There are strong evidences that the universe is almost exclusively made up of baryons: indeed, anti-matter is completely absent in our Solar System – except in the particle accelerators – and the study of the cosmic rays indicates that there are no strong γ-rays coming from the annihilation of nucleons with anti-nucleons. Thus, and unless one assumes that matter and anti-matter are separated from each other at very large scales, it seems that anti-baryons have completely vanished. This Baryon Asymmetry of the Universe (BAU) is well known for a long time, but its origin remains unclear. In 1967, Sakharov[1] gave three general conditions that must have been satisfied in the early universe in order to produce such an asymmetry:

- The baryon number had to be violated;
- C and CP had to be violated too;
- Departure of equilibrium had to occur.

(without the two last conditions, it is indeed possible to have processes which violate B, but any asymmetry can be generated).

Several scenarios have since been suggested. One of the most attractive explanations is that the BAU was created during the Electroweak Phase Transition (EWPT); it is also the main theory which starts with a symmetric universe, i.e. which explains the BAU dynamically. This is the subject of intensive search, partly because the physics entering such processes will be soon testable in particle accelerators. This theory is called the electroweak baryogenesis, and is the main subject of this essay.

In a first section, we introduce generalities about the EWPT, emphasizing properties of the broken phase. We see how B violation can occur, essentially at high enough temperatures, and explain how the BAU could have been generated during the phase transition. Imposing then that this BAU is not washed out, we find a constraint on the Higgs vacuum expectation value (VEV) at the phase transition. In a second section, we see how this constraint affects the less known sector of the Standard Model (the Higgs sector), essentially by giving an upper bound to the Higgs mass. To do this, we briefly introduce how the Higgs field appears in the Standard Model and how it couples other fields. This allows us to calculate the effective potential in a perturbation expansion, taking into account thermal effects. Finally, we discover that the Standard Model is ruled out by the current experiments. As a conclusion, we therefore review some candidates for new physics that should enter at electroweak energy scale.

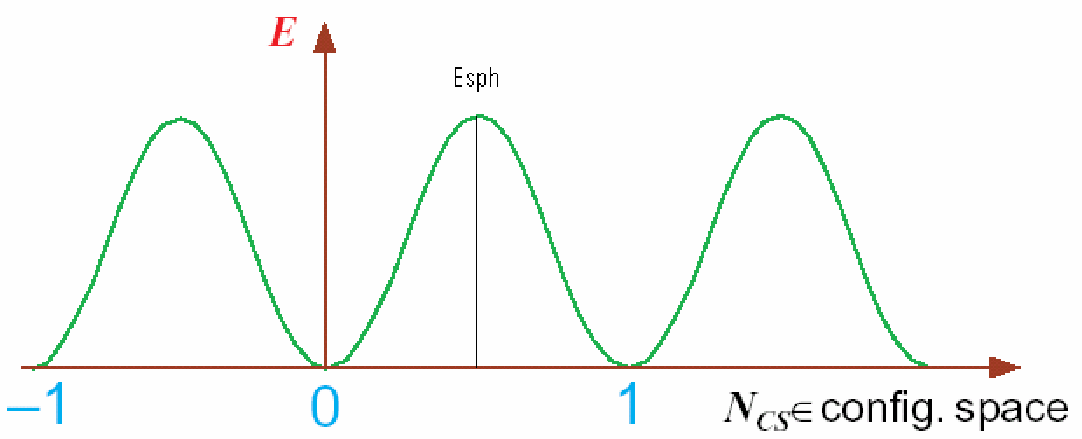
1) EWPT and BAU generation.

1)1. Anomalies and B violation.

T'Hooft [2] was the first to point out that in quantum mechanics, classically conserved currents may suffer from *anomalies*, and therefore that the associated charge could be non-conserved. Indeed it is the case of the GSW theory, i.e. based on a SU(2)xU(1) gauge theory, and the currents associated with baryon and lepton number have a non vanishing divergence: $\partial_\mu j^\mu_B = N_f \frac{1}{32\pi^2}\varepsilon^{\mu\nu\rho\sigma}(\frac{1}{2}g^2 F^a_{\mu\nu}F^a_{\rho\sigma} + \frac{1}{2}g'^2 f_{\mu\nu}f_{\rho\sigma})$ where N_f is the number of families of quarks and leptons, g, g' are the coupling constants associated with SU(2) and U(1), and F, f are the corresponding field strengths. The expression for the divergence of the lepton current is identical. It is then interesting to note that corresponding processes violate B and L in such a way that B-L is conserved. With this expression, one can integrate to find the variation of the baryon number:

$$B(t_f) - B(t_i) = N_f \frac{1}{32\pi^2}\varepsilon^{\mu\nu\rho\sigma}\int_{ti}^{tf} dt \int d^3x (\frac{1}{2}g^2 F^a_{\mu\nu}F^a_{\rho\sigma} + \frac{1}{2}g'^2 f_{\mu\nu}f_{\rho\sigma}) \equiv N_f(N_{CS}(t_f) - N_{CS}(t_i))$$

where we have introduced the Chern-Simon number N_{cs}. In the Weyl gauge, ie $A_0 = 0$, and in a pure gauge configuration, it takes integer values. Actually, by considering the configuration space, this non conservation implies the appearance of an infinite number of vacuaa (more precisely of an infinite number of classes of vacuua, because we group gauge equivalent vacuaa), characterized by the Chern-Simon number. These vacuua are topologically non-equivalent, and each of them has a definite baryon number.



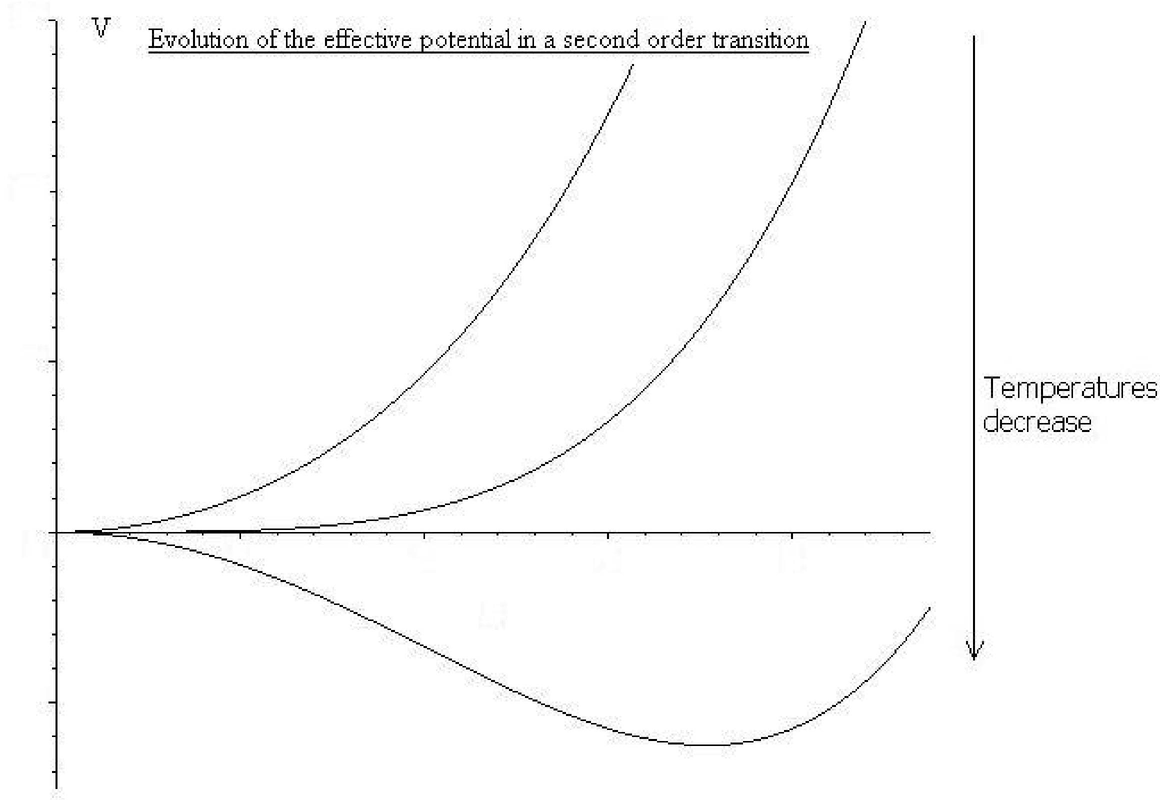
Baryon non conservation processes then correspond to a transition of the system from one vacuum to another. The key question is now to study the possibility of such transitions, and their probabilities. A very important result[3] tells us that the energy barrier between two different vacuua is finite; the top of the barrier is in fact a saddle point, and corresponds to an unstable static solution of the SU(2)xU(1) gauge theory, which is called a sphaleron (sphaleros = "ready to fall"). In the limit $\theta_W \longrightarrow 0$, ie a pure SU(2) gauge theory, its energy is given by[3]:

$E_{sph} = \frac{2M_W}{\alpha_W} B(\frac{\lambda}{g^2})$ where λ is the Higgs self coupling, M_W is the W mass, $\alpha_W = \frac{g^2}{4\pi}$ and B takes value in [1.5,2.7] (for $\theta_W \neq 0$ the static solution has been found[3a], and its energy is a bit smaller).

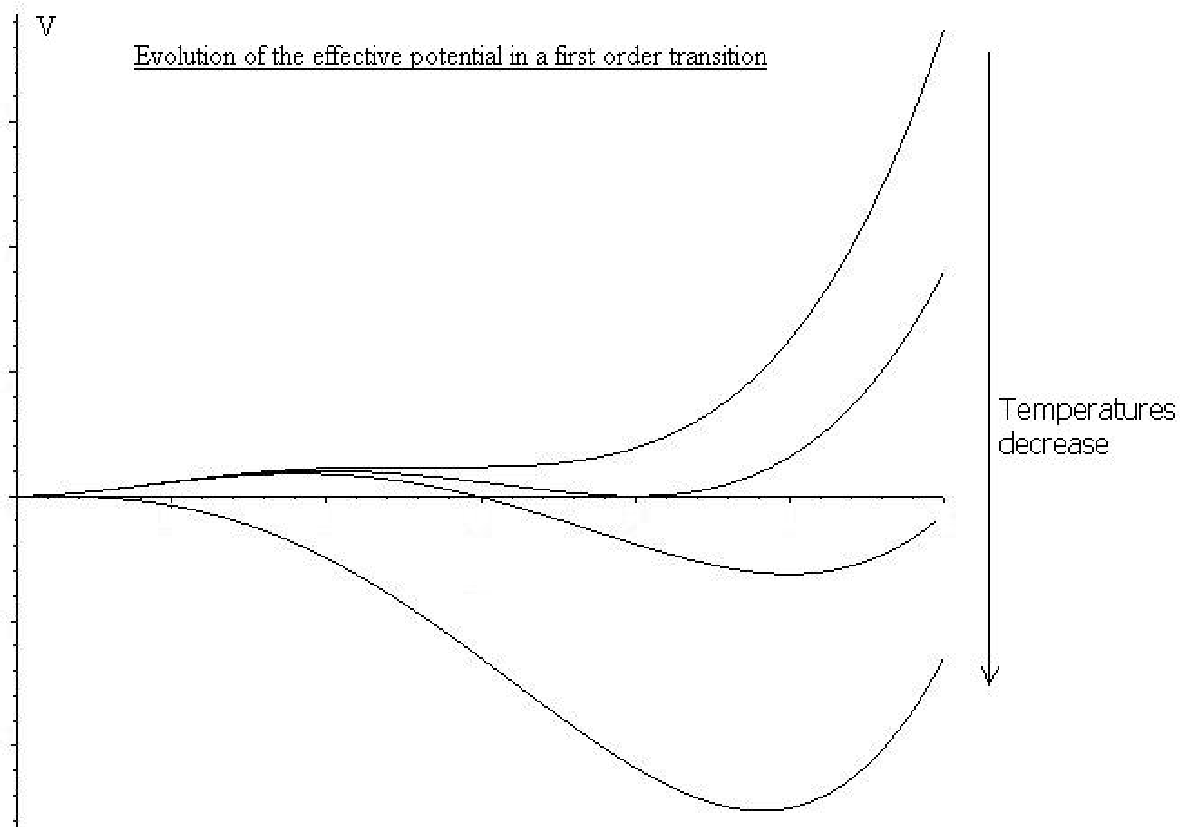
At zero temperature, one can calculate the probability for a field configuration to pass through this barrier, and one finds that the rate is exponentially suppressed: $\Gamma \approx 10^{-170}$, and a bit more if we take into account theories with Higgs mechanism[4]. This is unobservably small, and therefore baryon number can't be violated at zero temperature. Moreover, if such processes could be active, it would be in such a way that any asymmetry is erased (Le Chatellier law). The first problem – a too small rate- can be overcome with "thermal fluctuations" of the gauge and the Higgs fields: in the early universe, and as long as we stay in the broken phase, temperatures were higher, and therefore one can expect that the sphaleron rate is enhanced. Indeed, non zero temperature allows the system to pass over the barrier in a classical way. Thus, B violation could have occurred in the early universe. The second problem is more subtle; we have in fact to fulfil the third Sakharov condition, ie be out of the equilibrium. To study this possibility, we have to consider with more details scenarios of the EWPT.

1)2. Order of the Electro-Weak Phase Transition.

There are essentially two different scenarios for the phase transition, namely a first or a second order. They differ by the evolution and the general form of the effective potential, as shown below. In a second order EWPT, the initial vacuum becomes unstable, and the system falls in the degenerate new vacuum.



But in a first order EWPT, as temperatures decrease, another minimum appears and then becomes degenerate with the origin minimum. For lower temperatures, this new minimum becomes the true vacuum.



In this case, the system can not classically go in this new vacuum, because of the energy barrier. The only way to do this is by tunnelling, and it corresponds to the formation of bubbles in the universe, which expand at speed of light. Then, non equilibrium arises at the walls of these bubbles, which separate the symmetric and the broken phase.

A very important result has been shown by Kuzmin, Rubakov and Shaposhnikov [5] about the possibility of a second order transition: using a comparison between the baryon non conservation processes and the rate of the universe expansion, they proved that in such a scenario, any BAU must have been washed out at the electroweak scale, and the additional BAU that could have been generated by electroweak processes after the phase transition is smaller than the observed BAU – at least from 3 orders of magnitude. As a result of this, the transition must be of first order, and the BAU can be generated by B violation in the walls of the bubbles.

1)3. Sphaleron process at non zero temperature.

The next step is now to study the possibility for the generated BAU to survive after the EWPT: indeed, sphaleron processes must not wash out the asymmetry in the broken phase. The “risk” is higher just after the phase transition, because after it, the temperature decreases (universe expansion).The background of the calculation of the rate of sphaleron processes is based on the formalism developed by Affleck and Langer[6], in which:

$\Gamma \cong 2\,\mathrm{Im}F$ if $T < \frac{\omega_-}{2\pi}$ and $\Gamma \approx \frac{\omega_-}{\pi T}$ if $T > \frac{\omega_-}{2\pi}$. Here ω_- is the negative mode frequency (recall that the sphaleron corresponds to a saddle point), and F is the free energy (it is complex once again because of the saddle point). Many improvements were made to this calculation[7], and the result of all these, for the SU(2) gauge-Higgs system, is:

$$\Gamma_{sph} = kN_{tr}N_{rot}\frac{\omega_-}{2\pi}\left(\frac{\alpha_W(T)}{4\pi}\right)^3 \exp(-E_{sph}/T)$$

$\alpha_W(T)$ is the (temperature dependent) fine structure constant;
N_{tr} =26, N_{rot} =5.3x10^3 (they correspond to the zero modes contribution of the translations and rotations), and k is of order of unity.
Of course, this expression is only valid for temperatures below the temperature of the EWPT (above, there is no more sphaleron, but dimensional analysis checked by MC simulations gave[7,8]: $\Gamma \cong (\alpha_W(T)T)^4$).

1)4. The switch off of sphaleron processes.

To be meaningful, this has to be compared with the universe expansion rate: if this rate is greater than the sphaleron rate, such processes are at the equilibrium, and nothing happens. But if the expansion is slow enough, then these processes are not at the equilibrium, and they can violate the baryon number. Therefore, the current observed asymmetry tells us that just after the EWPT, the rate of sphaleron process was lower than the expansion of the universe[8] (in other words, these processes were inactive). This is the main constraint of this model: we only have to make a comparison between the sphaleron rate written above (multiplied by the particle density $\propto T^3$) and the Hubble parameter given by:

$H(T) \approx 1.7\sqrt{g^*}\frac{T^2}{M_{Pl}}$ where g* is of order 100 at T = T_c.

This allows us to derive a well known inequality that must have been satisfied at the phase transition:

$$\frac{\varphi_C}{T_C} \geq 1$$

where φ_C and T_C are the value of the Higgs field and the temperature just after the transition. Using this, we can derive a constraint on the still unknown parameter of the Standard Model: the Higgs mass. To do this, we have to study with more details how the EWPT is linked to the Higgs field, and how, using the couplings of this field with other particles of the Standard Model, we can compute the corresponding effective potential.

2) Standard Model and effective potential.

2)1. Generalities and definitions.

In the Standard Model, most of the fields couple with the Higgs to give particles a mass after spontaneously symmetry breaking and by the Higgs mechanism. The Lagrangian of the model is then the sum of kinetic terms and mass terms, involving this coupling. Starting with it, we can obtain the corresponding potential of the theory, which turns out to be the non kinetic part of the Lagrangian. But this potential is actually slightly modified by the contribution of radiative corrections, and the physically meaningful quantity is in fact what is called the effective potential. If we had it, we could calculate the position of the minimum, and thus the VEV of the Higgs field that appeared in the last section. It turns out that only 1PI Feynman diagrams will contribute to the corrections of the effective potential[10]. Thus, we have to introduce the generating functional for these diagrams, namely the effective action. We first start with the generating functional Z[J] representing the vacuum to vacuum amplitude in presence of a background source J(x) :

$$Z[J] \equiv \int D\varphi \exp(iS[\varphi] + i\int d^4xJ(x)\varphi(x))$$

We then introduce the generating functional W[J] corresponding to connected diagrams. One can show that:

$$\frac{Z[J]}{Z[0]} = \exp(iW[J])$$

Finally, we define the *effective action* to be:

$$\Gamma[\phi_C] \equiv W[J] - \int d^4xJ(x)\phi_C(x) \text{ where } \phi_C(x) \equiv \frac{\delta W}{\delta J(x)}.$$

This generating functional can be expanded in powers of the derivatives of ϕ_C:

$$\Gamma[\phi_C] = \int d^4x(A(\phi_C) + (\partial_\mu\phi_C)^2B(\phi_C) + ...),$$

but also in powers of ϕ_C itself:

$$\Gamma[\phi_C] = \int d^4x_1...d^4x_n\Gamma^{(n)}(x_1,...,x_n)\phi_C(x_1)...\phi_C(x_n)$$

The $\Gamma^{(n)}$s represent the 1PI correlation functions, and correspond to 1PI Feynman diagrams with n external legs.

By definition, the *effective potential* of the theory is:

$$V(\phi_C) \equiv -A(\phi_C)\text{ [1]} \qquad (*)$$

ie it is simply the ratio, up to a sign, of the effective action and a volume term, when the field ϕ_C is a constant.

[NB: spontaneously symmetry breaking appears if the field get a non zero VEV without any sources, ie J = 0. As $\frac{\delta\Gamma}{\delta\phi_C(x)} = -J(x)$, this term vanishes for non zero ϕ_C and therefore $\frac{dV}{d\phi_C} = 0$ for non zero ϕ_C]

[1] It can be shown that this definition gives the potential a physical meaning, i.e. it is the expectation value of the energy density. See for example [11]

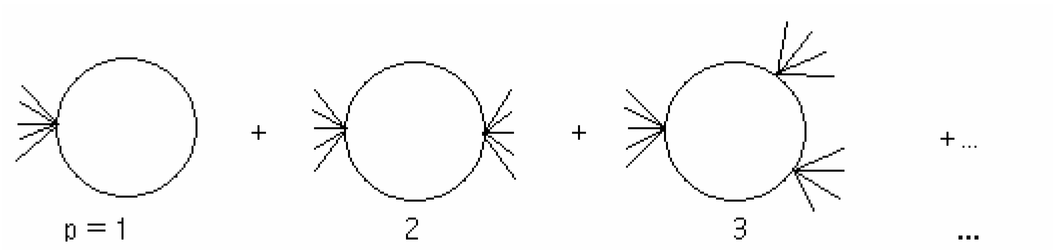
We now know how to compute V_{eff}: indeed, if we work in momentum space, it is equal to the sum of 1PI diagrams with vanishing external momenta (this in fact is the consequence that V is the non kinetic term of the Lagrangian). To the lowest order, we only have to consider tree diagrams, and therefore the effective potential is simply the classical potential in the Lagrangian (that is why we need a minus sign in the equation (*)); our task is now to find the contribution coming from radiative corrections which will modify the position of the vacuum.

2)2. Self interacting scalar field.

We first consider a self interacting scalar field, with a Lagrangian of the form:

$\mathcal{L} = (\partial_\mu \varphi)^2 - V(\varphi)$, where $V(\varphi) = g\dfrac{\varphi^n}{n!}$ .Then the zero loop effective potential, as already noticed, is just V (note that we can have n = 2)

At one loop, the most natural method consists in adding a term corresponding to the sum of all 1PI diagrams with one loop[12]:



The sum over all these diagrams is:

$$V_1(\phi_C) = i\int \frac{d^4k}{(2\pi)^4} \sum_{p=1}^{\infty} \frac{1}{2p} \left(\frac{\lambda \phi_C^{n-2}}{(n-2)!(k^2 + i\varepsilon)} \right)^p$$

Indeed, the p[th] diagram of this series has p vertices, (n-2)p external legs, p propagators, a symmetry group corresponding to rotations and reflections ($|G| = 2r$), and at each vertex, the interchange of the (n-2) external legs leave the graph invariant. Finally we have one loop integration and a i coming from the definition of the generating functional.
The one loop effective potential is then:

$$V = V_0 + V_1$$

V_1 can be evaluated by doing a Wick rotation and doing the sum: we obtain:

$$V_1(\phi_C) = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4} \ln\left(1 + \frac{g\phi_C^{n-2}}{(n-2)!k^2}\right)$$

If we add to the Lagrangian another term of the form $\lambda \dfrac{\phi_C^m}{m!}$, 1PI diagrams will contain two types of vertices, with (n-2) and (m-2) external legs. The new expression now contains a sum over two indices, and it is easy to see that we then have:

$$V_1(\phi_C) = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4} \ln(1 + \frac{g\phi_C^{n-2}}{(n-2)!k^2} + \frac{\lambda\phi_C^{m-2}}{(m-2)!k^2})$$

Therefore, the result for a general potential V (expressed as a polynomial of the field) is:

$V_1(\phi_C) = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4} \ln(1 + \frac{V''(\phi_C)}{k^2})$, where V'' is in fact the mass of the self interacting scalar field.

Introducing a cut-off Λ, and changing the variables $x = k^2$, we easily find:

$$V_1(\phi_C) = \frac{1}{64\pi^2}(2\Lambda^2 V'' + (V'')^2 \ln(\frac{V''}{\Lambda^2}) - \frac{1}{2}(V'')^2)$$

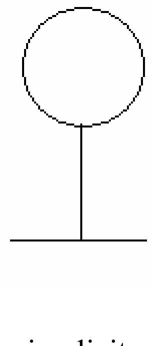
As we expected, this is cut-off dependent. This dependence can be absorbed by adding on counter-terms to the Lagrangian.

Actually, it should be mentioned that we could have obtained this result by another method[13], which consists in replacing ϕ_C by $\phi_C - a$ in the Lagrangian; expanding the effective action in terms of the shifted field, and remembering that V is the non kinetic term (we therefore set external momenta equal to zero), we have:

$V = -\sum_{n=1}^{\infty} \frac{1}{n!}\Gamma^{(n)}(p=0)(\phi_C - a)^n$ (diagrams here are different, because $\Gamma^{(n)}$ are 1PI correlation functions of the shifted theory…), and then:

$\frac{dV}{da}\Big|_{\phi=a} = \Gamma^{(1)}$., and this can be calculated by evaluating the following tadpole:

Tadpole of the self interacting scalar field



We mention this method because of its simplicity (which is clearer for higher loops calculations): only one diagram allows us to calculate the effective potential. In the theory considered above, i.e. a self interacting scalar field with a Lagrangian $\frac{1}{2}\mu^2\phi^2 + \frac{1}{4}\lambda\phi^4$, the masse term becomes $\mu^2 + 3\lambda\phi^2$, and we get the same expression as before with the second derivative of V replaced by this mass term. Absorbing the cut-off dependence, we can simply write:

$$V_1(\phi) = \frac{1}{64\pi^2}[(\mu^2 + 3\lambda\phi^2)^2 \ln(\frac{\mu^2 + 3\lambda\phi^2}{\mu^2 + 3\lambda\sigma^2}) + a(\mu^2 + 3\lambda\phi^2)^2 + b(\mu^2 + 3\lambda\sigma^2)(\mu^2 + 3\lambda\phi^2)]$$

a and b then depend on the renormalization conditions we have to choose[2]. The most natural choice is to preserve the mass of the scalar field (second derivative of the potential) and the minimum of V (we will write it σ); therefore, we impose:

$(\frac{dV_1}{d\phi})_{\phi=\sigma} = 0$ and $(\frac{d^2V_1}{d\phi^2})_{\phi=\sigma} = 0$. This determines a and b.

For a scalar field, with a mass $m^2 = A+B\varphi^2$ (as before), we find:

$$V_1(\phi) = \frac{1}{64\pi^2}(m^4(\phi)\ln(\frac{m^2(\phi)}{m^2(\sigma)}) - \frac{3}{2}m^4(\phi) + 2m^2(\phi)m^2(\sigma))$$

There is another set of renormalization conditions that is sometimes used: indeed, we can impose the renormalized coupling constant to be equal to the fourth derivative of the effective potential when the field vanishes. But in this particular case, a problem is encountered[12]: because of the logarithmic term in the one-loop contribution, the fourth derivative of V_{eff} is divergent at the origin. Thus, we must define the renormalized coupling constant at a different point, i.e. for a non zero value M of the field:

$$\lambda_R = \left.\frac{d^4V}{d\phi^4}\right|_{\phi=M}$$

We emphasize that M is an arbitrary mass scale. With this condition, the effective potential has a different expression (a and b are indeed different)

$$V(\phi) = V_0(\phi) + \frac{9\lambda_R^2}{64\pi^2}\phi^4[\ln(\frac{\phi^2}{M^2}) - \frac{25}{6}]$$

Consider with more details this expression: first, we see that the effective potential now depends on an arbitrary parameter M, which is quite unnatural; then, it shows the well known result that the perturbation expansion is an expansion in the coupling constant. More precisely, the n-loop contribution is proportional to λ_R^{n+1}. But we also see that it contains a logarithm: actually, the contribution of each additional loop will bring another $\ln(\frac{\phi^2}{M^2})$.

Therefore, the n-loop contribution generates a term proportional to: $\lambda_R^{n+1}(\ln(\frac{\phi^2}{M^2}))^n$. To be reliable, the quantity $\lambda_R \ln(\phi/M)$ must then be smaller than 1: it is not sufficient to have λ_R small. One can think that, with a judicious choice for M, the logarithm can be as small as we want. But M has a single value, and it is therefore not possible to fulfil this condition for all values of the field. Such a problem can actually be solved by the so-called renormalization group. Using this method, it is possible to obtain a perturbation expansion which is reliable for λ_R small, regardless of the logarithmic term. In the case of the effective potential, it simply states that:

$\frac{dV}{dM} = 0$, i.e., V must be M-independent. For more details, see [12].

[2] Indeed, the effective potential does depend on the renormalization conditions. However, physical quantities derived from it does not.

2)3. Gauge and fermion field contribution.

What we now want is the general expression of the one loop correction coming from scalar, gauge boson, and fermion fields in non abelian gauge theories (e.g. the GSW theory). Thus, we have now to study the contribution of gauge boson fields, which are coupled with the Higgs scalar field. The general coupling term in the Lagrangian can be written[14]:
$\frac{1}{2}\sum_{a,b} M^2_{a,b}(\phi)A_{\mu a}A^{\mu}_{b}$ where M is a real symmetric matrix, quadratic in the field ϕ.

In principle, in addition to this quadrilinear coupling, we also have a trilinear term which brings diagrams that are different from those we already draw. But, in the Landau gauge, these new diagrams do not contribute: in fact, with this choice, the propagator for the gauge fields can be expressed as:

$$D_{\mu\nu} = i\frac{g_{\mu\nu} - k_{\mu}k_{\nu}/k^2}{k^2}$$

But as the external leg carries zero momentum, the momentum of the internal scalar field is the opposite of the momentum of the gauge field, and thus gives zero when contracted with the propagator ($D_{\mu\nu}k_{\nu} = 0$): these diagrams have then no contribution to the one loop corrections.

We emphasize here that adding a gauge fixing term to the Lagrangian is necessary: otherwise, propagators are not well defined. Then, the calculation of the effective potential can be (and actually is) gauge dependent. However, physical quantities extracted from it are gauge independent, as we expect. There is also another advantage of the Landau gauge choice: by considering the propagators associated with unphysical scalars (e.g. Goldstone bosons) and ghost fields in this particular gauge, we can see that they do not contribute to the one loop corrections: in other words, they decouple, and we do not have to worry about them.

We have then to sum over diagrams almost identical as before, except that the loop corresponds to gauge fields. But now, we also have to sum over internal indices, and this is equivalent to multiply the mass matrices and take the trace. Taking the same renormalization conditions as before, we thus obtain (as in[14], but here we write explicitly the quadratic and quartic terms, by using our renormalization conditions):
$V_g = \frac{3}{64\pi^2}Tr[(M^2(\phi))^2\ln(M^2(\phi)) - \frac{3}{2}M^4(\phi) + 2M^2(\phi)M^2(\sigma)]$, where the 3 comes from the trace of the propagator.

This method is identical for all other types of fields. For fermions, as before, we first write the general coupling term between fermion fields and scalar fields:
$\sum_{a,b}\overline{\psi^a}m_{a,b}(\phi)\psi^b$ (here m is not real in general),
and we consider the same diagrams as before, with internal lines corresponding to fermions. Therefore, graphs are no longer symmetric under reflections (indeed, lines corresponding to fermions can be seen as carrying an arrow). This is compensated for by the non contribution of some graphs (only graphs with an even number of internal fermions do not vanish). Finally, we get a minus sign (Fermi statistics):

$$V_f = -\frac{1}{64\pi^2}Tr[(m(\phi)m^+(\phi))^2\ln(m\phi)m^+(\phi)) - \frac{3}{2}(m(\phi)m^+(\phi))^2 + 2m(\phi)m^+(\phi)m(\sigma)m^+(\sigma)],$$

where the trace is on internal indices as before, but also Dirac indices.

2)4. One loop effective potential in the GSW theory.

In fact, with the above results, we see that the formula obtained for scalar fields is also correct for each degree of freedom of vector and fermion fields, with an overall minus sign for fermions.
Now consider the particular case of the GSW theory: because the main corrections come from heavy particles, we only take into account the contributions of the W^+,W^-, Z and t. The couplings of these fields with the Higgs give:

$$m^2{}_W = \frac{1}{4}g^2\sigma^2 \qquad m^2{}_Z = \frac{1}{4}(g^2 + g'^2)\sigma^2 \quad m_t = \frac{1}{2}g_Y\sigma$$

If we plug this into the above result, the expressions simplify, because the mass matrices are diagonal. We finally obtain the one loop effective potential of the standard model:

$$V(\phi) = \frac{1}{2}\mu^2\phi^2 + \frac{1}{4}\lambda\phi^4 + B\phi^4 \ln(\frac{\phi^2}{\sigma^2}) + 2B\sigma^2\phi^2 - \frac{3}{2}B\phi^4 \ ,$$

where B = $\frac{1}{64\pi^2}\frac{1}{\sigma^4}(6m_W^4 + 3m_Z^4 - 12m_t^4)$ and the masses are given above.
This expression can be re-arranged by first noticing that $\mu^2 = -\lambda\sigma^2$ and then grouping quadratic and quartic terms[15]:

$$V(\phi) = -\frac{1}{2}(\lambda - 4B)\sigma^2\phi^2 + \frac{1}{4}\lambda'\phi^4 + B\phi^4 \ln(\frac{\phi^2}{\sigma^2}) \text{ with } \lambda' = \lambda - 6B$$

$$V(\phi) = -\frac{1}{2}(\lambda' + 2B)\sigma^2\phi^2 + \frac{1}{4}\lambda'\phi^4 + B\phi^4 \ln(\frac{\phi^2}{\sigma^2})$$

Thus, the Higgs mass is now expressed as: $m_H^2 = \lambda\sigma^2 = (\lambda' + 6B)\sigma^2$ (of course, it is the same as before, by our choice of renormalization conditions). One may be worried about the fact that B<0, and then the potential is unbounded, and the vacuum is unstable. But the use of the renormalization group shows that it is actually not the case. Below, we will be interested in the expression of the potential near its minimum, and this term will then vanishes (that's why we did not go through the calculation of the renormalization group: we do not need it).
Now, this potential is a zero temperature potential, i.e. it does not include contribution coming from the interaction between the Higgs field and the hot ambient plasma which has a finite temperature (finite here is non zero…). Thus we have now to study the contribution of thermal effects on the potential.

2)5. Effective potential at non zero temperature.

An operator at finite temperature is defined by the Gibbs average[12]:
$A_T(x_1,...,x_n) \equiv \frac{Tr(\exp(-\beta H)A(x_1,...,x_n))}{Tr(\exp(-\beta H))}$ where β = 1/T, and $Tr(\exp(-\beta H))$ = Z is the partition function. The Gibbs average is in fact the thermal expectation of the Green's functions.
As it is generally possible to write:

$$Tr(\exp(-\beta H)A(x_1,...,x_n)) = Tr(\exp(-\beta H)A_1(\underline{x_1},t)...A_n(\underline{x_n},t)) \ ,$$

we can see, inserting 1 = $\exp(-\beta H)\exp(\beta H)$ and using the cyclic property of the trace that a finite temperature operator is unchanged if one replaces it_n by $it_n + \beta$. We conclude that the only difference between Green's functions at finite temperature and at zero temperature is that they obey different boundary conditions; namely they are periodic in Euclidean time, with period β (for fermions, it is straightforward to see with the anti-commutation relations for the fields that the Green's functions are anti-periodic). Actually, there is a subtlety for gauge fields, for which this reasoning breaks down[16]. We have seen above that the very definition of the partition function ensures the periodic (or anti-periodic) property of Green's functions. With this definition, we can write the partition function as a functional integral over periodic (or anti-periodic) fields. But for gauge fields, some states are unphysical. Then, the trace of the operator $\exp(-\beta H)$, which can still be expressed as a functional integral over periodic gauge fields, is not exactly the partition function, because the trace takes into account the full Hilbert space which contains many unphysical states. To solve this problem, we first have to fix the gauge (e.g. $A_0 = 0$ is often used), define the physical states of the Hilbert space, and select them with a projection operator P. The partition function for gauge fields is then $Tr(P\exp(-\beta H))$. It can be shown that, with this definition, we can write Z as a functional integral over periodic gauge fields, in which A_0 is non zero but vanishes at spatial infinity. Therefore, we will ignore this subtlety from now on, and do as if there was no difference between gauge fields and scalar or fermion fields.

It follows that the prescription we have to use is the replacement:

- $\int d^4x_E \longrightarrow \int_0^\beta dt \int d^3x$
- $k_0 \longrightarrow 2\pi nT$ ((n+1) for fermions)
- $\int dk_0 \longrightarrow 2\pi T \sum_{n=-\infty}^{+\infty}$

Armed with this, we can now evaluate the contribution of thermal effects coming from bosons and fermions.

2)5.1 Boson and fermion contribution.

Using the above prescription, the boson contribution can be written as:

$$\frac{1}{2}T\sum_{n=-\infty}^{+\infty}\int\frac{d^3\underline{k}}{(2\pi)^3}\ln(\underline{k}^2+(2\pi nT)^2+m^2(\phi))$$

The calculation of this expression is a bit tricky[17]; first define:

$f(x) \equiv \sum_n \ln(4\pi^2 n^2 T^2 + x^2)$. We have: $\frac{df(x)}{dx} = \frac{1}{\pi T}\sum_n \frac{(x/2\pi T)}{n^2 + (x/2\pi T)^2}$

Using the fact that: $\sum_{n=1}^{\infty}\frac{x}{n^2+x^2} = -\frac{1}{2x} + \frac{\pi}{2}\coth(\pi x)$, we find:

$\frac{df(x)}{dx} = \frac{2}{T}(\frac{1}{2} + \frac{1}{\exp(x/t)-1})$ and, up to unimportant terms:

$$f(x) = \frac{2}{T}(\frac{x}{2} + T\ln(1-\exp(-k/T)))$$

Therefore, the effective potential has the following form ($x^2 = \underline{k}^2 + m^2(\phi)$):

$$V(\phi,T) = \int \frac{d^3\underline{k}}{(2\pi)^3}(\frac{(\underline{k}^2+m^2)^{1/2}}{2} + T\ln(1-\exp((\underline{k}^2+m^2(\phi))^{1/2}/T)))$$

The first term corresponds to the effective potential at zero temperature, and is in fact equal to the expression V we have previously obtained. Then, we write:

$V(\phi,T) = V(\phi) + V_1(\phi,T)$ with:

$$V_1(\phi,T) = \frac{T^4}{2\pi^2}\int_0^\infty dy(y^2\ln(1-\exp(-\sqrt{y^2+\frac{m^2(\phi)}{T^2}})))$$

(the last expression is obtained after 2 changes of variables, namely $|\underline{k}^2| = x$ and $y = \frac{x}{T}$).

Again, this result must be multiplied by the number of degrees of freedom.
For fermions, there is an overall minus sign and a plus sign in the logarithm.

2)5.2. Evaluation of thermal effects in the approximation $T >> m(\phi)$.

In general, this integral can be expressed in terms of modified Bessel functions[15]. However, in the special case where $T >> m(\phi)$, we can do an expansion in m/T[17].
Let x = $\frac{m}{T}$. It appears that the second derivatives of V_1, $\left.\frac{\partial^2 V_1(\phi,T)}{\partial(x^2)^2}\right|_{x=0}$ can be calculated. First, note that we have:

$$V_1(\phi,T)_{x=0} = \frac{T^4}{2\pi^2}\int_0^\infty dy(y^2\ln(1-\exp(-y))) = -\frac{\pi^2T^4}{90}$$

Taking the first derivative, we find:

$$\frac{\partial V_1(\phi,T)}{\partial x^2} = \frac{T^4}{4\pi^2}\int_0^\infty dy(y^2(\frac{1}{(y^2+x^2)^{1/2}})(\frac{1}{\exp((y^2+x^2)^{1/2})-1}))$$

and therefore:

$$\left.\frac{\partial V_1(\phi,T)}{\partial(x^2)}\right|_{x=0} = \frac{T^4}{4\pi^2}\int_0^\infty dy(y(\exp(y)-1)^{-1}) = \frac{T^4}{24}$$ (and not T², as in reference [17]).

The second derivative can be expressed in many ways. To write it in a convenient way, we first notice that:

$$\frac{\partial}{\partial(x^2)}[(\frac{1}{\sqrt{(y^2+x^2)}})(\frac{1}{\exp(\sqrt{(y^2+x^2)})-1})] = \frac{\partial}{\partial(y^2)}[(\frac{1}{\sqrt{(y^2+x^2)}})(\frac{1}{\exp(\sqrt{(y^2+x^2)})-1})]$$

Then, in the integral, we can replace $\frac{\partial}{\partial(x^2)}$ by $\frac{\partial}{\partial(y^2)}$. It allows us to integrate by parts, using the fact that: $y^2\frac{\partial}{\partial(y^2)} = \frac{y}{2}\frac{\partial}{\partial y}$. We obtain:

$$\frac{\partial^2 V_1(\phi,T)}{\partial(x^2)^2} = -\frac{T^4}{8\pi^2}\int_0^\infty dy\left(\frac{1}{\sqrt{(y^2+x^2)}}\right)\left(\frac{1}{\exp(\sqrt{(y^2+x^2)})-1}\right)$$

This integral can be calculated as a function of x by introducing a regulated expression:

$$f(x,\varepsilon) = \int_0^\infty dy\left(\frac{y^{-\varepsilon}}{\sqrt{(y^2+x^2)}}\right)\left(\frac{1}{\exp(\sqrt{(y^2+x^2)})-1}\right)$$

using the following relation(already used above):

$$\sum_{n=-\infty}^{+\infty}\frac{E}{4\pi^2 n^2+E^2} = \frac{1}{2}+\frac{1}{\exp(E)-1}$$, with here E = $\sqrt{x^2+y^2}$,

we can split f into two pieces:

$$f(x,\varepsilon) = \int_0^\infty dy y^{-\varepsilon}\sum_n \frac{1}{4\pi^2 n^2+x^2+y^2} - \frac{1}{2}\int_0^\infty dy\frac{y^{-\varepsilon}}{(x^2+y^2)^{1/2}} \equiv f_1(x,\varepsilon)+f_2(x,\varepsilon)$$

First consider f_1: after the change of variables $y \longrightarrow \frac{y}{\sqrt{4\pi^2 n^2+x^2}}$, we can write it:

$$f_1(x,\varepsilon) = \sum_n \frac{1}{(x^2+4\pi^2 n^2)^{(1+\varepsilon)/2}}\int_0^\infty dy\frac{y^{-\varepsilon}}{1+y^2}$$. The integral now can be done:

$$f_1(x,\varepsilon) = [x^{-1-\varepsilon}+2\sum_{n=1}^\infty(2\pi n)^{-1-\varepsilon}+2\sum_{n=1}^\infty(2\pi n)^{-1-\varepsilon}(\frac{1}{(1+x^2/(4\pi^2 n^2))^{(1+\varepsilon)/2}}-1)]\frac{\pi}{2\cos(\pi\varepsilon/2)}$$

In the first sum, we can introduce the Zeta function ξ; the second sum has a limit when $\varepsilon \longrightarrow 0$ which is O(x²). Using the relation:

$\xi(1+\varepsilon) = (2\pi)^\varepsilon(\frac{1}{\varepsilon}-\ln(2\pi)+\gamma+O(\varepsilon))$, we get:

$$f_1(x,\varepsilon) = \frac{\pi}{2x}+\frac{1}{2}\frac{1}{(2\pi)^\varepsilon}\xi(1+\varepsilon)+O(\varepsilon) = \frac{1}{2\varepsilon}+\frac{\pi}{2x}+\frac{1}{2}\gamma-\frac{1}{2}\ln(2\pi)+O(\varepsilon)$$, up to terms in x².

For f_2, we first change the variable $y \longrightarrow y/x$:

$$f_2(x,\varepsilon) = -\frac{1}{2x^\varepsilon}\int_0^\infty dy\frac{y^{-\varepsilon}}{(1+y^2)^{1/2}} = -\frac{1}{2x^\varepsilon}\frac{1}{2}B(\frac{1}{2}(1-\varepsilon),\frac{\varepsilon}{2})$$ [3]

As we have: $x^{-\varepsilon} \approx 1-\varepsilon\ln x$ and $B(\frac{1-\varepsilon}{2},\frac{\varepsilon}{2}) \approx \frac{2}{\varepsilon}+2\ln 2+O(\varepsilon)$, we find the expression of f_2:

$$f_2(x,\varepsilon) = -\frac{1}{2\varepsilon}+\frac{1}{2}\ln(\frac{x}{2})+O(\varepsilon)$$

Adding the two results, we see that the divergent parts cancel, and we finally obtain:

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[3] It seems that a factor $\frac{1}{2}$ is missing in [17]: actually, the B denotes the Beta function, defined by: $B(x,y) \equiv \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$. For Re(x) and Re(y) >0, this can be expressed as:

$B(x,y) = \int_0^\infty dt(\frac{t^{x-1}}{(1+t)^{x+y}}) = 2\int_0^\infty dt(\frac{t^{2x-1}}{(1+t^2)^{x+y}}$, so here $x = \frac{1}{2}(1-\varepsilon)$ and $y = \frac{\varepsilon}{2}$, and we have an extra factor of 2. Fortunately, it does not affect the result of [17].

$$\frac{\partial^2 V_1(\phi,T)}{\partial (x^2)^2} = -\frac{T^4}{8\pi^2}\left(\frac{\pi}{2x} + \frac{1}{2}\ln\left(\frac{x}{4\pi}\right) + \frac{1}{2}\gamma\right)$$ plus terms in $O(x^2)$.

This expression can now be integrated, using the "initial" values we have just found for the potential and its first derivative. We get:

$$\frac{\partial V_1(\phi,T)}{\partial (x^2)} = -\frac{T^4}{8\pi^2}\left(\pi x + \frac{1}{4}(x^2\ln(x^2) - x^2) + \frac{x^2}{2}(\gamma - \ln(4\pi))\right) + \frac{T^4}{24}$$

$$V_1(\phi,T) = -\frac{\pi^2 T^4}{90} + \frac{1}{24}m^2(\phi)T^2 - \frac{1}{12\pi}m^3(\phi)T - \frac{1}{64\pi^2}m^4(\phi)\ln\left(\frac{m^2(\phi)}{c_B T^2}\right)$$

where we have replaced x by $\frac{m(\phi)}{T}$ and $\ln(c_{B}) = \frac{3}{2} + 2\ln(4\pi) - 2\gamma$

The B indices denote the boson contribution.

Now, we have to evaluate the contribution of fermions, given by the integral previously given:

$V_1(\phi,T) = -\frac{T^4}{2\pi^2}\int_0^\infty dy\left(y^2\ln\left(1+\exp\left(-\sqrt{y^2+\frac{m^2(\phi)}{T^2}}\right)\right)\right)$, for each degree of freedom (4 here).

The process is completely similar as for bosons: we will calculate the second derivative of V_1 by introducing a regulated expression. We first find:

$$V_1(\phi,T)\Big|_{x=0} = -\frac{2T^4}{\pi^2}\int_0^\infty dy[y^2\ln(1+\exp(-y))] = -\frac{7\pi^2 T^4}{180} \text{ and}$$

$$\left.\frac{\partial V_1(\phi,T)}{\partial x^2}\right|_{x=0} = \frac{T^4}{\pi^2}\int_0^\infty dy\left(\frac{y}{1+\exp(y)}\right) = \frac{T^4}{12}$$

Then we have, using the same "trick" as before:

$$\frac{\partial^2 V_1(\phi,T)}{\partial (x^2)^2} = -\frac{T^4}{2\pi^2}\int_0^\infty dy\left(\frac{1}{\sqrt{y^2+x^2}}\right)\left(\frac{1}{(\exp(\sqrt{y^2+x^2})+1)}\right)$$

Thus, the regulated expression has the following form:

$$g(x,\varepsilon) = \int_0^\infty dy\left(\frac{y^{-\varepsilon}}{\sqrt{y^2+x^2}}\right)\left(\frac{1}{(\exp(\sqrt{y^2+x^2})+1)}\right)$$

We now use the relation: $\sum_{n=-\infty}^{\infty}\frac{\sqrt{y^2+x^2}}{(2n+1)^2\pi^2+x^2+y^2} = \frac{1}{2} - \frac{1}{1+\exp(\sqrt{y^2+x^2})}$ to split g into two pieces:

$g_1(x,\varepsilon) = -\int_0^\infty dy\left[y^{-\varepsilon}\sum_{n=-\infty}^{\infty}\frac{1}{(2n+1)^2\pi^2+x^2+y^2}\right]$ and g_2 which is identical to f_2 (up to an overall minus sign).

Using the same method as for bosons, we can write:

$$g_1(x,\varepsilon) = -\sum_{n=-\infty}^{\infty}((2n+1)^2+x^2)^{-(1+\varepsilon)/2}\int_0^\infty dy\frac{y^{-\varepsilon}}{1+y^2}$$

The integral is known; for the sum, we can split it as before into a piece which has a limit when $\varepsilon \longrightarrow 0$ (this limit is O(x²)) and the "useful" piece:

$$-2\sum_{n=0}^{\infty}\left(\frac{1}{(2n+1)\pi}\right)^{1+\varepsilon}$$

we can write this sum as a difference of Zeta functions; it becomes:

$$-2\pi^{-1-\varepsilon}(\xi(1+\varepsilon)-2^{-1-\varepsilon}\xi(1+\varepsilon))$$

This is equal to:

$-\frac{2}{\pi}2^{\varepsilon}(\frac{1}{\varepsilon}+\gamma-\ln(2\pi))+\frac{1}{\pi}(\frac{1}{\varepsilon}+\gamma-\ln(2\pi))$. Multiplying this by $\frac{\pi}{2}$ (coming from the integral), we get:

$$g_1(x,\varepsilon)=-\frac{1}{2\varepsilon}-\frac{1}{2}(\gamma-\ln(2\pi))-\ln 2+O(\varepsilon)+O(x^2)$$

The final expression for g_1 is then:

$g_1(x,\varepsilon)=-\frac{1}{2\varepsilon}+\frac{1}{2}\ln(\frac{\pi}{2})-\frac{1}{2}\gamma+O(\varepsilon)$ up to terms in O(x²)

Again, the divergent parts cancel when we sum g_1 and g_2. Integrating this sum with the initial conditions we calculated, we find the contribution of thermal effects coming from fermions:

$$V_1^F(\phi,T)=-\frac{7\pi^2T^4}{180}+\frac{1}{12}m^2(\phi)T^2+\frac{4}{64\pi^2}m^4(\phi)\ln(\frac{m^2(\phi)}{c_F T^2})$$

where $c_F=2\ln(\pi)+\frac{3}{2}-2\gamma$.

Note the (very) important difference between the fermionic and bosonic contribution, namely the appearance of a term in $m^3(\phi)T$ for bosons (with a minus sign).

2)5.3 One loop effective potential at finite temperature.

Eliminating the unimportant terms (e.g. field independent) and summing all the contributions to the effective potential (zero loop, one loop, and one loop thermal contribution), we can write it in the following way:

$$V(\phi,T)=-\frac{1}{2}(\lambda'+2B)\sigma^2\phi^2+\frac{1}{4}\lambda'\phi^4+B\phi^4\ln(\frac{\phi^2}{\sigma^2})+DT^2\phi^2-ET\phi^3-\frac{\phi^4}{64\pi^2\sigma^4}[6m_W^4(\sigma)\ln(\frac{m_W^2(\phi)}{c_B T^2})$$

$$+3m_Z^4(\sigma)\ln(\frac{m_Z^2(\phi)}{c_B T^2})-12m_t^4(\sigma)\ln(\frac{m_t^2(\phi)}{c_F T^2})]$$

with:

$D=\frac{1}{24\sigma^2}(6m_W^2(\sigma)+3m_Z^2(\sigma)+6m_t^2(\sigma))$ and $E=\frac{1}{12\pi\sigma^3}(6m_W^3+3m_Z^3)$

As we are interested in the case where $\phi\approx\sigma$, we can simplify this expression by replacing ϕ by σ in the logarithmic terms[15]. Then:

$$V(\phi,T)=-\frac{1}{2}(\lambda'+2B)\sigma^2\phi^2+\frac{1}{4}\lambda_T\phi^4+DT^2\phi^2-ET\phi^3,$$

with: $\lambda_T = \lambda' - \frac{1}{64\pi^2\sigma^4}[6m_W^4(\sigma)\ln(\frac{m_W^2(\sigma)}{c_B T^2}) + 3m_Z^4(\sigma)\ln(\frac{m_Z^2(\sigma)}{c_B T^2}) - 12m_t^4(\sigma)\ln(\frac{m_t^2(\sigma)}{c_F T^2})]$

Finally, we get:

$$\boxed{V(\phi,T) = D(T^2 - T_2^2)\phi^2 - E\phi^3 T + \frac{1}{4}\lambda_T\phi^4}$$ [4]

where: $T_2^2 = \frac{\lambda' + 2B}{2D}\sigma^2 = \frac{m_H^2 - 4B\sigma^2}{2D}$

If we study this potential, we can see that a second minimum appears for a temperature given by $T_0^2 = \frac{T_2^2}{1 - 9E^2/(8\lambda_T D)}$. This minimum becomes degenerate with the origin when:

$T^2 = T_1^2 = \frac{T_2^2}{1 - E^2/(\lambda_T D)}$. The corresponding value of the Higgs field is:

$\phi_C = \frac{2ET_C}{\lambda_{Tc}}$

The condition given above concerning the switching off of sphaleron process just after the phase transition can then be written:

$\lambda_{Tc} < 2E$ This gives an upper bound on the coupling λ' (indeed, all the other terms in the expression of λ_{Tc} are known), and therefore on the Higgs boson mass, namely:

$$\boxed{m_H < 57 GeV}$$

This is obviously completely ruled out by the experiments, which currently give m_H>115GeV.

At this level, we have to answer to essential features: the first one is about the contribution of higher loops in the effective potential; and the second one (they are actually linked) is the reliability of such a perturbation expansion (due to thermal effects).

If we take into account two loops contribution, calculations become messy, and before they were made rigorously, the effect on the cubic term remained unclear (even linear terms were found). Now, it appears that contribution of higher loops relaxes the condition on the Higgs mass by increasing the factor of E[18]. Now, in general, higher loops contribution will bring terms $\approx (\lambda^2 T/m)^N$. For mass proportional to the fields, we see that perturbation expansion breaks down for small value of the Higgs field, but this is precisely the case just after the phase transition! Such behaviour leads some physicists to wonder whether the phase transition (and therefore the restoration of the symmetry at high temperatures) really occurs. Why, indeed, believe that when our calculation breaks down, the Higgs field vanishes at some temperature? It may just have a small, non zero value. In such a context, non perturbative approaches are essential to solve these problems. A few years ago, methods based on Monte Carlo simulations gave a relaxed upper bound of the Higgs mass[19]: $m_H < 80 GeV$.

But such a condition is still ruled out by the experiments

[4] We now have to worry about the validity of our approximation $T >> m(\phi)$. Actually, by making a comparison with the exact expression, it turns out that this expansion is reliable even if m/T is not very small[15]. Quantitatively, for m/T<1.6, the accuracy is less than 5%.

Conclusion and possible extensions.

So, what can be done? If we remember the expression of the effective potential, and as we have already noticed, the strength of the first order transition is determined by the ϕ^3 term, and it comes from the zero mode contribution of bosons. A natural extension of this model is to consider more (heavy) bosons in the theory, which will enhance the cubic term and therefore makes the first order transition stronger. It is easy to see that this will directly increase the VEV of the Higgs field, and therefore relax the condition on the Higgs mass. Among these extensions is the super-symmetry, in which each fermion has a bosonic super-partner. But such theories faced with two main problems: the first one is that even with these additional heavy bosons, it seems to be difficult to derive an upper bound which is still compatible with the experiment. Secondly, none of these bosons have been discovered yet, despite intensive search!

Anyway, it is likely the next decade will be decisive: new accelerators will be soon able to reach electroweak energy scale. It is also likely the "old" standard model shall be at least modified. However, many physicists believe that the Higgs mechanism really occurred, and that there is no doubt that the Higgs boson will be (soon) discovered.

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