EE541 Homework #4, #5, and #6

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This paper collects Rob Schuler's answers to homework problems assigned before the second examination in EE541. The purpose of type setting these assignments is two fold. The first point is to learn how to typeset equations of this nature using LaTeX. The second purpose is to produce an electronic form that will be easy to keep for years after the course.

1 Numbering of Assigned Problems

The sections are numbered sequentially by the LyX editor starting from 1. Problems are numbered at the first tier title using a scheme of Assignment-dot-problem. For example Problem 1.1 is the first problem of the first homework assignment and Problem 2.3 is the third problem of the second homework assignment.

1.1 Homework #4

Homework #4 consists of six problems assigned from Chapter 5 of the textbook: 5.1, 5.5, 5.16, 5.17, 5.21, and 5.28.

Section in this Paper	Problem Number	Assigned Number
2	4.1	Balanis 5.1
3	4.2	Balanis 5.5
4	4.3	Balanis 5.16
5	4.4	Balanis 5.17
6	4.5	Balanis 5.21
7	4.6	Balanis 5.28

1.2 Homework #5

Homework #5 consists of a single derivation assigned in class. Specifically to derive the equations for the TM modes of a circular wave guide of cross section a, using cylindrical coordinates.

Section in this Paper	Problem Number	Assigned Number
8	5.1	Class Notes

1.3 Homework #6

Homework #6 consists of three problems assigned in class and three problems assigned from Chapter 6 of the textbook: 6.1, 6.17, and 6.21.

Section in this Paper	Problem Number	Assigned Number
9	6.1	Class Notes 1
10	6.2	Class Notes 2
11	6.3	Class Notes 3
12	6.4	Balanis 6.1
13	6.5	Balanis 6.17
14	6.6	Balanis 6.20
15	6.6	Balanis 6.21

2 Problem 4.1 (Balanis 5.1)

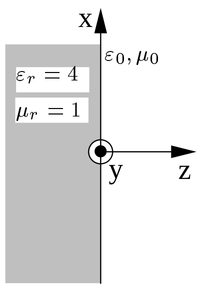
2.1 Statement

A uniform plane wave traveling in a dielectric medium with $\varepsilon_r = 4$ and $\mu_r = 1$ is incident normally upon a free-space medium. If the incident electric field is given by

$$\widetilde{E^i} = \widehat{a_y} 2 \times 10^{-3} e^{-j\beta z}$$

write:

(a) The corresponding incident magnetic field.
(b) The reflection and transmission coefficients.
(c) The reflected and transmitted electric and magnetic fields.
(d) The incident, reflected, and transmitted power densities.



2.2 Solution

First we calculate the intrinsic impedance η for the dielectric medium:

$$\begin{aligned} \eta &= \sqrt{\frac{\mu}{\varepsilon}} \\ &= \sqrt{\frac{1 \times \mu_0}{4 \times \varepsilon_0}} \\ &\doteq \sqrt{\frac{1 \times 4\pi \times 10^{-7}}{4 \times 8.85 \times 10^{-12}}} \\ &\doteq \sqrt{\frac{\pi}{8.85 \times 10^{-5}}} \\ &\doteq 188.4 \end{aligned}$$

Part (a) $\widetilde{H^i}$

$$H^i = \widehat{a_i} \times \frac{1}{\eta} \widetilde{E^i}$$

where $\widehat{a_i}$ is a unit vector in the direction of wave propagation (the direction of incidence in this case). In this case, $\widehat{a_i} = \widehat{a_z}$ and the direction of $\widetilde{E^i}$ is $\widehat{a_y}$, which means that $\widetilde{H^i}$ must be in the $-\widehat{a_x}$ direction since $\widehat{a_z} \times \widehat{a_y} = -\widehat{a_x}$. To verify this, note that the direction of $\widetilde{E^i}$ ($\widehat{a_y}$) crossed with the direction of $\widetilde{H^i}$ ($-\widehat{a_x}$) yields the direction of propagation $\widehat{a_z}$. This allows us to directly write:

$$\begin{aligned} \widetilde{H^i} &= -\widehat{a_x} \left(\frac{2 \times 10^{-3}}{\eta} \right) e^{-j\beta z} \\ &= -\widehat{a_x} \left(\frac{2 \times 10^{-3}}{188.4} \right) e^{-j\beta z} \\ &= -\widehat{a_x} \left(10.6 \times 10^{-6} \right) e^{-j\beta z} \end{aligned}$$

Part (b) R and T

$$\begin{aligned} R &= \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \\ &= \frac{377 - 188.4}{377 + 188.4} \\ &= 0.334 \end{aligned}$$

$$\begin{aligned} T &= \frac{2\eta_2}{\eta_2 + \eta_1} \\ &= \frac{2(377)}{377 + 188.4} \\ &= 1.334 \end{aligned}$$

Part (c) Reflected and Transmitted $\widetilde{E}$ and $\widetilde{H}$. In this case $\widetilde{E^i}$, $\widetilde{E^r}$ and $\widetilde{E^t}$ are all polarized in the same direction ($\widehat{a_y}$). This means that $\widetilde{H^i}$ and $\widetilde{H^t}$ will be in the $-\widehat{a_x}$ direction, but $\widetilde{H^r}$ will be in the $+\widehat{a_x}$ direction (So that $\widetilde{E} \times \widetilde{H}$ will point in the direction of wave propagation, which is $-\widehat{a_z}$ for the reflected wave). It is also important to consider what happens to β as we go from the dielectric to free-space. Recall that the wave constant $\beta = \omega\sqrt{\mu\varepsilon}$ or

$$\begin{aligned} \beta &= \omega\sqrt{\mu_r\mu_0\varepsilon_r\varepsilon_0} \\ &= \omega\sqrt{\mu_r\varepsilon_r}\sqrt{\mu_0\varepsilon_0} \\ &= \beta_0\sqrt{\mu_r\varepsilon_r} \end{aligned}$$

where $\beta_0 = \frac{\beta}{\sqrt{\mu_r\varepsilon_r}} = 0.5\beta$ is the wave constant in free space. From this we can write directly:

$$\begin{aligned} \widetilde{E^r} &= R\widetilde{E^i} \\ &= (0.334)\widehat{a_y}2 \times 10^{-3}e^{-j\beta z} \\ &= \widehat{a_y}0.668 \times 10^{-3}e^{-j\beta z} \end{aligned}$$

$$\begin{aligned} \widetilde{E^t} &= e^{-0.5\beta jz}T\widetilde{E^i} \\ &= (1.334)\widehat{a_y}2 \times 10^{-3}e^{-j0.5\beta z} \\ &= \widehat{a_y}2.668 \times 10^{-3}e^{-j0.5\beta z} \end{aligned}$$

$$\begin{aligned} \widetilde{H^r} &= -R\widetilde{H^i} \\ &= (0.334)\widehat{a_x}\left(10.6 \times 10^{-6}\right)e^{-j\beta z} \\ &= \widehat{a_x}\left(3.54 \times 10^{-6}\right)e^{-j\beta z} \end{aligned}$$

$$\begin{aligned} \widetilde{H^t} &= \widehat{a_z} \times \frac{1}{\eta_2}\widetilde{E^t} \\ &= -\widehat{a_x}\frac{1}{377}2.668 \times 10^{-3}e^{-j0.5\beta z} \\ &= -\widehat{a_x}\left(7.08 \times 10^{-6}\right)e^{-j0.5\beta z} \end{aligned}$$

Part (d) Power Densities Recall that the time average power density $= \frac{1}{2}RE\left(\widetilde{E} \times \widetilde{H^*}\right)$ and the instantaneous power density $= \overrightarrow{E} \times \overrightarrow{H}$. Since all of the fields are stated in the frequency domain, we only find the time average power densities here.

$$\begin{aligned}
transmitted\,time\,average\,power\,density &= \frac{1}{2}RE\left(\widetilde{E^t} \times \widetilde{H^{t*}}\right) \\
&= \frac{1}{2}RE\left(\widehat{a_y}2.668 \times 10^{-3}e^{-j0.5\beta z} \times -\widehat{a_x}\left(7.08 \times 10^{-6}\right)e^{j0.5\beta z}\right) \\
&= \frac{1}{2}RE\left(\begin{vmatrix} \widehat{a_x} & \widehat{a_y} & \widehat{a_z} \\ 0 & 2.668 \times 10^{-3}e^{-j0.5\beta z} & 0 \\ -\left(7.08 \times 10^{-6}\right)e^{j0.5\beta z} & 0 & 0 \end{vmatrix}\right) \\
&= \frac{1}{2}RE\left(\left(7.08 \times 10^{-6}\right)e^{j0.5\beta z}\widehat{a_z}2.668 \times 10^{-3}e^{-j0.5\beta z}\right) \\
&= \frac{1}{2}RE\left(\left(1.89 \times 10^{-8}\right)\widehat{a_z}\right) \\
&= 9.44 \times 10^{-9}\widehat{a_z}
\end{aligned}$$

$$\begin{aligned}
incident\,time\,average\,power\,density &= \frac{1}{2}RE\left(\widetilde{E^i} \times \widetilde{H^{i*}}\right) \\
&= \frac{1}{2}RE\left(\widehat{a_y}2 \times 10^{-3}e^{-j\beta z} \times -\widehat{a_x}\left(10.6 \times 10^{-6}\right)e^{j\beta z}\right) \\
&= \frac{1}{2}RE\left(\begin{vmatrix} \widehat{a_x} & \widehat{a_y} & \widehat{a_z} \\ 0 & 2 \times 10^{-3}e^{-j\beta z} & 0 \\ -\left(10.6 \times 10^{-6}\right)e^{j\beta z} & 0 & 0 \end{vmatrix}\right) \\
&= \frac{1}{2}RE\left(\left(10.6 \times 10^{-6}\right)e^{j\beta z}\widehat{a_z}2 \times 10^{-3}e^{-j\beta z}\right) \\
&= \frac{1}{2}RE\left(\left(2.12 \times 10^{-8}\right)\widehat{a_z}\right) \\
&= 1.06 \times 10^{-8}\widehat{a_z}
\end{aligned}$$

$$\begin{aligned}
reflected\,time\,average\,power\,density &= \frac{1}{2}RE\left(\widetilde{E^r} \times \widetilde{H^{r*}}\right) \\
&= \frac{1}{2}RE\left(\widehat{a_y}0.668 \times 10^{-3}e^{-j\beta z} \times -\widehat{a_x}\left(3.54 \times 10^{-6}\right)e^{+j\beta z}\right) \\
&= \frac{1}{2}RE\left(\begin{vmatrix} \widehat{a_x} & \widehat{a_y} & \widehat{a_z} \\ 0 & 0.668 \times 10^{-3}e^{-j\beta z} & 0 \\ -\left(3.54 \times 10^{-6}\right)e^{j\beta z} & 0 & 0 \end{vmatrix}\right) \\
&= \frac{1}{2}RE\left(\left(3.54 \times 10^{-6}\right)e^{j\beta z}\widehat{a_z}0.668 \times 10^{-3}e^{-j\beta z}\right) \\
&= \frac{1}{2}RE\left(\left(2.36 \times 10^{-9}\right)\widehat{a_z}\right) \\
&= 1.18 \times 10^{-9}\widehat{a_z}
\end{aligned}$$

NOTE: the reflected plus the transmitted average power densities equal the incident average power density:
$0.18 \times 10^{-8} + 0.944 \times 10^{-8} = 1.12 \times 10^{-8} \approx 1.06 \times 10^{-8}$

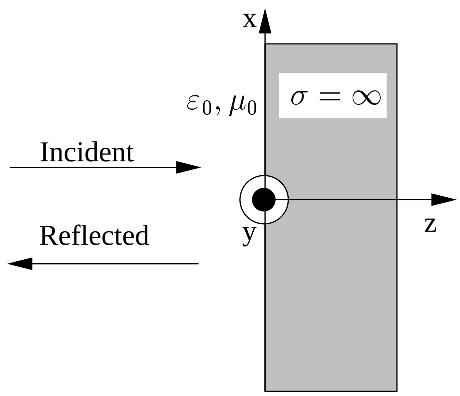
3 Problem 4.2 (Balanis 5.5)

3.1 Statement

A time-harmonic electromagnetic wave traveling in free space is incident normally upon a perfect conducting planar surface, as shown in Figure P5-5. Assuming the incident electric field is given by

$$\widetilde{E^i} = \widehat{a_x} E_0 e^{-j\beta_0 z}$$

find (a) the reflected electric field, (b) the incident and reflected magnetic fields, and (c) the current density $\widetilde{J_s}$ induced on the conducting surface.



3.2 Solution

The first step in this problem is to calculate the Reflection and Transmission Coefficients. In this case, we must account for the infinite conductivity in the P.E.C. when calculating η_2.

$$\begin{aligned}
\eta_1 &= 377 \\
\eta_2 &= \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} \\
&= \sqrt{\frac{j\omega\mu}{\infty + j\omega\varepsilon}} \\
&= 0
\end{aligned}$$

So

$$\begin{aligned}
R &= \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \\
&= \frac{0 - 377}{0 + 377} \\
&= -1
\end{aligned}$$

$$\begin{aligned}
T &= \frac{2\eta_2}{\eta_2 + \eta_1} \\
&= \frac{2(0)}{0 + 377} \\
&= 0
\end{aligned}$$

Part (a) reflected electric field The reflected electric field can be found using equation 5-48a on page 207 of Balanis:

$$\begin{aligned}\widetilde{E^r} &= \widehat{a_x}\Gamma^b E_0 e^{+\alpha_1 z} e^{+j\beta_1 z} \\ &= -\widehat{a_x} E_0 e^{+j\beta_0 z}\end{aligned}$$

Part (b) incident and reflected magnetic fields These fields can be written directly from the incident and reflected electric fields by simply dividing by the intrinsic impedance η, which is approximately 377 Ω in free space. The orientation of $\widetilde{H}$ will be perpendicular to both the direction of the wave propagation and the orientation of $\widetilde{E}$, and can be found using the right-hand rule.

$$\widetilde{H^i} = \widehat{a_y}\frac{E_0}{377} e^{-j\beta_0 z}$$

$$\widetilde{H^r} = \widehat{a_y}\frac{E_0}{377} e^{+j\beta_0 z}$$

Part (c) induced current Recall that $\widetilde{J}_s = \widehat{n} \times \widetilde{H}$. In this case, $\widehat{n} = -\widehat{a_z}$ and

$$\begin{aligned}\widetilde{H^{total}} &= \widetilde{H^i} + \widetilde{H^r} \\ &= \widehat{a_y}\frac{E_0}{377}\left[e^{+j\beta_0 z} + e^{-j\beta_0 z}\right] \\ \left.\widetilde{H^{total}}\right|_{z=0} &= \widehat{a_y}\frac{2E_0}{377}\end{aligned}$$

$$\begin{aligned}\widetilde{J}_s &= \widehat{n} \times \widetilde{H} \\ &= -\widehat{a_z} \times \widehat{a_y}\frac{2E_0}{377} \\ &= \widehat{a_x}\frac{2E_0}{377}\, Amps\end{aligned}$$

4 Problem 4.3 (Balanis 5.16)

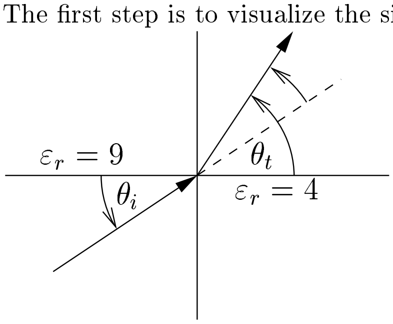
4.1 Statement

A perpendicularly polarized plane wave traveling in a dielectric medium with relative permittivity of 9 is obliquely incident on another dielectric with a relative permittivity of 4. Assuming that the permeabilities of both media are the same, find the incident angle (measured from the normal to the interface) that results in total reflection.

4.2 Solution

This problem is asking us to find the *critical angle* for the two given media.

The first step is to visualize the situation:



Here θ_i is the angle of incidence and θ_t is the angle of the transmitted wave. We know that the transmitted wave will "move" in the direction indicated because the permittivity in the second medium is less than that in the first ($\varepsilon_2 < \varepsilon_1$). In other words, we know that θ_t will always be greater than θ_i. Using equation 5-36 on page 197 of Balanis:

$$\theta_i \geq \theta_c = \sin^{-1}\left(\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\right)$$

We see that the critical angle is:

$$\begin{aligned} \theta_c &= \sin^{-1}\left(\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\right) \\ &= \sin^{-1}\left(\sqrt{\frac{4\varepsilon_0}{9\varepsilon_0}}\right) \\ &= \sin^{-1}(.666) \\ &= .729 rad \\ &= 41.8^\circ \end{aligned}$$

This means that for any incident angle greater than 0.729 radians or 41.8 degrees, all of the energy of the wave shall be reflected back into the first medium.

5 Problem 4.4 (Balanis 5.17)

5.1 Statement

Calculate the Brewster and critical angles for a parallel polarized wave when the plane interface is (a) water to air (ε_r of water is 81), (b) air to water, and (c) high density glass to air (ε_r of glass is 9).

5.2 Solution

Recall that the Brewster angle is that angle of incidence (from the normal) that results in total transmission of a wave from the first medium into the second. When both media have the same permeability (as is true in all 3 cases in this problem), the Brewster angle can be found by equation 5-33b on page 195 of Balanis:

$$\theta_i = \theta_B = \tan^{-1}\left(\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\right)$$

Also, recall equation 5-36 on page 197 of Balanis for finding the critical angle:

$$\theta_i \geq \theta_c = \sin^{-1}\left(\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\right)$$

NOTE: in the case of total transmission, θ_i must exactly equal the Brewster angle θ_B, which is different than the physical relationship for the critical angle in which θ_i must equal or be greater than the critical angle θ_c.

Part (a) water to air

$$\begin{aligned}
\theta_B &= \tan^{-1}\left(\sqrt{\frac{\varepsilon_0}{81\varepsilon_0}}\right) \\
&= \tan^{-1}\left(\frac{1}{9}\right) \\
&= 0.111 rad \\
&= 6.34^\circ
\end{aligned}$$

$$\begin{aligned}
\theta_c &= \sin^{-1}\left(\sqrt{\frac{\varepsilon_0}{81\varepsilon_0}}\right) \\
&= \sin^{-1}\left(\frac{1}{9}\right) \\
&= 0.111.rad \\
&= 6.38^\circ
\end{aligned}$$

Part (b) air to water

$$\begin{aligned}
\theta_B &= \tan^{-1}\left(\sqrt{\frac{81\varepsilon_0}{\varepsilon_0}}\right) \\
&= \tan^{-1}(9) \\
&= 1.46 rad \\
&= 83.66^\circ
\end{aligned}$$

$$\begin{aligned}
\theta_c &= \sin^{-1}\left(\sqrt{\frac{81\varepsilon_0}{\varepsilon_0}}\right) \\
&= \sin^{-1}(9) \\
&= \frac{\pi}{2} - j2.887 rad \\
&= 90^\circ - j2.887
\end{aligned}$$

There is no physically realizable critical angle when going from air to water!

Part (c) glass to air

$$
\begin{aligned}
\theta_B &= \tan^{-1}\left(\sqrt{\frac{\varepsilon_0}{9\varepsilon_0}}\right) \\
&= \tan^{-1}\left(\frac{1}{3}\right) \\
&= 0.322 rad \\
&= 18.43^\circ
\end{aligned}
$$

$$
\begin{aligned}
\theta_c &= \sin^{-1}\left(\sqrt{\frac{\varepsilon_0}{9\varepsilon_0}}\right) \\
&= \sin^{-1}\left(\frac{1}{3}\right) \\
&= 0.340 rad \\
&= 19.47^\circ
\end{aligned}
$$

6 Problem 4.5 (Balanis 5.21)

6.1 Statement

The heights above the earth of a transmitter and receiver are, respectively, 100 and 10 m, as shown in Figure P5-21. Assuming that the transmitter radiates both perpendicular and parallel polarizations, how far apart (in meters) should the transmitter and receiver be placed so that the reflected wave has no parallel polarization? Assume that the reflecting medium is a lossless flat earth with a dielectric constant of 16.

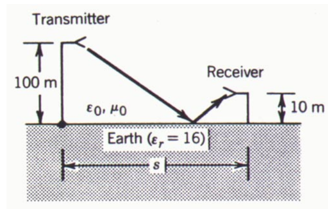


FIGURE P5-21

6.2 Solution

From page 195 of Balanis, "*The incidence angle θ_i, as given by (5-31a) or (5-33), which reduces the reflection coefficient for parallel polarization to zero, is referred to as the Brewster angle θ_B.* It should be noted that when $\mu_1 = \mu_2$, *the incidence Brewster angle $\theta_i = \theta_B$ of (5-33) exists only if the polarization of the wave is parallel (vertical).*"

For completeness:

$\sin\theta_i = \sqrt{\frac{\frac{\varepsilon_2}{\varepsilon_1} - \frac{\mu_2}{\mu_1}}{\frac{\varepsilon_2}{\varepsilon_1} - \frac{\varepsilon_1}{\varepsilon_2}}}$	(5-31a)
$\theta_i = \theta_B = \sin^{-1}\left(\sqrt{\frac{\varepsilon_2}{\varepsilon_1+\varepsilon_2}}\right)$	(5-33)

From this we can see that if we pick the distance between the transmitter and receiver such that the wave strikes the ground at the Brewster angle θ_B, then we know that the parallel part of the wave shall be completely transmitted into the earth, while the vertical part of the wave will be reflected (at least partially).

Brewster Angle First we calculate the Brewster angle using equation 5-33:

$$\begin{aligned}
\theta_B &= \sin^{-1}\left(\sqrt{\frac{\varepsilon_2}{\varepsilon_1 + \varepsilon_2}}\right) \\
&= \sin^{-1}\left(\sqrt{\frac{16\varepsilon_0}{\varepsilon_0 + 16\varepsilon_0}}\right) \\
&= \sin^{-1}\left(\sqrt{\frac{16}{17}}\right) \\
&= \sin^{-1}(.970) \\
&= 1.33 rad \\
&= 76.0^\circ
\end{aligned}$$

Distance s Using elementary trigonometry, we know that the distance d_1 from the transmitter to the point on the ground where the wave hits the ground at the Brewster angle can be found by

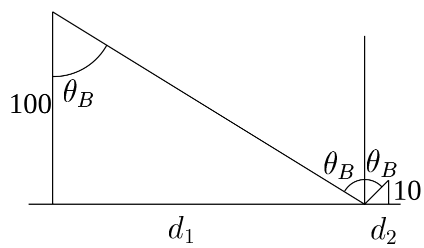
$$\begin{aligned}\tan\theta_B &= \frac{d_1}{100m}\\ d_1 &= 100m\tan(1.33)\\ &= 407.23m\end{aligned}$$

We also know from Snell that the angle of incidence equals the angle of reflection for the vertical part of the wave, so the distance d_2 from the point on the ground from which the wave hits to the receiver can be found by

$$\begin{aligned}\tan\theta_B &= \frac{d_2}{10m}\\ d_2 &= 10m\tan(1.33)\\ &= 40.723m\end{aligned}$$

The total distance s between the transmitter and receiver is then

$$s = d_1 + d_2 = 407.23m + 40.723m \doteq 448m$$



7 Problem 4.6 (Balanis 5.28)

7.1 Statement

At large observation distances the field radiated by a satellite antenna which is attempting to communicate with a submerged submarine is locally TEM (also assume uniform plane wave), as shown in Figure P5-28. Assuming the incident electric field before it impinges on the water is $1\,mV/m$ and the submarine is directly below the satellite, find at $1\,MHz$:

(a) The intensity of the reflected $\overrightarrow{E}$ field.
(b) The SWR created in air.
(c) The incident and reflected power densities.
(d) The intensity of the transmitted $\overrightarrow{E}$ field.
(e) The intensity of the transmitted power density.
(f) The depth d (in meters) of the submarine where the intensity of the transmitted electric field is 0.368 of its value immediately after it enters the water.
(g) The depth (in meters) of the submarine so that the distance from the surface of the ocean to the submarine is 20λ (λ in water).
(h) The time (in seconds) it takes the wave to travel from the surface of the ocean to the submarine at a depth of $100\,m$.
(i) The velocity of the wave in water to that in air (v/v_o).

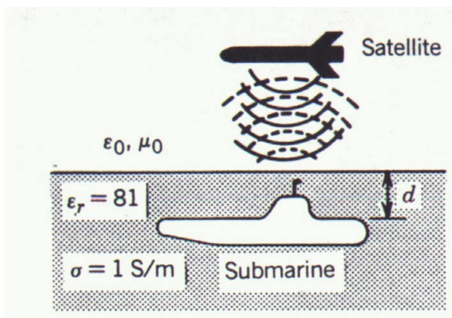


FIGURE P5-28

7.2 Solution

First, since $E_0 = 1mV = 1 \times 10^{-3}V$, and $\beta = \frac{\omega}{c} = \frac{2\pi 10^6}{3\times 10^8} = \frac{2}{3}\pi \times 10^{-2}$we can directly write the equation for the incident $\overrightarrow{E}$ field:

$$\begin{aligned}\widetilde{E_i} &= E_0 e^{-j\beta z} \\ &= 10^{-3} e^{-j\frac{2}{3}\pi\times 10^{-2} z}\end{aligned}$$

Next we calculate the coefficients of reflection and transmission.

In air, $\eta_1 \doteq 377\Omega$.

In this case, the Sea Water has a conductivity of 1 S/m, so:

$$\begin{aligned}\eta_2 &= \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} \\ &= \sqrt{\frac{j2\pi \times 10^6 \mu_0}{1 + j2\pi \times 10^6 \varepsilon_0\,(81)}} \\ &\doteq 377\Omega\sqrt{\frac{j2\pi \times 10^6}{1 + j162\pi \times 10^6}}\end{aligned}$$

$$
\begin{aligned}
&= 377\Omega\sqrt{\frac{j2\pi\times 10^6}{1+j162\pi\times 10^6}\times\frac{1-j162\pi\times 10^6}{1-j162\pi\times 10^6}} \\
&= 377\Omega\sqrt{\frac{j2\pi\times 10^6+2(162)\left(\pi\times 10^6\right)^2}{1+\left(162\pi\times 10^6\right)^2}} \\
&= 377\Omega\sqrt{\frac{j2\pi+3.2\times 10^9}{1+2.59\times 10^{11}}} \\
&= \frac{377}{5\times 10^5}\sqrt{3.2\times 10^9+j2\pi} \\
&= 7.54\times 10^{-4}\left(5.7\times 10^4+j5.5\times 10^{-5}\right) \\
&= 42.7+j4.2\times 10^{-8}
\end{aligned}
$$

NOTE: In this case, the real part is much much bigger than the imaginary part—we can consider $\eta_2 = 43$.

$$
\begin{aligned}
R &= \frac{\eta_2-\eta_1}{\eta_2+\eta_1} \\
&= \frac{43-377}{43+377} \\
&= -0.8
\end{aligned}
$$

$$
\begin{aligned}
T &= \frac{2\eta_2}{\eta_2+\eta_1} \\
&= \frac{2(43)}{43+377} \\
&= 0.2
\end{aligned}
$$

Part (a) Reflected Now we can find the reflected $\widetilde{E}$ field by multiplying the incident $\widetilde{E}$ field by R.

$$
\begin{aligned}
\widetilde{E_r} &= R\widetilde{E_i} \\
&= (-0.8)10^{-3}e^{-j\frac{2}{3}\pi\times 10^{-2}z} \\
&= 8\times 10^{-4}e^{-j\frac{2}{3}\pi\times 10^{-2}z}
\end{aligned}
$$

Part (b) SWR In this case $\varepsilon_2 > \varepsilon_1$, so we can use equation 5-9b on page 185 of Balanis to calculate the standing wave ratio (SWR):

$$
\begin{aligned}
SWR &= \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \\
&= \sqrt{\frac{81}{1}} \\
&= 9
\end{aligned}
$$

Part (c) Incident and Reflected Power Densities Using equation 5-6a on page 183 of Balanis, the incident power density is:

$$
\begin{aligned}
S_{av}^i &= \frac{1}{2}Re\left(\widetilde{E^i}\times\widetilde{H^{i*}}\right) = \widehat{a_z}\frac{|E_0|^2}{2\eta_1} \\
&= \widehat{a_z}\frac{10^{-6}}{754} \\
&= \widehat{a_z}1.326\times 10^{-9}
\end{aligned}
$$

Using equation 5-6b on page 183 of Balanis, the reflected power density is:

$$\begin{aligned} S_{av}^{r} &= \frac{1}{2}Re\left(\widetilde{E^{r}} \times \widetilde{H^{r*}}\right) = -\widehat{a_z}\left|\Gamma^{b}\right|^{2}\frac{\left|E_0\right|^{2}}{2\eta_1} \\ &= -\widehat{a_z}\left(-0.8\right)^{2}\frac{10^{-6}}{754} \\ &= -\widehat{a_z}8.49 \times 10^{-10} \end{aligned}$$

Part (d) Transmitted We can find the transmitted $\widetilde{E}$ field by multiplying the incident $\widetilde{E}$ field by T and accounting for the attenuation constant α.

$$\begin{aligned} \alpha &\doteq \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}} \\ &= 0.5\left(377\right)\sqrt{\frac{1}{81}} \\ &= 20.9 \end{aligned}$$

$$\begin{aligned} \widetilde{E_r} &= T\widetilde{E_i}e^{-21z} \\ &= (0.2)10^{-3}e^{-jz\frac{2}{3}\pi\times10^{-2}}e^{-21z} \\ &= 2 \times 10^{-4}e^{-\left(21+j\frac{2}{3}\pi\times10^{-2}\right)z} \end{aligned}$$

Part (e) Transmitted Power Density Using equation 5-51c on page 207 of Balanis, the transmitted power density is:

$$\begin{aligned} S_{av}^{t} &= \widehat{a_z}\left|T^{b}\right|^{2}\frac{\left|E_0\right|^{2}}{2}e^{-2\alpha_2 z}Re\left(\frac{1}{\eta_2^{*}}\right) \\ &= \widehat{a_z}\left(0.2\right)^{2}\frac{10^{-6}}{2}Re\left(\frac{1}{43}\right) \\ &= (.04)(1.16 \times 10^{-8}) \\ &= \widehat{a_z}4.7 \times 10^{-10} \end{aligned}$$

Part (f) Depth

$$\begin{aligned} 0.368 &= e^{-z21} \\ \ln(0.368) &= z21 \\ z &= \frac{-21}{\ln(0.368)} \\ z &= 21\,meters \end{aligned}$$

Part (g) Wave Length The wave length in water can be found by looking at the wave length in air and the wave number in water.

$$\omega\sqrt{\mu\varepsilon} = \beta = \frac{2\pi}{\lambda}$$

or

$$\lambda = \frac{2\pi}{\beta}$$

In water: $\beta_w = 2\pi \times 10^{6}\sqrt{\mu_0\varepsilon_0 81} = 9\beta_{air} = 9\frac{2}{3}\pi \times 10^{-2} = 6\pi \times 10^{-2}$, so the wave length in water is

$$\begin{aligned}\lambda &= \frac{2\pi}{\beta_w} \\ &= \frac{2\pi}{6\pi \times 10^{-2}} \\ &= \frac{1}{3} \times 10^2\, meters\end{aligned}$$

$20\lambda = 20(33.333) = 666.67\, meters$

Part (h) Wave Velocity in Water The speed of the wave is found by:

$$\begin{aligned}c &= \frac{\omega}{\beta} \\ &= \frac{2\pi \times 10^6}{6\pi \times 10^{-2}} \\ &= \frac{1}{3} \times 10^8\, \frac{m}{s}\end{aligned}$$

So the time for a wave to travel 100 meters is:

$$\begin{aligned}t &= \frac{d}{c} \\ &= \frac{100\, m}{0.333 \times 10^8\, m/s} \\ &= 300 \times 10^{-8}\, seconds\end{aligned}$$

Part (i) Relative Velocity

$$\frac{\frac{1}{3} \times 10^8}{3 \times 10^8} = \frac{1}{9}$$

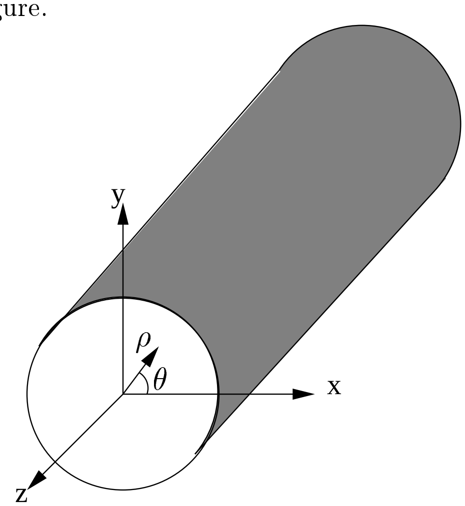
8 Problem (Class Notes)

8.1 Statement

Given a circular waveguide with cross section defined by radius a, find the TM^z modes.
NOTE: The answer will involve Bessel Functions.

8.2 Solution

The TM^z modes will be expressed in cylindrical coordinates matching the geometry shown in the following figure.



"Transverse magnetic modes (often also known as transverse magnetic fields) are field configurations whose magnetic field components lie in a plane that is transverse to a given direction. That direction is often chosen to be the path of wave propagation. For example, if the desired fields are TM to z (TM^z), this implies that $H_z = 0$. The other two magnetic field components (H_x and H_y) and the three electric field components (E_x, E_y, and E_z) may or may not all exist.

By examining (6-43) and (6-51) it is evident that to derive the field expressions that are TM to a given direction, independent of the coordinate system, it is sufficient to let the vector potential $\widetilde{A}$ have only a component in the direction in which the fields are desired to be TM. The remaining components of $\widetilde{A}$ as well as those of $\widetilde{F}$ are set equal to zero." (Balanis, p. 269).

$$\begin{aligned} \widetilde{A} &= \widehat{a_z} A_z(\rho, \phi, z) \\ \widetilde{F} &= 0 \end{aligned}$$

Once the vector potentials are known, we can find the $\widetilde{E}$ and $\widetilde{H}$ fields using the equations (6-70) on page 273 of Balanis, for TM^z Cylindrical Coordinates:

$$\begin{array}{cc} E_\rho = -j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 A_z}{\partial\rho\partial z} & H_\rho = \frac{1}{\mu}\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} \\ E_\phi = -j\frac{1}{\omega\mu\varepsilon}\frac{1}{\rho}\frac{\partial^2 A_z}{\partial\phi\partial z} & H_\phi = -\frac{1}{\mu}\frac{\partial}{\partial\rho} \\ E_z = -j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z} + \beta^2\right) A_z & H_z = 0 \end{array}$$

The vector potential $\widetilde{A}$ must satisfy the vector wave equation, so we end up with:

$$\nabla^2 A_z(\rho, \phi, z) + \beta^2 A_z(\rho, \phi, z) = 0$$

For a circular waveguide, A_z takes the form:

$$\begin{aligned}
A_z(\rho, \phi, z) &= [A_1 J_m(\beta_\rho \rho) + B_1 Y_m(\beta_\rho \rho)] \\
&\times [A_2 \cos(m\phi) + B_2 \sin(m\phi)] \\
&\times \left[A_3 e^{-j\beta_z z} + B_3 e^{+j\beta_z z}\right] \\
\beta^2 &= \beta_\rho^2 + \beta_z^2
\end{aligned}$$

Next we look at the boundary conditions:

- $\widetilde{E} = 0$ when $\rho = a$.
- The fields must be finite everywhere.
- The fields must repeat every 2π radians in ϕ.

Since, $Y_m(0) = \infty$, $B_1 = 0$, otherwise we violate the second boundary condition.

If we assume that the wave only propagates in the +z direction, then $B_3 = 0$.

We can collect constants—$A_{mn} = A_1 A_3$

So, now we can write:

$$A_z^+(\rho, \phi, z) = A_{mn} J_m(\beta_\rho \rho)[A_2 \cos(m\phi) + B_2 \sin(m\phi)] e^{-j\beta_z z}$$

Now we can look at $\widetilde{E_z}$:

$$\begin{aligned}
\widetilde{E_z}(\rho, \phi, z) &= -j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z} + \beta^2\right) A_z \\
&= -j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z} + \beta^2\right) A_{mn} J_m(\beta_\rho \rho)[A_2 \cos(m\phi) + B_2 \sin(m\phi)] e^{-j\beta_z z} \\
&= -j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z}\right) A_{mn} J_m(\beta_\rho \rho)[A_2 \cos(m\phi) + B_2 \sin(m\phi)] e^{-j\beta_z z} \\
&+ -j\frac{1}{\omega\mu\varepsilon}(\beta^2) A_{mn} J_m(\beta_\rho \rho)[A_2 \cos(m\phi) + B_2 \sin(m\phi)] e^{-j\beta_z z} \\
&= -j\frac{1}{\omega\mu\varepsilon}\left(\beta^2 + (j\beta_z)^2\right) A_{mn} J_m(\beta_\rho \rho)[A_2 \cos(m\phi) + B_2 \sin(m\phi)] e^{-j\beta_z z} \\
&= -j\frac{\beta_\rho^2}{\omega\mu\varepsilon} A_{mn} J_m(\beta_\rho \rho)[A_2 \cos(m\phi) + B_2 \sin(m\phi)] e^{-j\beta_z z}
\end{aligned}$$

Since $\widetilde{E} = 0$ when $\rho = a$, we can say:

$$-j\frac{\beta_\rho^2}{\omega\mu\varepsilon} A_{mn} J_m(\beta_\rho a)[A_2 \cos(m\phi) + B_2 \sin(m\phi)] e^{-j\beta_z z} = 0$$

This requires that (see equation 9-25 on page 478 of Balanis):

$$J_m(\beta_\rho a) = 0 \Rightarrow \beta_\rho a = X_{mn} \Rightarrow \beta_\rho = \frac{X_{mn}}{a}$$

X_{mn} is the nth zero (n = 1, 2, 3, . . .) of the Bessel function J_m of the first kind of order m (m = 0, 1, 2, 3, . . .).

This leads to a complex definition for β_z (see equation 9-26 on page 478 of Balanis):

$$(\beta_z)_{mn} = \begin{cases} \sqrt{\beta^2 - \left(\frac{X_{mn}}{a}\right)^2} & for\ \beta > \beta_\rho \\ 0 & for\ \beta = \beta_c = \beta_\rho \\ -j\sqrt{\left(\frac{X_{mn}}{a}\right)^2 - \beta^2} & for\ \beta < \beta_\rho \end{cases}$$

9 Problem (Class Notes 1)

9.1 Statement

Calculate the reflection coefficient and percent of incident energy reflected when a uniform plane wave is normally incident on a Plexiglas radome (assume a infinite flat plane) of thickness 3/8 inch, relative permittivity 2.8, with free space on both sides. Frequency corresponds to free-space wavelength of 20 cm.

9.2 Solution

Recall that:

$$
\begin{aligned}
R &= \frac{\Gamma_{12} + \Gamma_{23} e^{-j2\beta_2 d}}{1 + \Gamma_{12}\Gamma_{23} e^{-j2\beta_2 d}} \\
\Gamma_{12} &= \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \\
\Gamma_{23} &= \frac{\eta_3 - \eta_2}{\eta_3 + \eta_2}
\end{aligned}
$$

In this case

$$
\begin{aligned}
\eta_1 &= 377 \\
\eta_2 &= \sqrt{\frac{\mu_0}{\varepsilon_0 2.8}} \\
&= 377\sqrt{\frac{1}{2.8}} \\
&= 225 \\
\eta_3 &= 377
\end{aligned}
$$

$$
\begin{aligned}
\Gamma_{12} &= \frac{225 - 377}{225 + 377} \\
&= -0.25
\end{aligned}
$$

$$
\begin{aligned}
\Gamma_{23} &= \frac{377 - 225}{377 + 225} \\
&= +0.25
\end{aligned}
$$

$$
\begin{aligned}
\omega &= \frac{2\pi c}{\lambda} \\
&= \frac{2\pi(3 \times 10^8)}{.2} \\
&= 3\pi \times 10^9
\end{aligned}
$$

$$
\begin{aligned}
d &= \frac{3}{8} inch \\
&= .375 inch \\
&= 0.0095 meter
\end{aligned}
$$

$$\begin{aligned}\beta_2 &= \omega\sqrt{\mu_0\varepsilon_o 2.8}\\ &= 3\pi\times 10^9\sqrt{4\pi\times 10^{-7}\times 8.85\times 10^{-12}\times 2.8}\\ &= 52.6\end{aligned}$$

So

$$\begin{aligned}R &= \frac{-0.25+0.25e^{-j2(52.6)(.0095)}}{1+(-0.25)(0.25)e^{-j2(52.6)(.0095)}}\\ &= \frac{-0.25+0.25e^{-j}}{1+0.0625e^{-j}}\\ &= \frac{-0.25+0.25\left(\cos 1 - j\sin 1\right)}{1+0.0625\left(\cos 1 - j\sin 1\right)}\\ &= \frac{-0.25+0.25(0.54-j0.84)}{1+0.0625(0.54-j0.84)}\\ &= \frac{-0.115-j0.21}{1.034-j0.0525}\\ &= -0.101-j0.208\end{aligned}$$

Applying the principle of equation 5-6b on page 182 and equation 5-51 on page 207 of Balanis, the percentage of incident energy reflected is found by squaring the magnitude of R:

$$|R|^2 = (0.101)^2 + (0.208)^2 = 0.053 = 5.3\%$$

10 Problem (Class Notes 2)

10.1 Statement

What refractive index and what thickness do you need to make a quarter wave anti-reflection coating between air and silicon at 10 GHz? The relative permittivity of silicon is 3.5.

10.2 Solution

Recall that:

$$\begin{aligned} R &= \frac{\Gamma_{12} + \Gamma_{23} e^{-j2\beta_2 d}}{1 + \Gamma_{12}\Gamma_{23} e^{-j2\beta_2 d}} \\ \Gamma_{12} &= \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \\ \Gamma_{23} &= \frac{\eta_3 - \eta_2}{\eta_3 + \eta_2} \end{aligned}$$

For the slab to be anti-reflective, R must equal 0, which happens when the numerator equals 0.

$$R = 0 = \Gamma_{12} + \Gamma_{23} e^{-j2\beta_2 d}$$

For $f_0 = 10GHz$, $d = \frac{\lambda_2}{4}$, then

$$2\beta_2 d\big|_{f=10GHz} = 2\left(\frac{2\pi}{\lambda_2}\right)\left(\frac{\lambda_2}{4}\right) = \pi$$

$$e^{-j\pi} = -1$$

So we get (see example 5-10 on pages 225-226 of Balanis for the algebra):

$$\begin{aligned} R = 0 &= \Gamma_{12} + \Gamma_{23} e^{-j2\beta_2 d} \\ &= \Gamma_{12} - \Gamma_{23} \\ 0 &= \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} - \frac{\eta_3 - \eta_2}{\eta_3 + \eta_2} \\ \eta_2 &= \sqrt{\eta_1 \eta_2} \end{aligned}$$

$$\begin{aligned} \eta_1 &= 377 \\ \eta_3 &= \sqrt{\frac{\mu_0}{\varepsilon_0 3.5}} \\ &= 377\sqrt{\frac{1}{3.5}} \\ &= 202 \\ \eta_2 &= \sqrt{377 \times 202} \\ &= 275.6 \end{aligned}$$

This means that the dielectric constant of the slab is

$$\begin{aligned} 275.6 &= 377\sqrt{\frac{1}{\varepsilon_r}} \\ \varepsilon_r &= 1.870 \end{aligned}$$

$$
\begin{aligned}
Index\,of\,Refraction &= \frac{\frac{1}{\sqrt{\mu_0\varepsilon_0}}}{\frac{1}{\sqrt{\mu_0\varepsilon_0\varepsilon_r}}} \\
&= \sqrt{\varepsilon_r} \\
&= \sqrt{1.87} \\
&= 1.37
\end{aligned}
$$

We also know that

$$\frac{2\pi}{\lambda} = \beta = \omega\sqrt{\varepsilon\mu}$$

$$
\begin{aligned}
\frac{2\pi}{\lambda} &= 20\pi \times 10^9\sqrt{1.87 \times 8.85 \times 10^{-12} \times 4\pi \times 10^{-7}} \\
&= 45.6 \\
\lambda &= 2.2cm
\end{aligned}
$$

$$
\begin{aligned}
d &= \frac{\lambda}{4} \\
&= \frac{2.2cm}{4} \\
&= 0.55cm
\end{aligned}
$$

11 Problem (Class Notes 3)

11.1 Statement

Compare the currents that would be required between a half-wave dipole (0.5λ) antenna and a small dipole of length 0.05λ to produce 100 W of radiated power from each.

11.2 Solution

The power radiated by a half-wave dipole can be found from the equation:

$$\begin{aligned} P_{rad} &= I^2 R_{rad} \\ &= I^2 73.1 \end{aligned}$$

So it will take:

$$\begin{aligned} 100 &= I^2 73.1 \\ I &= \sqrt{\frac{100}{73.1}} \\ &= 1.17\, amps \end{aligned}$$

to radiate 100 W of power.

The power radiated by a hertzian dipole can be found from the equation (assuming free-space and far-field):

$$\begin{aligned} P_e &= \eta \frac{\pi}{3} \left| \frac{I\ell}{\lambda} \right|^2 \left[1 - \frac{j}{(kr)^3} \right] \\ &= 377 \frac{\pi}{3} \left| \frac{I.05\lambda}{\lambda} \right|^2 \left[1 - \frac{j}{(kr)^3} \right] \\ &\simeq 394.8(0.05I)^2 \\ &= 0.987 I^2 \end{aligned}$$

So it will take:

$$\begin{aligned} 100 &= .987 I^2 \\ I &= \sqrt{\frac{100}{.987}} \\ &= 10.07\, amps \end{aligned}$$

to radiate 100 W of power.

It takes 10 times more current to radiate the same power from the smaller dipole!

12 Problem (Balanis 6.1)

12.1 Statement

If $\widetilde{H_e} = j\omega\varepsilon\nabla \times \Pi_e$, where Π_e is the Hertzian potential, show that:
(a) $\nabla^2\Pi_e + \beta^2\Pi_e = j\,(1/\omega\varepsilon)\,\widetilde{J}$
(b) $\widetilde{E_e} = \beta^2\Pi_e + \nabla\,(\nabla \cdot \Pi_e)$
(c) $\Pi_e = -j\,(1/\omega\mu\varepsilon)\,\widetilde{A}$

12.2 Solution

NOTE: This is based on the discussion in section 6.2 on pages 256 and 257 of Balanis.
Substitute the given equation into Maxwell's curl equation:

$$\nabla \times \widetilde{E_e} = -j\omega\mu\widetilde{H_e}$$

reduces it to

$$\nabla \times \widetilde{E_e} = -j\omega\mu\,(j\omega\varepsilon\nabla \times \Pi_e) = \omega^2\varepsilon\mu\nabla \times \Pi_e$$

which can also be written as

$$\nabla \times \left[\widetilde{E_e} - \beta^2\Pi_e\right] = 0$$

From the vector identity

$$\nabla \times (-\nabla\phi_e) = 0$$

we can transform the previous equation to

$$\widetilde{E_e} - \beta^2\Pi_e = -\nabla\phi_e$$

or

$$\widetilde{E_e} = -\nabla\phi_e + \beta^2\Pi_e \tag{1}$$

ϕ_e represents an arbitrary electric scalar potential that is a function of position.
Taking the curl of both sides of the given equation and using the vector identiy

$$\nabla \times \nabla \times \Pi_e = \nabla(\nabla \cdot \Pi_e) - \nabla^2\Pi_e$$

leads to

$$\nabla \times \left(\frac{1}{j\omega\varepsilon}\widetilde{H_e}\right) = \nabla(\nabla \cdot \Pi_e) - \nabla^2\Pi_e$$

For a homogeneous medium, this reduces to

$$\frac{1}{j\omega\varepsilon}\nabla \times \widetilde{H_e} = \nabla(\nabla \cdot \Pi_e) - \nabla^2\Pi_e$$

Equating Maxwell's equation

$$\nabla \times \widetilde{H_e} = \widetilde{J} + j\omega\varepsilon\widetilde{E_e}$$

transforms the previous equation to

$$
\begin{aligned}
\frac{1}{j\omega\varepsilon}\left(\widetilde{J} + j\omega\varepsilon\widetilde{E_e}\right) &= \nabla(\nabla\cdot\Pi_e) - \nabla^2\Pi_e \\
\frac{\widetilde{J}}{j\omega\varepsilon} + \widetilde{E_e} &= \nabla(\nabla\cdot\Pi_e) - \nabla^2\Pi_e
\end{aligned}
$$

Now we substitute the previous value for $\widetilde{E_e}$ to get:

$$
\begin{aligned}
\frac{\widetilde{J}}{j\omega\varepsilon} + \widetilde{E_e} &= \nabla(\nabla\cdot\Pi_e) - \nabla^2\Pi_e \\
\frac{\widetilde{J}}{j\omega\varepsilon} - \nabla\phi_e + \beta^2\Pi_e &= \nabla(\nabla\cdot\Pi_e) - \nabla^2\Pi_e \\
\nabla^2\Pi_e + \beta^2\Pi_e &= -\frac{\widetilde{J}}{j\omega\varepsilon} + \nabla(\nabla\cdot\Pi_e) + \nabla\phi_e \\
\nabla^2\Pi_e + \beta^2\Pi_e &= \frac{j}{\omega\varepsilon}\widetilde{J} + \nabla\left(\nabla\cdot\Pi_e + \phi_e\right)
\end{aligned}
$$

So far, only the curl of Π_e has been specified. This means that we are free to specify its divergence. To make life easy and to satisfy the identity of Part a of the problem, we choose:

$$
\nabla\cdot\Pi_e = -\phi_e \tag{2}
$$

Part (a) This reduces the previous equation to:

$$
\nabla^2\Pi_e + \beta^2\Pi_e = \frac{j}{\omega\varepsilon}\widetilde{J}
$$

which demonstrates that (a) is true.

Part (b) Inserting (2) into (1) gives us:

$$
\begin{aligned}
E_e &= -\nabla\phi_e + \beta^2\Pi_e \\
&= \nabla\left(\nabla\cdot\Pi_e\right) + \beta^2\Pi_e \\
&= \beta^2\Pi_e + \nabla\left(\nabla\cdot\Pi_e\right)
\end{aligned}
$$

which demonstrates that (b) is true.

Part (c) Finally, if we equate equation (6-4a) on pate 256 of Balanis with the given equation in this problem, we see that:

$$
\begin{aligned}
\frac{1}{\mu}\nabla\times A = \widetilde{H_A} &= \widetilde{H_e} = j\omega\varepsilon\nabla\times\Pi_e \\
\nabla\times\frac{1}{\mu}A &= \nabla\times j\omega\varepsilon\Pi_e \\
\frac{1}{\mu}A &= j\omega\varepsilon\Pi_e \\
\frac{1}{\mu\omega\varepsilon}A &= j\Pi_e \\
\Pi_e &= \frac{-j}{\omega\mu\varepsilon}A
\end{aligned}
$$

which demonstrates (c) is true.

13 Problem (Balanis 6.17)

13.1 Statement

Show that for observations made at very large distance ($\beta r \gg 1$) the electric and magnetic fields of Example 6-3 reduce to:

$$\begin{aligned} E_\theta &= j\eta \frac{\beta I_e \ell e^{-j\beta r}}{4\pi r} \sin\theta \\ H_\phi &\simeq \frac{E_\theta}{\eta} \\ E_r &\simeq 0 \\ E_\phi &= H_r = H_\theta = 0 \end{aligned}$$

13.2 Solution

Example 6-3 defines the electric and magnetic fields for a very thin linear electric current element of very short length and a constant current to be:

$$\begin{aligned} &E_r = \eta \tfrac{I_e \ell \cos\theta}{2\pi r^2} \left(1 + \tfrac{1}{j\beta r}\right) e^{-j\beta r} && H_r = 0 \\ &E_\theta = j\eta \tfrac{\beta I_e \ell \sin\theta}{4\pi r} \left[1 + \tfrac{1}{j\beta r} - \tfrac{1}{(\beta r)^2}\right] e^{-j\beta r} && H_\theta = 0 \\ &E_\phi = 0 && H_\phi = j \tfrac{\beta I_e \ell \sin\theta}{4\pi r} \left(1 + \tfrac{1}{j\beta r}\right) e^{-j\beta r} \end{aligned}$$

When $\beta r \gg 1$ terms that have βr in the denominator will become negligible. Terms with any power (greater than 1) of r in the denominator will also become zero. These terms are highlighted in bold below.

$$\begin{aligned} &E_r = \eta \tfrac{I_e \ell \cos\theta}{\boldsymbol{2\pi r^2}} \left(1 + \tfrac{1}{\boldsymbol{j\beta r}}\right) e^{-j\beta r} && H_r = 0 \\ &E_\theta = j\eta \tfrac{\beta I_e \ell \sin\theta}{4\pi r} \left[1 + \tfrac{1}{\boldsymbol{j\beta r}} - \tfrac{1}{\boldsymbol{(\beta r)^2}}\right] e^{-j\beta r} && H_\theta = 0 \\ &E_\phi = 0 && H_\phi = j \tfrac{\beta I_e \ell \sin\theta}{4\pi r} \left(1 + \tfrac{1}{\boldsymbol{j\beta r}}\right) e^{-j\beta r} \end{aligned}$$

Zeroing these terms out, yields:

$$\begin{aligned} &E_r = 0 && H_r = 0 \\ &E_\theta = j\eta \tfrac{\beta I_e \ell \sin\theta}{4\pi r} e^{-j\beta r} && H_\theta = 0 \\ &E_\phi = 0 && H_\phi = j \tfrac{\beta I_e \ell \sin\theta}{4\pi r} e^{-j\beta r} = \tfrac{E_\theta}{\eta} \end{aligned}$$

14 Problem (Balanis 6.20)

14.1 Statement

The current distribution on a very thin wire dipole antenna of overall length ℓ is given by

$$I_e = \begin{cases} \hat{a}_z I_0 \sin\left[\beta\left(\frac{\ell}{2} - z'\right)\right] & 0 \leq z' \leq \frac{\ell}{2} \\ \hat{a}_z I_0 \sin\left[\beta\left(\frac{\ell}{2} + z'\right)\right] & -\frac{\ell}{2} \leq z' \leq 0 \end{cases}$$

where I_0 is a constant. Representing the distance R of (6-112) by the far-field approximations of (6-112a) through (6-112b), derive the far-zone electric and magnetic fields radiated by the dipole using (6-97a) and the far-field formulations of Section 6.7.

For completeness:

$R = \left[r^2 + (r')^2 - 2rr'\cos\psi\right]^{\frac{1}{2}}$		(6-112)
$R = r - r'\cos\psi$	for phase variations	(6-112a)
$R = r$	for amplitude variations	(6-112b)
$\widetilde{A} = \frac{\mu}{4\pi}\int_c I_e\left(x', y', z'\right)\frac{e^{-j\beta R}}{R}dl'$		(6-97)

14.2 Solution

Fitting (6-112a) and (6-112b) into (6-97) and using the Calculus shown on slides 13 through 15 of Lecture #8 gives:

$$\begin{aligned} \widetilde{A} &= \hat{a}_z \frac{\mu}{4\pi}\int_c I_0 \sin\left(\beta\left(\frac{\ell}{2} - |z'|\right)\right)\frac{e^{-j\beta(r - r'\cos\psi}}{r}dz' \\ &= \hat{a}_z \frac{\mu}{4\pi r} I_0 e^{-j\beta r}\int_{-\frac{\ell}{2}}^{+\frac{\ell}{2}} \sin\left(\beta\left(\frac{\ell}{2} - |z'|\right)\right) e^{j\beta r'\cos\psi}dz' \\ &= \hat{a}_z \frac{\mu}{4\pi r} I_0 e^{-j\beta r}\left[\frac{2\left[\cos\left(\beta\frac{\ell}{2}\cos\psi\right) - \cos\left(\beta\frac{\ell}{2}\right)\right]}{\beta\sin^2\psi}\right] \end{aligned}$$

Now we can use the short-cut equations for far-field approximation from slide 17 of Lecture #8 (see also equations 6-101 on page 281 of Balanis).

$$\begin{array}{cc} \widetilde{E_r} \simeq 0 & \widetilde{H_r} \simeq 0 \\ \widetilde{E_\theta} \simeq -j\omega A_\theta & \widetilde{H_\theta} \simeq \frac{j\omega}{\eta}A_\phi \\ \widetilde{E_\phi} \simeq -j\omega A_\phi & \widetilde{H_\phi} \simeq -\frac{j\omega}{\eta}A_\theta \end{array}$$

This requires converting $\widetilde{A}$ into spherical coordinates. Since we have only an $\hat{a}_z$ component, this means equation (II-12a) on page 924 of Balanis reduces to:

$$\begin{aligned} A_r &= A_z\cos\theta \\ A_\theta &= -A_z\sin\theta \\ A_\phi &= 0 \end{aligned}$$

$$\begin{array}{cc} \widetilde{E_r} \simeq 0 & \widetilde{H_r} \simeq 0 \\ \widetilde{E_\phi} \simeq 0 & \widetilde{H_\theta} \simeq 0 \\ \widetilde{E_\theta} \simeq j\omega\sin\theta\frac{\mu}{4\pi r}I_0 e^{-j\beta r}\left[\frac{2\left[\cos\left(\beta\frac{\ell}{2}\cos\psi\right) - \cos\left(\beta\frac{\ell}{2}\right)\right]}{\beta\sin^2\psi}\right] & \widetilde{H_\phi} \simeq \frac{j\omega}{\eta}\sin\theta\frac{\mu}{4\pi r}I_0 e^{-j\beta r}\left[\frac{2\left[\cos\left(\beta\frac{\ell}{2}\cos\psi\right) - \cos\left(\beta\frac{\ell}{2}\right)\right]}{\beta\sin^2\psi}\right] \end{array}$$

15 Problem (Balanis 6.21)

15.1 Statement

Show that the radiated far-zone electric and magnetic fields derived in Problem 6.20 reduce for a half-wavelength dipole ($\ell = \lambda/2$) to

$$\begin{aligned} E_\theta &\simeq j\eta \frac{I_0 e^{-j\beta r}}{2\pi r} \left[\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \right] \\ H_\phi &\simeq \frac{E_\theta}{\eta} \\ E_r &\simeq E_\phi \simeq H_r \simeq H_\theta \simeq 0 \end{aligned}$$

15.2 Solution

Recall from the solution to problem 6.20 (see section 14 of this paper):

$$\widetilde{E_r} \simeq 0 \qquad \widetilde{H_r} \simeq 0$$

$$\widetilde{E_\phi} \simeq 0 \qquad \widetilde{H_\theta} \simeq 0$$

$$\widetilde{E_\theta} \simeq j\omega \sin\theta \frac{\mu}{4\pi r} I_0 e^{-j\beta r} \left[\frac{2\left[\cos\left(\beta\frac{\ell}{2}\cos\psi\right) - \cos\left(\beta\frac{\ell}{2}\right)\right]}{\beta \sin^2\psi} \right] \qquad \widetilde{H_\phi} \simeq \frac{j\omega}{\eta} \sin\theta \frac{\mu}{4\pi r} I_0 e^{-j\beta r} \left[\frac{2\left[\cos\left(\beta\frac{\ell}{2}\cos\psi\right) - \cos\left(\beta\frac{\ell}{2}\right)\right]}{\beta \sin^2\psi} \right]$$

From this, we can see by direct observation that the last two lines of the stated problem are true. That is:

$$\begin{aligned} H_\phi &\simeq \frac{E_\theta}{\eta} \\ E_r &\simeq E_\phi \simeq H_r \simeq H_\theta \simeq 0 \end{aligned}$$

This leaves us with proving that

$$E_\theta \simeq j\eta \frac{I_0 e^{-j\beta r}}{2\pi r} \left[\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \right]$$

when

$$\ell = \frac{\lambda}{2}$$

$$\begin{aligned} \widetilde{E_\theta} &\simeq j\omega \sin\theta \frac{\mu}{4\pi r} I_0 e^{-j\beta r} \left[\frac{2\left[\cos\left(\beta\frac{(\lambda/2)}{2}\cos\psi\right) - \cos\left(\beta\frac{(\lambda/2)}{2}\right)\right]}{\beta \sin^2\psi} \right] \\ &\simeq j\omega \sin\theta \frac{\mu}{4\pi r} I_0 e^{-j\beta r} \left[\frac{2\left[\cos\left(\frac{2\pi}{\lambda}\frac{\lambda}{4}\cos\psi\right) - \cos\left(\frac{2\pi}{\lambda}\frac{\lambda}{4}\right)\right]}{\beta \sin^2\psi} \right] \\ &\simeq j\omega \sin\theta \frac{\mu}{4\pi r} I_0 e^{-j\beta r} \left[\frac{2\left[\cos\left(\frac{\pi}{2}\cos\psi\right) - \cos\left(\frac{\pi}{2}\right)\right]}{\beta \sin^2\psi} \right] \\ &\simeq j\omega \sin\theta \frac{\mu}{4\pi r} I_0 e^{-j\beta r} \left[\frac{2\left[\cos\left(\frac{\pi}{2}\cos\psi\right)\right]}{\beta \sin^2\psi} \right] \end{aligned}$$

At this point we note that $\psi = \theta = azimuthal$ angle.

$$
\begin{aligned}
\widetilde{E_\theta} &\simeq j\omega \sin\theta \frac{\mu}{4\pi r} I_0 e^{-j\beta r} \left[\frac{2\left[\cos\left(\frac{\pi}{2}\cos\theta\right)\right]}{\beta \sin^2\theta} \right] \\
&\simeq j\omega \frac{\mu}{2\beta\pi r} I_0 e^{-j\beta r} \left[\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \right] \\
&\simeq \frac{j\omega\mu}{\beta} \frac{I_0 e^{-j\beta r}}{2\pi r} \left[\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \right] \\
&\simeq \frac{j\omega\mu}{\omega\sqrt{\mu\varepsilon}} \frac{I_0 e^{-j\beta r}}{2\pi r} \left[\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \right] \\
&\simeq \frac{j\mu\sqrt{\mu\varepsilon}}{\mu\varepsilon} \frac{I_0 e^{-j\beta r}}{2\pi r} \left[\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \right] \\
&\simeq \frac{j\sqrt{\mu\varepsilon}}{\varepsilon} \frac{I_0 e^{-j\beta r}}{2\pi r} \left[\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \right] \\
&\simeq \frac{j\sqrt{\varepsilon}}{\sqrt{\varepsilon}} \sqrt{\frac{\mu}{\varepsilon}} \frac{I_0 e^{-j\beta r}}{2\pi r} \left[\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \right] \\
&\simeq j\eta \frac{I_0 e^{-j\beta r}}{2\pi r} \left[\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \right]
\end{aligned}
$$