GENERAL GEOMETRY AND GEOMETRY OF ELECTROMAGNETISM

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**Abstract**

It is shown that Electromagnetism creates geometry different from Riemannian geometry. General geometry including Riemannian geometry and geometry underlying Electromagnetism as special cases is constructed. Action for electromagnetic field and Maxwell equations are derived from curvature function of geometry underlying Electromagnetism.

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# 1 Introduction

After it was realized that the underlying geometry for gravitation is Riemannian geometry the action functional for gravitational field has been derived from curvature characteristics of Riemannian geometry, namely the Lagrangian for gravitational field is scalar curvature of Riemannian geometry. And equation for geodesics coincides with the equation of motion for a particle interacting with gravitational field [1],[2].

After this discovery, many physicists and mathematicians tried to find underling geometry for electromagnetism. The requirements for this geometry are the following; equation of motion for a particle interacting with electromagnetic field must coincide with the equation for geodesics and Lagrangian for electromagnetic field must be related to curvature characteristics of the underlying geometry as in the case of Riemannian geometry and gravitation. All attempts to geometrize electromagnetism has been done in the framework of Riemannian geometry in a variety of different approaches [3]-[11]. Unfortunately, these attempts failed to satisfy the above requirements completely [12], [13] and the problem of geometrization of electromagnetism remained open.

In the present paper we show that electromagnetism can not be geometrized in the framework of Riemannian geometry. Therefore, for geometrization of electromagnetism we need different type of geometry. We construct a new geometry for this aim and call it General Geometry. This geometry includes already known Riemannian geometry as a special case. We introduce notion of curvature function which serves as a source for defining curvature characteristics of geometry. And prove that the most simplest particular case of General Geometry is geometry underlying Electromagnetism.

We show that equation for geodesics in geometry underlying electromagnetism coincides with the equation of motion for a charged particle interacting with electromagnetic field and construct Maxwell equations and action functional for electromagnetic field from curvature function.

In the next section we show that electromagnetism creates geometry different from Riemannian geometry and therefore cannot be geometrized in the framework of Riemannian geometry.

We develop General Geometry including Riemannian geometry as a special case in section 3.

In section 4, we derive action functional for electromagnetic field and Maxwell equations from curvature function of the underlying geometry.

Section 5 is devoted to discussion of correspondence between physical properties of fields and mathematical properties of corresponding geometries. We also discuss the problem of geometry and matter.

## 2 Interacting Classical Particle

The action for a free particle is

$$S = \frac{1}{2} m \int du \eta_{\mu\nu} x_u^\mu x_u^\nu,$$

where $x_u = dx/du, \eta_{\mu\nu} = diag(1-1-1-1)$, and $m$ is mass parameter. Consider a particle which interacts with gravitational field $h_{\mu\nu}$ with coupling constant $\lambda$:

$$S = \frac{1}{2} m \int du \eta_{\mu\nu} x_u^\mu x_u^\nu + \frac{1}{2} \lambda \int du h_{\mu\nu} x_u^\mu x_u^\nu. \tag{1}$$

We can represent (1) as

$$S = \frac{1}{2} \int du g_{\mu\nu} x_u^\mu x_u^\nu, \tag{2}$$

$$g_{\mu\nu} = m\eta_{\mu\nu} + \lambda h_{\mu\nu}. \tag{3}$$

This has been interpreted as a free particle moving in curved spacetime with metric $g_{\mu\nu}$ in General Relativity.

We go one step further and consider a charged particle, with charge $q$ which interacts with electromagnetic field $A_\mu$:

$$S = \frac{1}{2} m \int du \eta_{\mu\nu} x_u^\mu x_u^\nu + \frac{q}{c} \int du A_\mu x_u^\mu, \tag{4}$$

and represent (4) as

$$S = \frac{1}{2} \int du {}^A g_{\mu\nu} x_u^\mu x_u^\nu, \tag{5}$$

$${}^A g_{\mu\nu} = m\eta_{\mu\nu} + \frac{q}{c}(A_\mu f_\nu + A_\nu f_\mu), \tag{6}$$

where $f_\nu x_u^\nu = 1$. It easy to see that $f_\nu$ are functions of $du/dx^\nu$ and therefore spoil locality. It is clearly seen that we cannot represent (4) in the form of (5) with local functions of $x$ and $x_u$. This indicates that electromagnetism creates geometry different from Riemannian one. Hence, it cannot be geometrized in the framework of Riemannian geometry. The appropriate geometry for electromagnetism is presented in sections 3 and 4.

For the general case $S = \frac{1}{2} \int du \mathbf{g}_{\mu\nu}(x, x_u) x_u^\mu x_u^\nu$ when $\mathbf{g}_{\mu\nu}$ are considered as a function of $x$ and $x_u$ we obtain equation of motion

$$\frac{d^2 x^\nu}{du^2} G_{\lambda\nu} + \Gamma_{\lambda,\mu\nu} x_u^\mu x_u^\nu = 0, \tag{7}$$

where

$$G_{\lambda\nu}(\mathbf{g}_{\mu\nu}) = \frac{1}{2} \frac{\partial^2 \mathbf{g}_{\mu\sigma}}{\partial x_u^\nu \partial x_u^\lambda} x_u^\mu x_u^\sigma + \frac{\partial \mathbf{g}_{\mu\nu}}{\partial x_u^\lambda} x_u^\mu + \frac{\partial \mathbf{g}_{\lambda\sigma}}{\partial x_u^\nu} x_u^\sigma + \mathbf{g}_{\lambda\nu},$$

$$2\Gamma_{\lambda,\mu\nu}(\mathbf{g}_{\mu\nu}) = \frac{\partial^2 \mathbf{g}_{\mu\nu}}{\partial x^\sigma \partial x_u^\lambda} x_u^\sigma + \frac{\partial \mathbf{g}_{\lambda\nu}}{\partial x^\mu} + \frac{\partial \mathbf{g}_{\lambda\mu}}{\partial x^\nu} - \frac{\partial \mathbf{g}_{\mu\nu}}{\partial x^\lambda}.$$

Function $G_{\lambda\nu}(\mathbf{g}_{\mu\sigma})$ plays the role of a tensor for raising and lowering indices and $\Gamma_{\lambda,\mu\nu}$ connection.

$$G_{\lambda\nu}({}^A g_{\mu\sigma}) = \eta_{\lambda\nu}, \quad G_{\lambda\nu}(g_{\mu\sigma}) = g_{\lambda\nu}. \tag{8}$$

# 3 General Geometry

In this section we construct a new geometry. This geometry includes Riemannian geometry, geometry underlying Electromagnetism (see next section), geometry underlying a unified model of Electromagnetism and Gravitation [13], and infinite number of geometries, physical interpretation of which is not known at the present time, as special cases. Because of this we call it General Geometry.

Let M be a manifold with coordinates $x^\lambda, \lambda = 1, ..., n$. Consider a curve on this manifold $x^\lambda(u)$. Vector field

$$V = \xi^\lambda \frac{\partial}{\partial x^\lambda}$$

has coordinates $\xi^\lambda$. In Riemannian geometry it is accepted that

$$\frac{d\xi^\lambda}{du} = -\Gamma'^\sigma_{\lambda\nu}(x) x^\nu_u \xi^\lambda, \tag{9}$$

where $\Gamma'^\sigma_{\lambda\nu}(x)$ are functions of $x$ only.

To construct General Geometry we assume that

$$\frac{d\xi^\sigma}{du} = -\Gamma^\sigma_\lambda(x, x_u)\xi^\lambda. \tag{10}$$

$\Gamma^\sigma_\lambda(x, x_u)$ are general functions of $x$ and $x_u$. The next step is to consider $x$ as a function of two parameters $u, \upsilon$ and find $\lim\limits_{\substack{\Delta\upsilon\to 0\\ \Delta u\to 0}} \Delta\xi^\sigma/\Delta u \Delta\upsilon$. In order to do that we need

$$\frac{d\xi^\sigma}{du} = -\Gamma^\sigma_\lambda \xi^\lambda, \quad \frac{d\xi^\sigma}{d\upsilon} = -\tilde{\Gamma}^\sigma_\lambda \xi^\lambda,$$

$$\Gamma^\sigma_\lambda = \Gamma^\sigma_\lambda(x, x_u, x_\upsilon), \quad \tilde{\Gamma}^\sigma_\lambda = \tilde{\Gamma}^\sigma_\lambda(x, x_u, x_\upsilon).$$

After simply calculations we arrive at

$$\lim_{\substack{\Delta\upsilon\to 0\\ \Delta u\to 0}} \frac{\Delta\xi^\sigma}{\Delta u \Delta\upsilon} = R^\sigma_\lambda \xi^\lambda,$$

where

$$R^\sigma_\lambda = \frac{d}{d\upsilon}\Gamma^\sigma_\lambda - \frac{d}{du}\tilde{\Gamma}^\sigma_\lambda + \tilde{\Gamma}^\sigma_\rho \Gamma^\rho_\lambda - \Gamma^\sigma_\rho \tilde{\Gamma}^\rho_\lambda.$$

We call $R^\sigma_\lambda$ curvature function.

Representing $\Gamma^\sigma_\lambda(x, x_u)$ as

$$\Gamma^\sigma_\lambda(x, x_u) = F^\sigma_\lambda(x) + \Gamma^\sigma_{\lambda\nu}(x) x^\nu_u + \Gamma^\sigma_{\lambda\nu\mu}(x) x^\nu_u x^\mu_u + ...$$

and considering each order in $x_u$ separately we define a set of new geometries. Only the first order in $x_u$ is already known Riemannian geometry. It is easy to demonstrate that. Let

$$\Gamma^\sigma_\lambda(x, x_u, x_\upsilon) = \Gamma^\sigma_{\lambda\nu}(x) x^\nu_u, \quad \tilde{\Gamma}^\sigma_\lambda(x, x_u, x_\upsilon) = \Gamma^\sigma_{\lambda\nu}(x) x^\nu_\upsilon.$$

Curvature function for this case is

$$R^\sigma_\lambda = R^\sigma_{\lambda\mu\nu}(x^\nu_u x^\mu_\upsilon - x^\nu_\upsilon x^\mu_u),$$

where

$$R^\sigma_{\lambda\mu\nu} = \partial_\mu \Gamma^\sigma_{\lambda\nu} - \partial_\nu \Gamma^\sigma_{\lambda\mu} + \Gamma^\sigma_{\rho\mu}\Gamma^\rho_{\lambda\nu} - \Gamma^\sigma_{\rho\nu}\Gamma^\rho_{\lambda\mu}$$

is the curvature tensor of Riemannian geometry. Hence, Riemannian geometry is a particular case of General Geometry.

# 4 Geometry of Electromagnetism

In this section we consider the most simplest case of General Geometry

$$\Gamma^\sigma_\lambda(x, x_u, x_\upsilon) = F^\sigma_\lambda(x(u,\upsilon)), \quad \tilde{\Gamma}^\sigma_\lambda(x, x_u, x_\upsilon) = F^\sigma_\lambda(x(u,\upsilon)),$$

when $\Gamma^\sigma_\lambda(x, x_u)$ does not depend on $x_u$. For this geometry we have

$$\frac{d\xi^\sigma}{du} = -F^\sigma_\lambda(x)\xi^\lambda. \tag{11}$$

Geometry defined by (11) has different properties than Riemannian geometry defined by (9). We do not get into details here. We simply mention that in this geometry the notion of parallel transport is not defined. As we show in the sequel this makes it be underlying geometry for Electromagnetism. We call it Geometry of Electromagnetism.

To obtain equations for geodesics we substitute $\xi^\lambda$ in (11) by $x^\lambda_u$ and arrive at

$$\frac{d^2 x_\sigma}{du^2} = -F_{\sigma\lambda}(x)x^\lambda_u.$$

This is exactly equation of motion for a charged particle moving in electromagnetic field $A_\mu$, if we choose

$$F_{\sigma\lambda} = \frac{q}{cm}(\partial_\sigma A_\lambda - \partial_\lambda A_\sigma),$$

where $c$ is the velocity of the light. Then we find curvature function

$$R^\sigma_\lambda = R^\sigma_{\mu\lambda}(x^\mu_\upsilon - x^\mu_u),$$

where

$$R^\sigma_{\mu\lambda} = \partial_\mu F^\sigma_\lambda.$$

After summing by two of the three indices we obtain

$$R_\lambda = R^\mu_{\mu\lambda} = \partial_\mu F^\mu_\lambda.$$

Equations $R_\lambda = 0$ coincide with the Maxwell equations. We choose the length of a curve as

$$ds = \sqrt{\eta_{\mu\nu} dx^\mu dx^\nu} + \frac{q}{cm} A_\mu dx^\mu$$

and indices are raised and lowered by $\eta_{\mu\nu}$ because of (8). We can construct from $R_\lambda$ and $A^\lambda$ a Lagrangian

$$R = A^\lambda R_\lambda = \partial_\mu (A^\lambda F^\mu_\lambda) - \frac{1}{2} F_{\mu\lambda} F^{\mu\lambda}.$$

This coincides with the Lagrangian of electromagnetic field up to total derivative.

We see that as in the case of Riemannian geometry and gravitation we can find equations and action functional for electromagnetic field from geometric characteristics of Geometry of Electromagnetism. And equation for geodesics coincides with the equation of motion for a particle interacting with electromagnetic field.

From the geometrical point of view a charged particle interacting with electromagnetic field can be considered as a free particle in the spacetime with the length of a curve $ds = \sqrt{\eta_{\mu\nu} dx^\mu dx^\nu} + \frac{q}{cm} A_\mu dx^\mu$ and equation for geodesic

$$\frac{d^2 x_\sigma}{du^2} = \frac{q}{cm} (\partial_\lambda A_\sigma - \partial_\sigma A_\lambda) x_u^\lambda,$$

where $A_\mu$ is a solution to equation $R_\lambda = 0$.

# 5 Discussion

We note that the property of gravitational field that we can choose reference frame where gravitational field is absent corresponds to the property of Riemannian geometry that we can perform change of variables so that the right hand side of (9) will be equal to zero.

For electromagnetic field we cannot find a reference frame where it is absent. This property demonstrates again that electromagnetism creates geometry different from Riemannian and corresponds to property of (11) that we cannot eliminate its right hand side by changing coordinates. Therefore, geometrization of electromagnetism in geometries like Riemannian, where notion of parallel transport is defined must fail.

Because of the above mentioned correspondences we see that Riemannian geometry and Geometry of Electromagnetism are well suited for gravitation and electromagnetism respectively. It is worth expecting that we can construct geometries of weak and strong interactions using correspondence between Physics and Mathematics. If we know underlying geometries for weak and strong interactions we can look for a geometry containing all geometries as special cases. Then from its curvature function we may construct an action unifying all fundamental interactions. In this way we will be able to find electroweak model without Higgs fields and unify all interactions.

As it follows from the results of previous section geometry underlying electromagnetism is defined by

$$\frac{d\xi^\sigma}{du} = -\frac{q}{cm} (\partial_\sigma A_\lambda - \partial_\lambda A_\sigma) \xi^\lambda.$$

And the length of a curve is $ds = \sqrt{\eta_{\mu\nu} dx^\mu dx^\nu} + \frac{q}{cm} A_\mu dx^\mu$. We see that geometry and the length of a curve depend on characteristics of interacting particles $q$, $m$ and sources for $A_\mu$ in contrast with the case of gravitation where geometry depends on characteristics of sources for gravitational field only. Geometry of electromagnetism gives us a new understanding of problem of geometry and matter with conclusion that geometry is determined by interaction. If there is no interaction geometry is flat as it follows if we choose neutral particle ($q = 0$). In this case in the presence of electromagnetic field $A_\mu$, $\frac{d\xi^\sigma}{du} = 0$.

We choose $\Gamma^\sigma_\lambda(x, x_u, x_v) = F^\sigma_\lambda(x) + \Gamma^\sigma_{\lambda\nu}(x)x^\nu_u$ for geometry underlying unified model of Electromagnetism and Gravitation [13]. In this case we construct action from curvature function which is the sum of the action for Electromagnetic field in spacetime with (3) and gravitational field $g_{\mu\nu}$. This model predicts that electromagnetic field is a source for gravitational field.

In conclusion we note that Kaluza -Klein theory gives Maxwell equations in weak fields approximation only. For the full theory, without any approximations, it fails to reproduce Maxwell equations.

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