying it to ψ in the consideration of eq (83), I get

$$Б\psi = \frac{i\hbar}{2}\left\{5\frac{\partial\psi}{\partial t} - 4\frac{i\hbar}{m}\nabla^2\psi - 3\frac{d\psi}{dt}\right\} \tag{102}$$

and in the consideration of eq (89), eq (9) and eq (66), I get

$$\hat{V} = \frac{i\hbar}{4}\left\{\frac{d}{dt} - \frac{i\hbar}{m}\nabla^2 + 3\frac{\partial}{\partial t}\right\} \tag{103}$$

and applying it to ψ in the consideration of eq (83), I get

$$Б\psi = \frac{i\hbar}{4}\left\{\frac{d\psi}{dt} - \frac{i\hbar}{m}\nabla^2\psi + 3\frac{\partial\psi}{\partial t}\right\} \tag{104}$$

and in the consideration of eq (89), eq (42) and eq (66), I get

$$\hat{V} = \frac{i\hbar}{4}\left\{5\frac{d}{dt} + 3\frac{i\hbar}{m}\nabla^2 - \frac{\partial}{\partial t}\right\} \tag{105}$$

and applying it to ψ in the consideration of eq (83), I get

$$Б\psi = \frac{i\hbar}{4}\left\{5\frac{d\psi}{dt} + 3\frac{i\hbar}{m}\nabla^2\psi - \frac{\partial\psi}{\partial t}\right\} \tag{106}$$

And considering eq (10) in the consideration of linear operator analysis, I get

$$\hat{V} = \frac{1}{2}\{\hat{H} - \hat{L}\} \tag{107}$$

and in the consideration of eq (107), eq (42) and eq (51), I get

$$\hat{V} = \frac{i\hbar}{4}\left\{3\frac{d}{dt} + \frac{i\hbar}{m}\nabla^2 + \frac{\partial}{\partial t}\right\} \tag{108}$$

and applying it to ψ in the consideration of eq (108) and eq (83), I get

$$Б\psi = \frac{i\hbar}{4}\left\{3\frac{d\psi}{dt} + \frac{i\hbar}{m}\nabla^2\psi + \frac{\partial\psi}{\partial t}\right\} \tag{109}$$

and in the consideration of eq (90), eq (7) and eq (64), I get

$$\hat{V} = i\hbar\left\{3\frac{d}{dt} + \frac{5}{2}\frac{i\hbar}{m}\nabla^2 - 2\frac{\partial}{\partial t}\right\} \tag{110}$$

and applying it to ψ in the consideration of eq (83), I get

$$Б\,\psi = i\hbar\left\{3\frac{d\psi}{dt} + \frac{5}{2}\frac{i\hbar}{m}\nabla^2\psi - 2\frac{\partial\psi}{\partial t}\right\} \tag{111}$$

and in the consideration of eq (90), eq (64) and eq (51), I get

$$\hat{V} = \frac{i\hbar}{2}\left\{5\frac{d}{dt} + 4\frac{i\hbar}{m}\nabla^2 - 3\frac{\partial}{\partial t}\right\} \tag{112}$$

and applying it to ψ in the consideration of eq (83), I get

$$Б\,\psi = \frac{i\hbar}{2}\left\{5\frac{d\psi}{dt} + 4\frac{i\hbar}{m}\nabla^2\psi - 3\frac{\partial\psi}{\partial t}\right\} \tag{113}$$

and in the consideration of eq (90), eq (66) and eq (48), I get

$$\hat{V} = \frac{i\hbar}{4}\left\{5\frac{\partial}{\partial t} - 3\frac{i\hbar}{m}\nabla^2 - \frac{d}{dt}\right\} \tag{114}$$

and applying it to ψ in the consideration of eq (83), I get

$$Б\,\psi = \frac{i\hbar}{4}\left\{5\frac{\partial\psi}{\partial t} - 3\frac{i\hbar}{m}\nabla^2\psi - \frac{d\psi}{dt}\right\} \tag{115}$$

10. Fundamental equations of quantum mechanics

With the help of given equations and results I have got some fundamental equations of quantum mechanics which are clever enough to describe the quantum phenomena's.
Now applying eq (11) to eq (43), I get the time independent Schrödinger equation,

$$E\psi = -\frac{\hbar^2}{2m}\nabla^2\psi + \hat{V}\psi \tag{116}$$

and in the consideration of eq (116) and eq (83), I get the new time independent Schrödinger equation,

$$E\psi = -\frac{\hbar^2}{2m}\nabla^2\psi + Б\psi \tag{117}$$

and in the consideration of eq (117), eq (68), eq (69) and eq (70), I get

$$E\psi = \frac{\Pi^2}{2m}\psi + Б\psi \tag{118}$$

And with the help of Saurav equation (15) in the consideration of eq (40) and eq (7), I get the time independent Saurav equation,

$$E\psi = -\frac{\hbar^2}{m}\nabla^2\psi - \hat{L}\psi \tag{119}$$

and in the consideration of eq (119) and eq (45), I get the new time independent Saurav equation,

$$E\psi = -\frac{\hbar^2}{m}\nabla^2\psi - \lambda\psi \tag{120}$$

and in the consideration of eq (120), eq (68), eq (69) and eq (70), I get

$$E\psi = \frac{\Pi^2}{m}\psi - \lambda\psi \tag{121}$$

and writing eq (120) in its generalized form in the consideration of eq (40), I get

$$i\hbar\frac{\partial\psi(q_\alpha,t)}{\partial t} + \frac{\hbar^2}{m}\nabla^2\psi(q_\alpha,t) + L(q_\alpha,\dot{q}_\alpha,t)\psi(q_\alpha,t) = 0 \tag{122}$$

which is generalized Saurav equation,
And in the consideration of eq (46) and eq (85), I get another time independent Saurav equation,

$$\lambda\psi = -\frac{\hbar^2}{2m}\nabla^2\psi - Б\psi \tag{123}$$

and in the consideration of eq (123) ,eq (68), eq (69), and eq (70), I get

$$\lambda\psi = \frac{\Pi^2}{2m}\psi - Б\psi \tag{124}$$

and writing eq (123) in the generalized form in the consideration of eq (46), I get

$$i\hbar\frac{d\psi(q_\alpha,t)}{dt} - \frac{\hbar^2}{2m}\nabla^2\psi(q_\alpha,t) - V(q_\alpha t)\psi(q_\alpha,t) = 0 \tag{125}$$

which is another generalized Saurav equation.
And writing eq (41) in its generalized form, I get

$$i\hbar\frac{d\psi(q_\alpha,t)}{dt} - \frac{\hbar^2}{m}\nabla^2\psi(q_\alpha,t) - E(q_\alpha,p_\alpha)\psi(q_\alpha,t) = 0 \tag{126}$$

which is another generalized Saurav equation.
And writing eq (117) in its generalized form, I get

$$E(q_\alpha,p_\alpha)\psi(q_\alpha,t) + \frac{\hbar^2}{2m}\nabla^2\psi(q_\alpha,t) - V(q_\alpha,t)\psi(q_\alpha,t) = 0 \tag{127}$$

which is another generalized Schrödinger equation.
And writing eq (120) in its generalized form, I get

$$E(q_\alpha,p_\alpha)\psi(q_\alpha,t) + \frac{\hbar^2}{m}\nabla^2\psi(q_\alpha,t) + L(q_\alpha,\dot{q}_\alpha,t)\psi(q_\alpha,t) = 0 \tag{128}$$

which is another generalized Saurav equation.
And writing eq (123) in its generalized form, I get

$$L(q_\alpha,\dot{q}_\alpha,t)\psi(q_\alpha,t) + \frac{\hbar^2}{2m}\nabla^2\psi(q_\alpha,t) + V(q_\alpha,t)\psi(q_\alpha,t) = 0 \tag{129}$$

which is another generalized Saurav equation.
Now writing eq (118) in its generalized form, I get

$$E(q_\alpha,p_\alpha)\psi(q_\alpha,t) - \frac{p_\alpha^2}{2m}\psi(q_\alpha,t) - V(q_\alpha,t)\psi(q_\alpha,t) = 0 \tag{130}$$

and writing eq (121) in its generalized form, I get

$$E(q_\alpha, p_\alpha)\psi(q_\alpha, t) - \frac{p_\alpha^2}{m}\psi(q_\alpha, t) + L(q_\alpha, \dot{q}_\alpha . t)\psi(q_\alpha, t) = 0 \qquad (131)$$

and writing eq (124) in its generalized form, I get

$$L(q_\alpha, \dot{q}_\alpha, t)\psi(q_\alpha, t) - \frac{p_\alpha^2}{2m}\psi(q_\alpha, t) + V(q_\alpha, t)\psi(q_\alpha, t) = 0 \qquad (132)$$

and in the consideration of eq (130) and eq (40), I get

$$i\hbar\frac{\partial\psi(q_\alpha, t)}{\partial t} - \frac{p_\alpha^2}{2m}\psi(q_\alpha, t) - V(q_\alpha, t)\psi(q_\alpha, t) = 0 \qquad (133)$$

and in the consideration of eq (131) and eq (40), I get

$$i\hbar\frac{\partial\psi(q_\alpha, t)}{\partial t} - \frac{p_\alpha^2}{m}\psi(q_\alpha, t) + L(q_\alpha, \dot{q}_\alpha, t)\psi(q_\alpha, t) = 0 \qquad (134)$$

and in the consideration of eq (132) and eq (46), I get

$$i\hbar\frac{d\psi(q_\alpha, t)}{dt} + \frac{p_\alpha^2}{2m}\psi(q_\alpha, t) - V(q_\alpha, t)\psi(q_\alpha, t) = 0 \qquad (135)$$

and in the consideration of eq (131) and eq (46), I get

$$i\hbar\frac{d\psi(q_\alpha, t)}{dt} + \frac{p_\alpha^2}{m}\psi(q_\alpha, t) - E(q_\alpha, p_\alpha)\psi(q_\alpha, t) = 0 \qquad (136)$$

and considering eq (131) in the consideration of eq (40) and eq (46), I get

$$i\hbar\frac{\partial\psi(q_\alpha, t)}{\partial t} - \frac{p_\alpha^2}{m}\psi(q_\alpha, t) - i\hbar\frac{d\psi(q_\alpha, t)}{dt} = 0 \qquad (137)$$

11. Laplacian eigenoperator equation

Let assuming

$$\hat{\mathcal{A}}\psi = \mathcal{A}\psi \qquad (138)$$

which is eigenoperator equation where $\hat{\mathcal{A}}$ is a quantum mechanical operator and $\mathcal{A}$ is a classical operator.
and considering eq (138) for Laplacian operator in the consideration of eq (19), I get

$$\hat{\nabla}_{\alpha}^{2}\psi = \nabla_{\alpha}^{2}\psi \tag{139}$$

which is Laplacian eigenoperator equation where $\hat{\nabla}_{\alpha}^{2}$ is quantum mechanical Laplacian operator (20) and ∇_{α}^{2} is classical Laplacian operator.
eigenoperator equation (138) can be used to describe all the operators having both quantum mechanical and classical operator values.
Now considering Laplacian eigenoperator equation (139) in the consideration of eq (20), I get

$$\nabla_{\alpha}^{2}\psi = \frac{i}{\hbar}\left\{[\nabla_{\alpha}^{2}s] + \frac{i}{\hbar}[\nabla_{\alpha}s]^{2}\right\}\psi \tag{140}$$

which is Laplacian eigenoperator equation. And considering eq (140) in the consideration of eq (13) and eq (70), I get

$$[\nabla_{\alpha}s]^{2} - i\hbar[\nabla_{\alpha}^{2}s] - p_{\alpha}^{2} = 0 \tag{141}$$

and as a limiting case $(\hbar \to 0)$ it reduces to classical one.

12. Saurav equation and action operator

With a simple treatment of given equations and results I have arrived at the fact that all the operators in quantum mechanics may be derived with the help of quantum mechanical action (action operator) in the consideration of their corresponding relation.
Let

$$L(q_{\alpha}, \dot{q}_{\alpha}, t) = \frac{ds}{dt} \tag{142}$$

and putting it in eq (28), I get

$$\frac{\partial s}{\partial t}\psi(q_{\alpha},t) + \frac{[\nabla_{\alpha}s]^{2}}{m}\psi(q_{\alpha},t) - \frac{i\hbar}{m}[\nabla_{\alpha}^{2}s]\psi(q_{\alpha},t) - \frac{ds}{dt}\psi(q_{\alpha},t) = 0 \tag{143}$$

and applying the quantum mechanical Laplacian operator (20) to generalized Schrödinger equation (21) and considering eq (8) and Hamilton-Jacobi equation (3), I get

$$\frac{\partial s}{\partial t}\psi(q_\alpha,t)+\frac{[\nabla_\alpha s]^2}{2m}\psi(q_\alpha,t)-\frac{i\hbar}{2m}[\nabla_\alpha^2 s]\psi(q_\alpha,t)+V(q_\alpha,t)\psi(q_\alpha,t)=0 \tag{144}$$

and from eq (143) and eq (144), I get

$$\frac{ds}{dt}-\frac{[\nabla_\alpha s]^2}{2m}+\frac{i\hbar}{2m}[\nabla_\alpha^2 s]+V(q_\alpha,t)=0 \tag{145}$$

and

$$2\frac{\partial s}{\partial t}+\frac{3}{2}\frac{[\nabla_\alpha s]^2}{m}-\frac{3}{2}\frac{i\hbar}{m}[\nabla_\alpha^2 s]+V(q_\alpha,t)-\frac{ds}{dt}=0 \tag{146}$$

and

$$\frac{\partial s}{\partial t}\psi(q_\alpha,t)+\frac{ds}{dt}\psi(q_\alpha,t)+2V(q_\alpha,t)\psi(q_\alpha,t)=0 \tag{147}$$

and canceling $\psi(q_\alpha,t)$ in eq (147), I get

$$\frac{\partial s}{\partial t}+2V(q_\alpha,t)+\frac{ds}{dt}=0 \tag{148}$$

which is potential energy in the terms of action.
And now we need to know all the observables in the terms of action to get their corresponding operator values in the consideration of action operator.
Let introducing a new type of action which may be called Hamiltonian action.

$$\varphi=\int H(q_\alpha,p_\alpha,t)\,dt \tag{149}$$

and in the consideration of

$$H(q_\alpha,p_\alpha,t)=p_\alpha\dot{q}_\alpha-L(q_\alpha,\dot{q}_\alpha,t) \tag{150}$$

and

$$p_\alpha\dot{q}_\alpha=2T(\dot{q}_\alpha,t) \tag{151}$$

and in the consideration of eq (142), I get

$$\frac{ds}{dt} - 2T(\dot{q}_\alpha, t) + \frac{d\varphi}{dt} = 0 \tag{152}$$

and in the consideration of eq (149) and Hamilton-Jacobi equation (3), I get

$$\frac{d\varphi}{dt} + \frac{\partial s}{\partial t} = 0 \tag{153}$$

and in the consideration of eq (152) and eq (153), I get the kinetic energy in the terms of action,

$$\frac{\partial s}{\partial t} + 2T(\dot{q}_\alpha, t) - \frac{ds}{dt} = 0 \tag{154}$$

Now applying eq (154) for eq (146), I get

$$\frac{\partial s}{\partial t} + \frac{3}{2}\frac{[\nabla_\alpha s]^2}{m} - \frac{3}{2}\frac{i\hbar}{m}[\nabla_\alpha^2 s] - 2T(\dot{q}_\alpha, t) + V(q_\alpha, t) = 0 \tag{155}$$

and in the consideration of eq (3), I get

$$2T(\dot{q}_\alpha, t) - \frac{3}{2}\frac{[\nabla_\alpha s]^2}{m} + \frac{3}{2}\frac{i\hbar}{m}[\nabla_\alpha^2 s] + H(q_\alpha, p_\alpha, t) - V(q_\alpha, t) = 0 \tag{156}$$

and in the consideration of $(H = T + V)$, I get

$$\frac{[\nabla_\alpha s]^2}{2m} - \frac{i\hbar}{2m}[\nabla_\alpha^2 s] - T(\dot{q}_\alpha, t) = 0 \tag{157}$$

and putting eq (154) in eq (157), I get

$$\frac{\partial s}{\partial t} + \frac{[\nabla_\alpha s]^2}{m} - \frac{i\hbar}{m}[\nabla_\alpha^2 s] - \frac{ds}{dt} = 0 \tag{158}$$

and in the consideration of eq (157) and $(T = \frac{p^2}{2m})$, I get

$$[\nabla_\alpha s]^2 - i\hbar[\nabla_\alpha^2 s] - p_\alpha^2 = 0 \tag{159}$$

And now in the consideration of quantum mechanical Hamilton-Jacobi equation (22) and $(H = T + V)$ and $T = \frac{p^2}{2m}$ and eq (4), I get the quantum mechanical Hamilton-Jacobi equation,

$$\frac{\partial s}{\partial t} + H(q_\alpha, p_\alpha, t) - \frac{i\hbar}{2m}[\nabla_\alpha^2 s] = 0 \tag{160}$$

and as a limiting case $(\hbar \to 0)$, eq (157), eq (159) and eq (160) reduces to classical one.
Now considering the operator values of s and $V(q_\alpha, t)$ in eq (147) in the consideration of eq (11), I get

$$\hat{S} = -i\hbar \hat{I} \tag{161}$$

where $\hat{I}$ is some unitary operator defined as,

$$\hat{I}\psi = I\psi \tag{162}$$

which is unitary eigenvalue equation where I is the unit matrix defined as,

$$I = \begin{pmatrix} 1 & 0 & 0 & . & 0 \\ 0 & 1 & 0 & . & 0 \\ 0 & 0 & 1 & . & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & . & 1 \end{pmatrix} \tag{163}$$

now we have the action operator (quantum mechanical action). And derivating all the operators in the consideration of eq (161), we need to know the corresponding relationships between desired observable and action.
Let assuming

$$H(q_\alpha, p_\alpha, t)\psi(q_\alpha, t) = -\frac{\partial s}{\partial t}\psi(q_\alpha, t) \tag{164}$$

and for the operator values of $H(q_\alpha, p_\alpha, t)$ and s (161), I get

$$\hat{H}(q_\alpha, p_\alpha t) = -\frac{\partial \hat{s}}{\partial t} = i\hbar\frac{\partial}{\partial t} \tag{165}$$

and

$$L(q_\alpha, \dot{q}_\alpha, t)\psi(q_\alpha, t) = \frac{ds}{dt}\psi(q_\alpha, t) \tag{166}$$

and for the operator values of $L(q_\alpha, \dot{q}_\alpha, t)$ and s (161), I get

$$\hat{L}(q_\alpha, \dot{q}_\alpha, t) = \frac{d\hat{s}}{dt} = -i\hbar\frac{d}{dt} \tag{167}$$

and

$$p_\alpha \psi(q_\alpha, t) = [\nabla_\alpha s]\psi(q_\alpha, t) \tag{168}$$

and for the operator values of p_α and s (161), I get

$$\hat{p}_\alpha = \nabla_\alpha \hat{s} = -i\hbar\nabla_\alpha \tag{169}$$

and considering eq (147), I get

$$V(q_\alpha, t)\psi(q_\alpha, t) = -\frac{1}{2}\left\{\frac{\partial s}{\partial t} + \frac{ds}{dt}\right\}\psi(q_\alpha, t) \tag{170}$$

and for the operator values of $V(q_\alpha, t)$ and s (161), I get

$$\hat{V}(q_\alpha, t) = -\frac{1}{2}\left\{\frac{\partial \hat{s}}{\partial t} + \frac{d\hat{s}}{dt}\right\} = \frac{i\hbar}{2}\left\{\frac{\partial}{\partial t} + \frac{d}{dt}\right\} \tag{171}$$

and considering eq (154), I get

$$T(\dot{q}_\alpha, t)\psi(q_\alpha, t) = \frac{1}{2}\left\{\frac{ds}{dt} - \frac{\partial s}{\partial t}\right\}\psi(q_\alpha, t) \tag{172}$$

and for the operator values of $T(\dot{q}_\alpha, t)$ and s (161), I get

$$\hat{T}(\dot{q}_\alpha, t) = \frac{1}{2}\left\{\frac{d\hat{s}}{dt} - \frac{\partial \hat{s}}{\partial t}\right\} = -\frac{i\hbar}{2}\left\{\frac{d}{dt} - \frac{\partial}{\partial t}\right\} \tag{173}$$

13. Saurav equation in modern notations

Let the Saurav equations (15) and (16) may be written in Dirac notations of state vectors. The wave function in Dirac notations is given by $|\psi\rangle$ and called the *Ket* and vector $\langle\psi|$ is called a *Bra* which is the complex conjugate of Ket. And writing eq (15) in Dirac notations, I get

$$i\hbar\frac{\partial|\psi\rangle}{\partial t}+\frac{\hbar^2}{m}\nabla^2|\psi\rangle-i\hbar\frac{d|\psi\rangle}{dt}=0 \tag{174}$$

and getting the complex conjugate of eq (174), I get

$$i\hbar\frac{d\langle\psi|}{dt}+\frac{\hbar^2}{m}\nabla^2\langle\psi|-i\hbar\frac{\partial\langle\psi|}{\partial t}=0 \tag{175}$$

and writing Saurav equation (16) in Dirac notations, I get

$$\frac{d|\psi\rangle}{dt}+\frac{i\hbar}{m}\nabla^2|\psi\rangle-\frac{\partial|\psi\rangle}{\partial t}=0 \tag{176}$$

and getting the complex conjugate of eq (176), I get

$$\frac{d\langle\psi|}{dt}-\frac{i\hbar}{m}\nabla^2\langle\psi|-\frac{\partial\langle\psi|}{\partial t}=0 \tag{177}$$

eq (174) to eq (177) are Saurav equations in modern notations (Dirac notations) where $|\psi\rangle(Ket)$ is given by $\psi(q_\alpha,t)$ and $\langle\psi|(Bra)$ is given by $\psi^*(q_\alpha,t)$.

14. Eigenvalue equations in modern notations

Let the eigenvalue equations may be written in Dirac notations. The complex conjugate of hermitian operators $\hat{\mathcal{A}}$ is its transposed operator $\tilde{\hat{\mathcal{A}}}$ (See appendix (1), eq (241)) and the complex conjugate of their eigenvalues are unchanged as they are real quantities and it is well known that all the operators in quantum mechanics are hermitian, So

$$\hat{\mathrm{H}}|\psi\rangle=i\hbar\frac{\partial|\psi\rangle}{\partial t}=E|\psi\rangle \tag{178}$$

$$\tilde{\mathrm{H}}\langle\psi| = -i\hbar\frac{\partial\langle\psi|}{\partial t} = E\langle\psi| \tag{179}$$

$$\hat{L}|\psi\rangle = -i\hbar\frac{d|\psi\rangle}{dt} = \lambda|\psi\rangle \tag{180}$$

$$\tilde{\hat{L}}\langle\psi| = i\hbar\frac{d\langle\psi|}{dt} = \lambda\langle\psi| \tag{181}$$

$$\hat{P}_\alpha|\psi\rangle = -i\hbar\nabla_\alpha|\psi\rangle = \Pi|\psi\rangle \tag{182}$$

$$\tilde{\hat{P}}_\alpha\langle\psi| = i\hbar\nabla_\alpha\langle\psi| = \Pi\langle\psi| \tag{183}$$

$$\hat{V}|\psi\rangle = \frac{i\hbar}{2}\left\{\frac{\partial|\psi\rangle}{\partial t} + \frac{d|\psi\rangle}{dt}\right\} = \text{Б}|\psi\rangle \tag{184}$$

$$\tilde{\hat{V}}\langle\psi| = -\frac{i\hbar}{2}\left\{\frac{\partial\langle\psi|}{\partial t} + \frac{d\langle\psi|}{dt}\right\} = \text{Б}\langle\psi| \tag{185}$$

$$\hat{\mathrm{T}}|\psi\rangle = -\frac{i\hbar}{2}\left\{\frac{d|\psi\rangle}{dt} - \frac{\partial|\psi\rangle}{\partial t}\right\} = \tau|\psi\rangle \tag{186}$$

$$\tilde{\mathrm{T}}\langle\psi| = \frac{i\hbar}{2}\left\{\frac{d\langle\psi|}{dt} - \frac{\partial\langle\psi|}{\partial t}\right\} = \tau\langle\psi| \tag{187}$$

$$\hat{S}|\psi\rangle = -i\hbar\hat{I}|\psi\rangle = \Omega|\psi\rangle \tag{188}$$

$$\tilde{\hat{S}}\langle\psi| = i\hbar\tilde{\hat{I}}\langle\psi| = \Omega\langle\psi| \tag{189}$$

where

$$\alpha = (E, \lambda, \Pi,\ \text{Б}, \tau, \Omega) \tag{190}$$

are the eigenvalues of their corresponding operators.

15. On the quantum description of systems

The expactation value of an operator is given by,

$$\left\langle \hat{\mathcal{A}} \right\rangle = \langle \psi | \hat{\mathcal{A}} | \psi \rangle \tag{191}$$

Let derivating eq (191) w.r.t. time, I get

$$\frac{\partial \left\langle \hat{\mathcal{A}} \right\rangle}{\partial t} = \left\langle \frac{\partial \psi}{\partial t} \middle| \hat{\mathcal{A}} | \psi \right\rangle + \langle \psi | \frac{\partial \hat{\mathcal{A}}}{\partial t} | \psi \rangle + \left\langle \psi | \hat{\mathcal{A}} \middle| \frac{\partial \psi}{\partial t} \right\rangle \tag{192}$$

and in the consideration of eq (8), I get

$$\frac{\partial \psi}{\partial t} = -\frac{i}{\hbar} \hat{\mathrm{H}} \psi \qquad ; \qquad \left| \frac{\partial \psi}{\partial t} \right\rangle = -\frac{i}{\hbar} \left| \hat{\mathrm{H}} \psi \right\rangle \tag{193}$$

and getting its complex conjugate, I get

$$\left(\frac{\partial \psi}{\partial t} \right)^{*} = \frac{i}{\hbar} \left(\tilde{\hat{\mathrm{H}}}^{*} \psi \right)^{*} \qquad ; \qquad \left\langle \frac{\partial \psi}{\partial t} \right| = \frac{i}{\hbar} \left\langle \tilde{\hat{\mathrm{H}}}^{*} \psi \right| \tag{194}$$

and putting eq (193) and eq (194) in eq (192), I get

$$\frac{\partial \left\langle \hat{\mathcal{A}} \right\rangle}{\partial t} = \left\langle \frac{\partial \hat{\mathcal{A}}}{\partial t} \right\rangle + \frac{i}{\hbar} \left[\left\langle \tilde{\hat{\mathrm{H}}}^{*} \psi \middle| \hat{\mathcal{A}} | \psi \right\rangle - \langle \psi | \hat{\mathcal{A}} \hat{\mathrm{H}} | \psi \rangle \right] \tag{195}$$

and assuming,

$$\left\langle \tilde{\hat{\mathrm{H}}}^{*} \psi \middle| \hat{\mathcal{A}} | \psi \right\rangle = \langle \psi | \hat{\mathrm{H}} \hat{\mathcal{A}} | \psi \rangle \tag{196}$$

(See appendix (1), eq (234)) and considering of eq (196) in eq (195), I get

$$\frac{\partial \left\langle \hat{\mathcal{A}} \right\rangle}{\partial t} = \left\langle \frac{\partial \hat{\mathcal{A}}}{\partial t} \right\rangle + \frac{i}{\hbar} \left\langle [\hat{\mathrm{H}}, \hat{\mathcal{A}}]_{-} \right\rangle \tag{197}$$

which is the *quantum mechanical equation of motion*.
Now getting the exact time derivative of eq (191), I get

$$\frac{d\langle\hat{\mathcal{A}}\rangle}{dt}=\left\langle\frac{d\psi}{dt}\middle|\hat{\mathcal{A}}\middle|\psi\right\rangle+\langle\psi|\frac{d\hat{\mathcal{A}}}{dt}|\psi\rangle+\left\langle\psi\middle|\hat{\mathcal{A}}\middle|\frac{d\psi}{dt}\right\rangle \tag{198}$$

and in the consideration of eq (6), I get

$$\frac{d\psi}{dt}=\frac{i}{\hbar}\hat{L}\psi \qquad ; \qquad \left|\frac{d\psi}{dt}\right\rangle=\frac{i}{\hbar}\left|\hat{L}\psi\right\rangle \tag{199}$$

and getting its complex conjugate, I get

$$\left(\frac{d\psi}{dt}\right)^{*}=-\frac{i}{\hbar}\left(\tilde{\hat{L}}^{*}\psi\right)^{*} \qquad ; \qquad \left\langle\frac{d\psi}{dt}\right|=-\frac{i}{\hbar}\left\langle\tilde{\hat{L}}^{*}\psi\right| \tag{200}$$

and putting eq (199) and eq (200) in eq (198), I get

$$\frac{d\langle\hat{\mathcal{A}}\rangle}{dt}=\left\langle\frac{d\hat{\mathcal{A}}}{dt}\right\rangle+\frac{i}{\hbar}\left[\langle\psi|\hat{\mathcal{A}}\hat{L}|\psi\rangle-\left\langle\tilde{\hat{L}}^{*}\psi\middle|\hat{\mathcal{A}}\middle|\psi\right\rangle\right] \tag{201}$$

and assuming,

$$\left\langle\tilde{\hat{L}}^{*}\psi\middle|\hat{\mathcal{A}}\middle|\psi\right\rangle=\langle\psi|\hat{L}\hat{\mathcal{A}}|\psi\rangle \tag{202}$$

and in the consideration of eq (202) in eq (201), I get

$$\frac{d\langle\hat{\mathcal{A}}\rangle}{dt}=\left\langle\frac{d\hat{\mathcal{A}}}{dt}\right\rangle-\frac{i}{\hbar}\left\langle[\hat{L},\hat{\mathcal{A}}]_{-}\right\rangle \tag{203}$$

which is the *quantum mechanical equation of motion*.
And now getting the gradient of $\langle\hat{\mathcal{A}}\rangle$ in the consideration of eq (191), I get

$$\nabla_{\alpha}\langle\hat{\mathcal{A}}\rangle=\langle\nabla_{\alpha}\psi|\hat{\mathcal{A}}|\psi\rangle+\langle\psi|\nabla_{\alpha}\hat{\mathcal{A}}|\psi\rangle+\langle\psi|\hat{\mathcal{A}}|\nabla_{\alpha}\psi\rangle \tag{204}$$

and in the consideration of eq (12), I get

$$\nabla_{\alpha}\psi=\frac{i}{\hbar}\hat{\mathrm{P}}_{\alpha}\psi \qquad ; \qquad |\nabla_{\alpha}\psi\rangle=\frac{i}{\hbar}\left|\hat{\mathrm{P}}_{\alpha}\psi\right\rangle \tag{205}$$

and getting its complex conjugate, I get

$$(\nabla_\alpha \psi)^* = -\frac{i}{\hbar}\left(\tilde{\hat{\mathrm{P}}}_\alpha^* \psi\right)^* \qquad ; \qquad \langle \nabla_\alpha \psi| = -\frac{i}{\hbar}\left\langle \tilde{\hat{\mathrm{P}}}_\alpha{}^* \psi\right| \tag{206}$$

and putting eq (205) and eq (206) in eq (204), I get

$$\nabla_\alpha \left\langle \hat{\mathcal{A}} \right\rangle = \left\langle \nabla_\alpha \hat{\mathcal{A}} \right\rangle + \frac{i}{\hbar}\left[\langle \psi | \hat{\mathcal{A}} \hat{\mathrm{P}}_\alpha | \psi \rangle - \left\langle \tilde{\hat{\mathrm{P}}}_\alpha^* \psi \middle| \hat{\mathcal{A}} \middle| \psi \right\rangle \right] \tag{207}$$

and assuming,

$$\left\langle \tilde{\hat{\mathrm{P}}}_\alpha^* \psi \middle| \hat{\mathcal{A}} \middle| \psi \right\rangle = \langle \psi | \hat{\mathrm{P}}_\alpha \hat{\mathcal{A}} | \psi \rangle \tag{208}$$

and in the consideration of eq (208) in eq (207), I get

$$\nabla_\alpha \left\langle \hat{\mathcal{A}} \right\rangle = \left\langle \nabla_\alpha \hat{\mathcal{A}} \right\rangle - \frac{i}{\hbar}\left\langle [\hat{\mathrm{P}}_\alpha , \hat{\mathcal{A}}]_- \right\rangle \tag{209}$$

which is the *quantum mechanical equation of motion*.
Now considering the eigenvalue equation,

$$\hat{\mathcal{A}} | \psi \rangle = \alpha | \psi \rangle \tag{210}$$

and in the consideration of eq (191), I get

$$\alpha = \frac{\langle \psi | \hat{\mathcal{A}} | \psi \rangle}{\langle \psi | \hat{I} | \psi \rangle} \tag{211}$$

which is the *quantum description of systems*.
Now writing eigenvalue equations in the view to eq (211), I get

$$E = \frac{\langle \psi | \hat{\mathrm{H}} | \psi \rangle}{\langle \psi | \hat{I} | \psi \rangle} \tag{212}$$

$$\lambda = \frac{\langle \psi | \hat{L} | \psi \rangle}{\langle \psi | \hat{I} | \psi \rangle} \tag{213}$$

$$\Pi = \frac{\langle \psi | \hat{\mathrm{P}} | \psi \rangle}{\langle \psi | \hat{I} | \psi \rangle} \tag{214}$$

$$\text{Б} = \frac{\langle \psi | \hat{V} | \psi \rangle}{\langle \psi | \hat{I} | \psi \rangle} \tag{215}$$

$$\tau = \frac{\langle \psi | \hat{\mathrm{T}} | \psi \rangle}{\langle \psi | \hat{I} | \psi \rangle} \tag{216}$$

$$\Omega = \frac{\langle \psi | \hat{S} | \psi \rangle}{\langle \psi | \hat{I} | \psi \rangle} \tag{217}$$

Now considering an operator defined as,

$$\hat{f} = \hat{\mathcal{A}}\hat{\mathcal{B}} \tag{218}$$

and getting its expectation value in the consideration of eq (191), I get

$$\langle \hat{f} \rangle = \langle \hat{\mathcal{A}}\hat{\mathcal{B}} \rangle = \langle \psi | \hat{\mathcal{A}}\hat{\mathcal{B}} | \psi \rangle \tag{219}$$

and considering eigenvalue equation (210) for operator $\hat{\mathcal{B}}$ in eq (219), I get

$$\langle \psi | \hat{\mathcal{A}}\hat{\mathcal{B}} | \psi \rangle = \beta \langle \psi | \hat{\mathcal{A}} | \psi \rangle \tag{220}$$

and

$$\beta = \frac{\langle \psi | \hat{\mathcal{A}}\hat{\mathcal{B}} | \psi \rangle}{\langle \psi | \hat{\mathcal{A}} | \psi \rangle} \tag{221}$$

where β is eigenvalue of $\hat{\mathcal{B}}$ operator.
And considering an operator defined as,

$$\hat{f} = \hat{\mathcal{A}}_1 \hat{\mathcal{A}}_2 \ldots\ldots \hat{\mathcal{A}}_i \ldots\ldots \hat{\mathcal{A}}_n \tag{222}$$

and considering it in eq (211) in the consideration of eigenvalue equation (210) for operator $\hat{\mathcal{A}}_i$, I get

$$\alpha_1\alpha_2....\alpha_n = \frac{\langle\psi|\hat{\mathcal{A}}_1\hat{\mathcal{A}}_2....\hat{\mathcal{A}}_n|\psi\rangle}{\langle\psi|\hat{I}|\psi\rangle} \tag{223}$$

where α_i is eigenvalue of $\hat{\mathcal{A}}_i$ operator and considering eq (222) in eq (221) in the consideration of eigenvalue equation (210) for operator $\hat{\mathcal{A}}_n$, I get

$$\alpha_n = \frac{\langle\psi|\hat{\mathcal{A}}_1\hat{\mathcal{A}}_2.......\hat{\mathcal{A}}_n|\psi\rangle}{\langle\psi|\hat{\mathcal{A}}_1\hat{\mathcal{A}}_2....\hat{\mathcal{A}}_{n-1}|\psi\rangle} \tag{224}$$

where α_n is eigenvalue of operator $\hat{\mathcal{A}}_n$.
Now considering eq (221) for action operator, I get

$$\alpha = \frac{\langle\psi|\hat{S}\hat{\mathcal{A}}|\psi\rangle}{\langle\psi|\hat{S}|\psi\rangle} \tag{225}$$

and in the consideration of eq (225) and eq (161), I get

$$\alpha = \frac{i}{\hbar}\frac{\langle\psi|\hat{S}\hat{\mathcal{A}}|\psi\rangle}{\langle\psi|\hat{I}|\psi\rangle} \tag{226}$$

and in the consideration of eq (161) and eq (211), I get

$$\alpha = -i\hbar\frac{\langle\psi|\hat{\mathcal{A}}|\psi\rangle}{\langle\psi|\hat{S}|\psi\rangle} \tag{227}$$

which is the quantum description of systems with an interpretation of action operator. And in the consideration of eq (227) and eq (191), I get

$$\left\langle\hat{\mathcal{A}}\right\rangle = \frac{i}{\hbar}\alpha\left\langle\hat{S}\right\rangle \tag{228}$$

eq (225), eq (226), eq (227) and eq (228) can be written for all the operators (observables) in quantum theory.

Appendix

1. Hermitian operators-hermitian conjugate

Let considering eigenvalue equation (210) for hermitian operators (with real eigenvalue), I get

$$\hat{\mathcal{A}}|\psi\rangle = \alpha|\psi\rangle \tag{229}$$

and

$$\hat{\mathcal{A}}^{*}\langle\varphi| = \alpha\langle\varphi| \tag{230}$$

where ψ and φ are two different state vectors. And with a simple treatment of eq (229) and eq (230), I get

$$\left\langle\varphi\middle|\hat{\mathcal{A}}\psi\right\rangle - \hat{\mathcal{A}}^{*}\langle\varphi|\psi\rangle = 0 \tag{231}$$

and assuming,

$$\hat{\mathcal{A}}^{*}\langle\varphi|\psi\rangle = \left\langle\hat{\mathcal{A}}^{!}\varphi\middle|\psi\right\rangle \tag{232}$$

where $\hat{\mathcal{A}}^{!}$ is hermitian conjugate of $\hat{\mathcal{A}}$ operator and in the consideration of eq (232) and eq (231), I get

$$\left\langle\hat{\mathcal{A}}^{!}\varphi\middle|\psi\right\rangle = \left\langle\varphi\middle|\hat{\mathcal{A}}\psi\right\rangle \tag{233}$$

and for the transpose properties of $\hat{\mathcal{A}}$, assuming

$$\left\langle\tilde{\hat{\mathcal{A}}}^{*}\varphi\middle|\psi\right\rangle = \left\langle\varphi\middle|\hat{\mathcal{A}}\psi\right\rangle \tag{234}$$

and in the consideration of eq (233) and eq (234), I get

$$\hat{\mathcal{A}}^{!} = \tilde{\hat{\mathcal{A}}}^{*} \tag{235}$$

Now considering eq (231) for transposed operator $\tilde{\hat{\mathcal{A}}}$, I get

$$\left\langle\varphi\middle|\tilde{\hat{\mathcal{A}}}\psi\right\rangle = \tilde{\hat{\mathcal{A}}}^{*}\langle\varphi|\psi\rangle \tag{236}$$

and in the consideration of eq (235), I get

$$\left\langle \varphi \middle| \tilde{\hat{\mathcal{A}}}\psi \right\rangle = \hat{\mathcal{A}}^{!} \langle \varphi | \psi \rangle \tag{237}$$

or

$$\left\langle \varphi \middle| \tilde{\hat{\mathcal{A}}}\psi \right\rangle = \left\langle \hat{\mathcal{A}}^{*!}\varphi \middle| \psi \right\rangle \tag{238}$$

and in the consideration of eq (233) for transposed operator $\tilde{\hat{\mathcal{A}}}$, I get

$$\left\langle \varphi \middle| \tilde{\hat{\mathcal{A}}}\psi \right\rangle = \left\langle \tilde{\hat{\mathcal{A}}}^{!}\varphi \middle| \psi \right\rangle \tag{239}$$

and in the consideration of eq (238) and eq (239), I get

$$\tilde{\hat{\mathcal{A}}}^{!} = \hat{\mathcal{A}}^{!*} \tag{240}$$

and canceling for hermitian conjugate in eq (240), I get

$$\tilde{\hat{\mathcal{A}}} = \hat{\mathcal{A}}^{*} \tag{241}$$

Remark

In the given treatment I have used some assumptions as,

$$\hat{f}\langle \varphi | \psi \rangle = \left\langle \hat{f}^{*}\varphi \middle| \psi \right\rangle = \int \psi\, \hat{f}\varphi^{*}\, dq \tag{242}$$

where $\hat{f}$ is some operator.
and I have also used,

$$\int \varphi^{*}\, \hat{f}\psi\, dq = \int \psi\, \tilde{\hat{f}}\varphi^{*}\, dq \tag{243}$$

which can be written as eq (234) in the consideration of eq (242) assuming $\hat{f} = \hat{\mathcal{A}}$.
So it would be easier to understand this appendix with the help and consideration of eq (242) and eq (243).

2. Hermitian operators-quantum description of systems

The ratio of eigenvalues of two corresponding operators may be written in the consideration of eq (211), as

$$\frac{\alpha}{\beta} = \frac{\langle \psi | \hat{\mathcal{A}} | \psi \rangle}{\langle \psi | \hat{\mathcal{B}} | \psi \rangle} \tag{244}$$

where α and β are eigenvalues of their corresponding operators $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$.
Now considering hermitian properties of $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ (i.e., α and β are real) and with a simple treatment of eq (244) in the consideration of eq (229) and eq (230), I get

$$\hat{\mathcal{A}}^{*} \langle \psi | \hat{\mathcal{B}} | \psi \rangle = \hat{\mathcal{B}}^{*} \langle \psi | \hat{\mathcal{A}} | \psi \rangle \tag{245}$$

and in the consideration of eq (232) in eq (245), I get

$$\left\langle \hat{\mathcal{A}}' \psi \middle| \hat{\mathcal{B}} \middle| \psi \right\rangle = \left\langle \hat{\mathcal{B}}' \psi \middle| \hat{\mathcal{A}} \middle| \psi \right\rangle \tag{246}$$

which is the corresponding equation of tow hermitian operators.
Now considering eq (245) for transposed properties of hermitian operators in the consideration of eq (241), I get

$$\tilde{\hat{\mathcal{A}}} \langle \psi | \hat{\mathcal{B}} | \psi \rangle = \tilde{\hat{\mathcal{B}}} \langle \psi | \hat{\mathcal{A}} | \psi \rangle \tag{247}$$

which is the transposed equation of two corresponding hermitian operators.
Now considering eq (247) in the consideration of eq (242) and eq (234) or considering eq (233), I get

$$\langle \psi | \hat{\mathcal{A}} \hat{\mathcal{B}} | \psi \rangle = \langle \psi | \hat{\mathcal{B}} \hat{\mathcal{A}} | \psi \rangle \tag{248}$$

and it can be written in the commutator form as,

$$\left\langle [\hat{\mathcal{A}}, \hat{\mathcal{B}}]_{-} \right\rangle = 0 \tag{249}$$

which holds for two corresponding hermitian operators.

Conclusion

Here is where I am ending my vision. But does it really end? Of course not, it just breaks off…. And one is tempted to ask what happened next. If there was an end was it followed by a descent?

But there are no descent in our understanding of nature. Time steadily moves and so does science, not as monotonously as time but in jumps or steps. These steps can be small or large but they never descent-knowledge cannot diminish, it can only increase. Understanding is accumulated but never depleted. In the periods of scientific revolutions the steps are steep and therefore difficult to follow.

Then what is an end of such a never-descending step-wise line? It is merely a roomy clearing before a new ascent whose outlines in the surrounding mist are still indistinct. One gets a false impression that one cannot go any higher. But it is just the steepness of the climb that creates for a time the illusion that an end has been conquered. In fact, no such end exists…. There is no end.

Any one can go higher. One can use my equations to arrive at his own good vision. One can apply my equations to get the energy eigenvalues of those systems for which the determination of potential energy is impossible. My equations can be used to extend many branches of natural science.

Thanks for the advances of science.

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