GENERAL GEOMETRY AND GEOMETRY OF ELECTROMAGNETISM

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Abstract

It is shown that Electromagnetism creates geometry different from Riemannian one. General geometry containing Riemannian and geometry underlying Electromagnetism as special cases is constructed. Action for electromagnetic field and Maxwell equations are derived from curvature function of Geometry of Electromagnetism.

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1 Introduction

In this paper we show that electromagnetism can not be geometrized in the framework of Riemannian geometry[1]. We develop General Geometry containing Riemannian as a special case and introduce notion of curvature function which serves as a source for defining curvature characteristics of geometry. A particular case of General geometry is geometry underlying Electromagnetism, Geometry of Electromagnetism.

As it is well known, equation for geodesics in Riemannian geometry coincides with the equation of motion of a classical particle interacting with gravitational field. Action for gravitational field is constructed from curvature tensor of the underlying geometry.

We show that equation for geodesics in Geometry of Electromagnetism coincides with the equation of motion of a charged particle interacting with electromagnetic field and construct Maxwell equations and action functional for electromagnetic field from curvature function.

In the next section we show that electromagnetism creates geometry different from Riemannian one and therefore cannot be geometrized in the framework of Riemannian geometry.

We develop General Geometry including Riemannian as a particular case in section 3.

In section 4, we derive action functional for electromagnetic field and Maxwell equations from curvature function of the underlying geometry.

Section 5 is devoted to discussion of correspondence between physical properties of fields and mathematical properties of corresponding geometries.

2 Interacting Classical Particle

The action for a free particle is

$$S = \frac{1}{2} m \int du \eta_{\mu\nu} x_u^\mu x_u^\nu,$$

where $x_u = dx/du, \eta_{\mu\nu} = (+---)$. Consider a particle which interacts with gravitational field $h_{\mu\nu}$:

$$S = \frac{1}{2} m \int du \eta_{\mu\nu} x_u^\mu x_u^\nu + \frac{1}{2} \lambda \int du h_{\mu\nu} x_u^\mu x_u^\nu. \tag{1}$$

We can represent (1) as

$$S = \frac{1}{2} \int du g_{\mu\nu} x_u^\mu x_u^\nu, \tag{2}$$

$$g_{\mu\nu} = m\eta_{\mu\nu} + \lambda h_{\mu\nu}. \tag{3}$$

[1]For the sake of simplicity we do not review the well known history of geometrization problem in Physics.

This has been interpreted as a free particle moving in curved spacetime with metric $g_{\mu\nu}$ in General Relativity.

We will go one step further and consider charged particle which interacts with electromagnetic field A_μ:

$$S = \frac{1}{2}m \int du \eta_{\mu\nu} x^\mu_u x^\nu_u + \frac{e}{c}\int du A_\mu x^\mu_u, \tag{4}$$

and represent (4) as

$$S = \frac{1}{2}\int du {}^A g_{\mu\nu} x^\mu_u x^\nu_u, \tag{5}$$

$${}^A g_{\mu\nu} = m\eta_{\mu\nu} + \frac{e}{c}(A_\mu f_\nu + A_\nu f_\mu), \tag{6}$$

where $f_\nu x^\nu_u = 1$. It easy to see that f_ν are functions of du/dx^ν and therefore spoil locality. It is clearly seen that we cannot represent (4) in the form of (5) with local functions of x and x_u. This indicates that electromagnetism creates geometry different from Riemannian one. Hence, it cannot be geometrized in the framework of Riemannian geometry. The appropriate geometry for electromagnetism will be presented in sections 3 and 4.

For the general case $S = \frac{1}{2}\int du \mathbf{g}_{\mu\nu}(x, x_u) x^\mu_u x^\nu_u$ when $\mathbf{g}_{\mu\nu}$ are considered as a function of x and x_u we obtain equation of motion

$$\frac{d^2 x^\nu}{du^2} G_{\lambda\nu} + \Gamma_{\lambda,\mu\nu} x^\mu_u x^\nu_u = 0, \tag{7}$$

where

$$G_{\lambda\nu}(\mathbf{g}_{\mu\nu}) = \frac{1}{2}\frac{\partial^2 \mathbf{g}_{\mu\sigma}}{\partial x^\nu_u \partial x^\lambda_u} x^\mu_u x^\sigma_u + \frac{\partial \mathbf{g}_{\mu\nu}}{\partial x^\lambda_u} x^\mu_u + \frac{\partial \mathbf{g}_{\lambda\sigma}}{\partial x^\nu_u} x^\sigma_u + \mathbf{g}_{\lambda\nu},$$

$$2\Gamma_{\lambda,\mu\nu}(\mathbf{g}_{\mu\nu}) = \frac{\partial^2 \mathbf{g}_{\mu\nu}}{\partial x^\sigma \partial x^\lambda_u} x^\sigma_u + \frac{\partial \mathbf{g}_{\lambda\nu}}{\partial x^\mu} + \frac{\partial \mathbf{g}_{\lambda\mu}}{\partial x^\nu} - \frac{\partial \mathbf{g}_{\mu\nu}}{\partial x^\lambda}.$$

Function $G_{\lambda\nu}(\mathbf{g}_{\mu\sigma})$ plays the role of a tensor for raising and lowering indices and $\Gamma_{\lambda,\mu\nu}$ connection.

$$G_{\lambda\nu}({}^A g_{\mu\sigma}) = \eta_{\lambda\nu}, \quad G_{\lambda\nu}(g_{\mu\sigma}) = g_{\lambda\nu}. \tag{8}$$

3 General Geometry

Let M be a manifold with coordinates $x^\lambda, \lambda = 1, ..., n$. Consider a curve on this manifold $x^\lambda(u)$. Vector field

$$V = \xi^\lambda \frac{\partial}{\partial x^\lambda}$$

has coordinates ξ^λ. In Riemannian geometry it is accepted that

$$\frac{d\xi^\lambda}{du} = -\Gamma'^\sigma_{\lambda\nu}(x) x^\nu_u \xi^\lambda, \tag{9}$$

where $\Gamma'^{\sigma}_{\lambda\nu}(x)$ are functions of x only. To construct General Geometry we assume that

$$\frac{d\xi^{\sigma}}{du} = -\Gamma^{\sigma}_{\lambda}(x, x_u)\xi^{\lambda}(x, x_u). \tag{10}$$

$\Gamma^{\sigma}_{\lambda}(x, x_u)$ are general functions of x and x_u. The next step is to consider x as a function of two parameters u, υ and find $\lim\limits_{\substack{\Delta\upsilon\to 0\\ \Delta u\to 0}} \Delta\xi^{\sigma}/\Delta u\Delta\upsilon$. In order to do that we need

$$\frac{d\xi^{\sigma}}{du} = -\Gamma^{\sigma}_{\lambda}\xi^{\lambda}, \quad \frac{d\xi^{\sigma}}{d\upsilon} = -\tilde{\Gamma}^{\sigma}_{\lambda}\xi^{\lambda},$$

$$\Gamma^{\sigma}_{\lambda} = \Gamma^{\sigma}_{\lambda}(x, x_u, x_\upsilon), \quad \tilde{\Gamma}^{\sigma}_{\lambda} = \tilde{\Gamma}^{\sigma}_{\lambda}(x, x_u, x_\upsilon).$$

After simply calculations we arrive at

$$\lim_{\substack{\Delta\upsilon\to 0\\ \Delta u\to 0}} \frac{\Delta\xi^{\sigma}}{\Delta u\Delta\upsilon} = R^{\sigma}_{\lambda}\xi^{\lambda},$$

where

$$R^{\sigma}_{\lambda} = \frac{d}{d\upsilon}\Gamma^{\sigma}_{\lambda} - \frac{d}{du}\tilde{\Gamma}^{\sigma}_{\lambda} + \tilde{\Gamma}^{\sigma}_{\rho}\Gamma^{\rho}_{\lambda} - \Gamma^{\sigma}_{\rho}\tilde{\Gamma}^{\rho}_{\lambda}.$$

We call R^{σ}_{λ} curvature function.

It is easy to demonstrate that for the special case when

$$\Gamma^{\sigma}_{\lambda}(x, x_u, x_\upsilon) = \Gamma^{\sigma}_{\lambda\nu}(x)x^{\nu}_u, \quad \tilde{\Gamma}^{\sigma}_{\lambda}(x, x_u, x_\upsilon) = \Gamma^{\sigma}_{\lambda\nu}(x)x^{\nu}_\upsilon$$

we obtain

$$R^{\sigma}_{\lambda} = R^{\sigma}_{\lambda\mu\nu}(x^{\nu}_u x^{\mu}_\upsilon - x^{\nu}_\upsilon x^{\mu}_u),$$

where

$$R^{\sigma}_{\lambda\mu\nu} = \partial_\mu\Gamma^{\sigma}_{\lambda\nu} - \partial_\nu\Gamma^{\sigma}_{\lambda\mu} + \Gamma^{\sigma}_{\rho\mu}\Gamma^{\rho}_{\lambda\nu} - \Gamma^{\sigma}_{\rho\nu}\Gamma^{\rho}_{\lambda\mu}$$

is the curvature tensor of Riemannian geometry. Hence, Riemannian geometry is a particular case of General Geometry.

4 Geometry of Electromagnetism

We choose functions Γ and $\tilde{\Gamma}$ as follows

$$\Gamma^{\sigma}_{\lambda}(x, x_u, x_\upsilon) = F^{\sigma}_{\lambda}(x(u, \upsilon)), \quad \tilde{\Gamma}^{\sigma}_{\lambda}(x, x_u, x_\upsilon) = F^{\sigma}_{\lambda}(x(u, \upsilon)),$$

then (10) becomes

$$\frac{d\xi^{\sigma}}{du} = -F^{\sigma}_{\lambda}(x)\xi^{\lambda}(x, x_u). \tag{11}$$

To obtain equations for geodesics we substitute ξ^{λ} by x^{λ}_u and arrive at

$$\frac{d^2x_\sigma}{du^2} = F_{\lambda\sigma}(x)x^{\lambda}_u.$$

This is exactly equation of motion for a charged particle moving in electromagnetic field A_μ with $F^\sigma_\lambda = \partial^\sigma A_\lambda - \partial_\lambda A^\sigma$. Then we find curvature function

$$R^\sigma_\lambda = R^\sigma_{\mu\lambda}(x^\mu_v - x^\mu_u),$$

where

$$R^\sigma_{\mu\lambda} = \partial_\mu F^\sigma_\lambda.$$

After summing by two of the three indices we obtain

$$R_\lambda = R^\mu_{\mu\lambda} = \partial_\mu F^\mu_\lambda.$$

Equations $R_\lambda = 0$ coincide with the Maxwell equations. For geometry (11) the length of a curve is

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + A_\mu dx^\mu$$

and indices are raised and lowered by $\eta_{\mu\nu}$ because of (8). We can construct from R_λ and A^λ a Lagrangian

$$R = A^\lambda R_\lambda = \partial_\mu(A^\lambda F^\mu_\lambda) - \frac{1}{2} F_{\mu\lambda} F^{\mu\lambda}.$$

We see that as in the case of Riemannian geometry and gravitation we can find equations and action functional for electromagnetic field from geometric characteristics of corresponding geometry.

From the geometrical point of view a charged particle interacting with electromagnetic field can be considered as a free particle in the spacetime with the length of a curve $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + A_\mu dx^\mu$ and equation for geodesic

$$\frac{d^2 x_\sigma}{du^2} = F_{\lambda\sigma}(x) x^\lambda_u,$$

where A_μ is a solution to equation $R_\lambda = 0$.

5 Discussion

We note that the property of gravitational field that we can choose reference frame where gravitational field is absent corresponds to the property of Riemannian geometry that we can perform change of variables so that the right hand side of (9) will be equal to zero.

For electromagnetic field we cannot find a reference frame where it is absent. This property demonstrates again that electromagnetism creates geometry different from Riemannian and corresponds to property of (11) that we cannot eliminate its right hand side by changing coordinates.

Because of the above mentioned correspondences we see that Riemannian geometry and Geometry of Electromagnetism are well suited for gravitation and electromagnetism respectively. It is worth expecting that we can construct geometries of weak and strong interactions using correspondence between Physics and Mathematics. If we know underlying geometries for weak and strong interactions we can look for a geometry containing all geometries as special cases. Then from its curvature function we may construct an action unifying all fundamental interaction. In this way we will be able to find electroweak model without Higgs fields and unify all interactions.

We choose $\Gamma^{\sigma}_{\lambda}(x, x_u, x_v) = F^{\sigma}_{\lambda}(x) + \Gamma^{\sigma}_{\lambda\nu}(x)x^{\nu}_{u}$ for geometry underlying unified model of Electromagnetism and Gravitation [1]. In this case we construct action from curvature function which is the sum of the action for Electromagnetic field in spacetime with (3) and gravitational field $g_{\mu\nu}$.

In conclusion we note that Kaluza -Klein theory gives Maxwell equations in weak fields approximation only. For the full theory, without any approximations, it fails to reproduce Maxwell equations.

References

[1] S. S. Shahverdiyev, Unification of Electromagnetism and Gravitation, in preparation