EE541 Homework #1, #2, & #3

Robert W. Schuler

October 5, 2002

This paper collects Rob Schuler's answers to homework problems assigned before the first examination in EE541. The purpose of type setting these assignments is two fold. The first point is to learn how to typeset equations of this nature using LaTeX. The second purpose is to produce an electronic form that will be easy to keep for years after the course.

1 Numbering of Assigned Problems

The sections are numbered sequentially by the LyX editor starting from 1. Problems are numbered at the first tier title using a scheme of Assignment-dot-problem. For example Problem 1.1 is the first problem of the first homework assignment and Problem 2.3 is the third problem of the second homework assignment.

1.1 Homework #1

Homework #1 consists of two problems assigned from the class notes.

Section in this Paper	Problem Number	Assigned Number
2	1.1	#3 of Page 19 of Week 1 Notes
3	1.2	Page 21 of Week 1 Notes

1.2 Homework #2

Homework #2 consists of four problems assigned from the textbook: 2.1, 3.1, 3.2, and 3.5.

Section in this Paper	Problem Number	Assigned Number
4	2.1	Balanis 2.1
5	2.2	Balanis 3.1
6	2.3	Balanis 3.2
7	2.4	Balanis 3.5

1.3 Homework #3

Homework #3 consists of seven problems assigned from Chapter 4 of the textbook: 4.1, 4.3, 4.5, 4.8, 4.22, 4.23, and 4.24.

Section in this Paper	Problem Number	Assigned Number
8	3.1	Balanis 4.1
9	3.2	Balanis 4.3
10	3.3	Balanis 4.5
11	3.4	Balanis 4.8
12	3.5	Balanis 4.22
13	3.6	Balanis 4.23
14	3.7	Balanis 4.24

2 Problem 1.1 (Class Notes)

2.1 Statement

Given the scalar function

$V(x, y) = \sin(\frac{\pi}{2}x)\sin(\frac{\pi}{2}y)$

Determine $\nabla \times (\nabla V)$

2.2 Solution

$\nabla V = \overrightarrow{a_x}\frac{\partial}{\partial x}V + \overrightarrow{a_y}\frac{\partial}{\partial y}V + \overrightarrow{a_z}\frac{\partial}{\partial z}V$

In this case, we drop the Z term:

$\nabla V = \overrightarrow{a_x}\frac{\partial}{\partial x}V + \overrightarrow{a_y}\frac{\partial}{\partial y}V$

Substituting:

$\nabla V = \overrightarrow{a_x}\frac{\partial}{\partial x}\sin(\frac{\pi}{2}x)\sin(\frac{\pi}{2}y) + \overrightarrow{a_y}\frac{\partial}{\partial y}\sin(\frac{\pi}{2}x)\sin(\frac{\pi}{2}y)$

Or:

$\nabla V = \overrightarrow{a_x}\frac{\pi}{2}\cos(\frac{\pi}{2}x)\sin(\frac{\pi}{2}y) + \overrightarrow{a_y}\frac{\pi}{2}\sin(\frac{\pi}{2}x)\cos(\frac{\pi}{2}y)$

Let $\overrightarrow{B} = \nabla V$. We get:

$B_x = \frac{\pi}{2}\cos(\frac{\pi}{2}x)\sin(\frac{\pi}{2}y)$ and $B_y = \frac{\pi}{2}\sin(\frac{\pi}{2}x)\cos(\frac{\pi}{2}y)$

NOTE: $B_z = 0$

We can now look at the definition for Curl in terms of $\overrightarrow{B}$.

$\nabla \times \overrightarrow{B} = \begin{vmatrix} \overrightarrow{a_x} & \overrightarrow{a_y} & \overrightarrow{a_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix} =$

$\overrightarrow{a_x}\frac{\partial}{\partial y}B_z + \overrightarrow{a_y}\frac{\partial}{\partial z}B_x + \overrightarrow{a_z}\frac{\partial}{\partial x}B_y - \overrightarrow{a_z}\frac{\partial}{\partial y}B_x - \overrightarrow{a_x}\frac{\partial}{\partial z}B_y - \overrightarrow{a_y}\frac{\partial}{\partial x}B_z$

Substituting for $\overrightarrow{B}$:

$+\overrightarrow{a_x}\frac{\partial}{\partial y}0 + \overrightarrow{a_y}\frac{\partial}{\partial z}\frac{\pi}{2}\cos(\frac{\pi}{2}x)\sin(\frac{\pi}{2}y) + \overrightarrow{a_z}\frac{\partial}{\partial x}\frac{\pi}{2}\sin(\frac{\pi}{2}x)\cos(\frac{\pi}{2}y)$

$-\overrightarrow{a_z}\frac{\partial}{\partial y}\frac{\pi}{2}\cos(\frac{\pi}{2}x)\sin(\frac{\pi}{2}y) - \overrightarrow{a_x}\frac{\partial}{\partial z}\frac{\pi}{2}\sin(\frac{\pi}{2}x)\cos(\frac{\pi}{2}y) - \overrightarrow{a_y}\frac{\partial}{\partial x}0$

Which reduces to:

$0 + 0 + \overrightarrow{a_z}\frac{\pi}{4}\cos(\frac{\pi}{2}x)\cos(\frac{\pi}{2}y) - \overrightarrow{a_z}\frac{\pi}{4}\cos(\frac{\pi}{2}x)\cos(\frac{\pi}{2}y) - 0 - 0$

or simply:

0

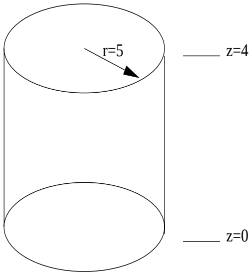
So, we can see that:$\nabla \times (\nabla \sin(\frac{\pi}{2}x)\sin(\frac{\pi}{2}y)) = 0$

This is to be expected since $\nabla \times (\nabla\psi) = 0$ according to II-41 on page 927 of Balanis.

3 Problem 1.2 (Class Notes)

3.1 Statement

Verify the Divergence Theorem for $\overrightarrow{A(r,z)} = \widehat{a_r} r^2 + \widehat{a_z} 2z$ on a cylindrical region enclosed by $r = 5$, $z = 0$, and $z = 4$.



3.2 Solution

3.2.1 Divergence Theorem

Recall that the Divergence Theorem states:

$$\int\!\!\int_v\!\!\int \nabla \cdot \overrightarrow{A} dv = \oint_s\!\!\oint \overrightarrow{A} \cdot \widehat{ds}$$

So, to verify the Divergence Theorem for the given vector field, we simply calculate both sides of the preceding equation and verify that they yield the same value. The following two subsections of this paper show the calculations for the volume and closed-surface integrals respectively. This is then followed by a comparison and analysis.

3.2.2 Volume Integral

First we find the the divergence in cylindrical coordinates:

$$\begin{aligned}
\nabla \cdot \overrightarrow{A} &= \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial}{\partial \Phi}(A_\Phi) + \frac{\partial}{\partial z}(A_z) \\
&= \frac{1}{r}\frac{\partial}{\partial r}(r^3) + \frac{1}{r}\frac{\partial}{\partial \Phi}(0) + \frac{\partial}{\partial z}(2z) \\
&= \frac{1}{r}3r^2 + 2 \\
&= 3r + 2
\end{aligned}$$

Next we find the volume integral. Recall that $dv = r dr d\Phi dz$ in cylindrical coordinates.

$$\begin{aligned}
\int\!\!\int_v\!\!\int \nabla \cdot \overrightarrow{A} dv &= \int_{z=0}^{4}\int_{\Phi=0}^{2\pi}\int_{r=0}^{5} 3r^2 + 2r dr d\Phi dz \\
&= \int_{z=0}^{4}\int_{\Phi=0}^{2\pi} r^3 + r^2\Big|_0^5 d\Phi dz \\
&= \int_{z=0}^{4}\int_{\Phi=0}^{2\pi} 125 + 25 - 0 d\Phi dz \\
&= \int_{z=0}^{4} 150\Phi\Big|_0^{2\pi} dz \\
&= \int_{z=0}^{4} 300\pi dz
\end{aligned}$$

$$
\begin{aligned}
&= 300\pi z|_0^4 \\
&= 1200\pi
\end{aligned}
$$

3.2.3 Closed Surface

Next we find the total flux through the surfaces enclosing V. In this case, we have three individual surfaces to consider: the top circle, the bottom circle, and the wall of the cylinder.

First we find the dot product of $\overrightarrow{A}$and $\overrightarrow{ds}$:

$$
\begin{aligned}
\overrightarrow{A}\cdot\overrightarrow{ds} &= (\widehat{ar}r^2+\widehat{az}2z)\cdot(\widehat{ar}rd\Phi dz+\widehat{a\Phi}drdz+\widehat{az}rd\Phi dr) \\
&= r^3 d\Phi dz + 0 + 2rzd\Phi dr
\end{aligned}
$$

Since the top and bottom circles are similar they can be treated similarly as follows:

$$
\begin{aligned}
\oint_s\oint \overrightarrow{A}\cdot\widehat{ds} &= \int_{\Phi=0}^{2\pi}\int_{r=0}^{5} 2rzd\Phi dr \\
&= \int_{\Phi=0}^{2\pi} zr^2|_0^5 \, d\Phi \\
&= \int_{\Phi=0}^{2\pi} 25zd\Phi \\
&= 25z\Phi|_0^{2\pi} \\
&= 50\pi z
\end{aligned}
$$

For the top circle, $z = 4$ and we get $50\pi * 4 = 200\pi$.

For the bottom circle, $z = 0$ and we get 0.

For the side we have:

$$
\begin{aligned}
\oint_s\oint \overrightarrow{A}\cdot\widehat{ds} &= \int_{z=0}^{4}\int_{\Phi=0}^{2\pi} r^3 d\Phi dz \\
&= \int_{z=0}^{4} \Phi r^3|_0^{2\pi} \, dz \\
&= \int_{z=0}^{4} 2\pi r^3 dz \\
&= 2\pi r^3 z|_0^4 \\
&= 8\pi r^3
\end{aligned}
$$

and since $r = 5$ for the side, we get $125 * 8\pi = 1000\pi$.

Finally we add all three sides together to get:

$200\pi + 0 + 1000\pi = 1200\pi$

3.2.4 Conclusion

Since the left and right had side of the Divergence Theorem both yield 1200π for the stated vector and boundary geometry, we can see that the theorem has been verified for this case.

4 Problem 2.1 (Balanis 2.1)

4.1 Statement

A dielectric slab, shown in Figure P2-1, exhibits an electric polarization vector of

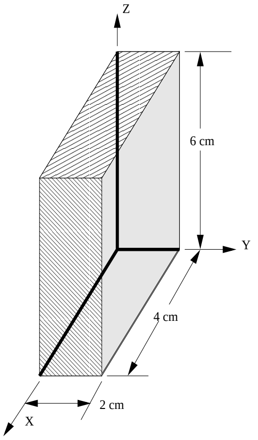
$$\overrightarrow{P} = \widehat{a_y} 2.762 \times 10^{-11} C/m^2$$

when it is subjected to an electric field of

$$\overrightarrow{E} = \widehat{a_y} 2V/m$$

Determine

(a) The bound surface charge density q_{sp} in each of its six faces.
(b) The net bound charge Q, associated with the slab.
(c) The volume bound charge density q_{vp} *within* the dielectric slab.
(d) The dielectric constant of the material.



4.2 Solution

4.2.1 Part (a) bound surface charge density

Let us name the the six surfaces according to their dimensions: X=0, X=2cm, Y=0, Y=2cm, Z=0, and Z=6cm. Since $\overrightarrow{P}$ and $\overrightarrow{E}$ are both aligned with the Y-axis, we know that the bound surface charge density will only be present on the two surfaces Y=0 and Y=2cm. There will be no bound surface charge density on X=0, X=2cm, Z=0, or Z=6cm.

According to equation (2-6) on page 49 of Balanis $P = q_{sp}$, or the magnitude of P gives us the bound surface charge density q_{sp}.

This means that the bound surface charge density on Y=0 is $-2.762 \times 10^{-11} C/m^2$, and the bound surface charge density on Y=2cm is $+2.762 \times 10^{-11} C/m^2$.

4.2.2 Part (b) net bound charge

In this case, the language in the problem statement is a little ambiguous. In discussing the solution with Dr. Mirotznik, one possible interpretation would be to integrate the surface charge across the volume of the dielectric slab. This solution, however, seems to disagree with the fact stated in the solution to part (c) of this problem (see below) that the bound charge within the slab is zero. Upon reflection, it seems reasonable that the only bound charge present in the slab exists as bound surface charge, which means that the calculation of net bound charge can be done by multiplying the six bound surface charge densities from part (a) by the six areas of the sides of the dielectric slab in question.

Since the only non-zero bound surface charge density is found on the surfaces Y=0 and Y=2cm, these are the only surfaces considered as part of this solution. Both of these surfaces have the same area:

$$Area = 4cm * 6cm = 24cm^2 = 2.4 \times 10^{-5} m^2$$

So, for Y=0

$$Q_{y=0cm} = -2.762 \times 10^{-11} C/m^2 * 2.4 \times 10^{-5} m^2 = -6.6288 \times 10^{-16} Coulombs$$

and for Y=2cm

$$Q_{y=2cm} = +2.762 \times 10^{-11} C/m^2 * 2.4 \times 10^{-5} m^2 = +6.6288 \times 10^{-16} Coulombs$$

Since these two values cancel each other, one possible interpretation is that the net bound charge is zero; $Q = 0 Coulombs$. Another interpretation is that the net bound charge is the sum (ignoring the sign) $Q = 1.33576 \times 10^{-15} Coulombs$. A final interpretation is to assume that the surface charge applies to the entire volume (even though the positive and negative charges cancel out) and to multiply the surface charge by the thickness. This is the same as integrating the surface charge over the volume of the slab and yields $Q = 6.6288 \times 10^{-16} * 2 \times 10^{-2} = 1.32576 \times 10^{-17} Coulombs$.

4.2.3 Part (c) volume charge density

"3. The volume charge density q, inside the material is zero because the positive and negative charges of adjacent dipoles cancel each other."(Balanis, p.47). The same situation that applies to the material in the quote applies to the dielectric slab under consideration, so the volume charge density *within* the the dielectric slab is zero.

4.2.4 Part (d) dielectric constant

First we observe that:

$$\begin{aligned}
\overrightarrow{D} &= \varepsilon_o \overrightarrow{E} + \overrightarrow{P} \\
&= (8.854 \times 10^{-12} F/m)(2V/m)\widehat{a_y} + (2.762 \times 10^{-12} C/m^2)\widehat{a_y} \\
&= 4.5328 \times 10^{-11} C/m^2 \widehat{a_y}
\end{aligned}$$

next:

$$\begin{aligned}
\overrightarrow{D} &= \varepsilon_0 \varepsilon_r \overrightarrow{E} \\
4.5328 \times 10^{-11} C/m^2 \widehat{a_y} &= (8.854 \times 10^{-12} F/m)\varepsilon_r (2V/m)\widehat{a_y} \\
\varepsilon_r &= 2.560
\end{aligned}$$

- NOTE: from Table 2-1 on page 50 of Balanis, a dielectric constant of 2.56 is found in the material "cross-linked polystyrene (unreinforced).

5 Problem 2.2 (Balanis 3.1)

5.1 Statement

Derive the vector wave equations 3-16a and 3-16b for time-harmonic fields using the Maxwell equations of Table 1-4 for time-harmonic fields.

Equations 3-16a and 3-16b on page 107 are:

$\nabla^2 \overrightarrow{E} = \nabla \times \overrightarrow{M_i} + j\omega\mu \overrightarrow{J_i} + \frac{1}{\varepsilon}\nabla q_{ev} + j\omega\mu\sigma\overrightarrow{E} - \omega^2\mu\varepsilon\overrightarrow{E}$	(3-16a)
$\nabla^2 \overrightarrow{H} = -\nabla \times \overrightarrow{J_i} + \sigma\overrightarrow{M_i} + j\omega\varepsilon\overrightarrow{M_i} + \frac{1}{\mu}\nabla q_{mv} + j\omega\mu\sigma\overrightarrow{H} - \omega^2\mu\varepsilon\overrightarrow{H}$	(3-16b)

5.2 Solution

3-16a Starting with with the Maxwell equation

$\nabla \times \overrightarrow{E} = -\overrightarrow{M_i} - \mu j\omega\overrightarrow{H}$ we take the curl of each side:

$\nabla \times [\nabla \times \overrightarrow{E}] = \nabla \times -\overrightarrow{M_i} + \nabla \times (-\mu j\omega\overrightarrow{H})$ which (due to the independence of $\overrightarrow{H}$ from time) can be rewritten as

$\nabla \times [\nabla \times \overrightarrow{E}] = -\nabla \times \overrightarrow{M_i} - \mu j\omega\nabla \times \overrightarrow{H}$. Next we substitute the curl of $\overrightarrow{H}$ term in the previous equation using another of Maxwell's equations: $\nabla \times \overrightarrow{H} = \varepsilon j\omega\overrightarrow{E} + \sigma\overrightarrow{E} + \overrightarrow{J_i}$; we get

$\nabla \times [\nabla \times \overrightarrow{E}] = -\nabla \times \overrightarrow{M_i} - \mu j\omega \left[\varepsilon j\omega\overrightarrow{E} + \sigma\overrightarrow{E} + \overrightarrow{J_i}\right]$. Next we invoke the vector identity $\nabla \times \nabla \times \overrightarrow{A} = \nabla(\nabla \cdot \overrightarrow{A}) - \nabla^2\overrightarrow{A}$, to change the left hand side of the previous equation, which results in

$\nabla(\nabla \cdot \overrightarrow{E}) - \nabla^2\overrightarrow{E} = -\nabla \times \overrightarrow{M_i} - \mu j\omega(\varepsilon j\omega\overrightarrow{E}) - \mu j\omega\sigma\overrightarrow{E} - \mu j\omega\overrightarrow{J_i}$. Here we recognize that $j^2 = -1$ so we have

$\nabla(\nabla \cdot \overrightarrow{E}) - \nabla^2\overrightarrow{E} = -\nabla \times \overrightarrow{M_i} + \mu\varepsilon\omega^2\overrightarrow{E} - \mu j\omega\sigma\overrightarrow{E} - \mu j\omega\overrightarrow{J_i}$. Next, we use the Maxwell equation $\nabla \cdot \overrightarrow{E} = \frac{q_{ev}}{\varepsilon}$ to get:

$\nabla(\frac{q_{ev}}{\varepsilon}) - \nabla^2\overrightarrow{E} = -\nabla \times \overrightarrow{M_i} + \mu\varepsilon\omega^2\overrightarrow{E} - \mu j\omega\sigma\overrightarrow{E} - \mu j\omega\overrightarrow{J_i}$. If we assume a homogeneous material, we can pull the ε out of the first term of this equation

$\frac{1}{\varepsilon}\nabla q_{ev} - \nabla^2\overrightarrow{E} = -\nabla \times \overrightarrow{M_i} + \mu\varepsilon\omega^2\overrightarrow{E} - \mu j\omega\sigma\overrightarrow{E} - \mu j\omega\overrightarrow{J_i}$. Finally this reduces to

$\nabla^2\overrightarrow{E} = \nabla \times \overrightarrow{M_i} + j\omega\mu\overrightarrow{J_i} + \frac{1}{\varepsilon}\nabla q_{ev} + j\omega\mu\sigma\overrightarrow{E} - \omega^2\mu\varepsilon\overrightarrow{E}$.

3-16b Starting with with the Maxwell equation

$\nabla \times \overrightarrow{H} = \overrightarrow{J_i} + \sigma\overrightarrow{E} + \varepsilon j\omega\overrightarrow{E}$and taking the curl of each side yields:

$\nabla \times [\nabla \times \overrightarrow{H}] = \nabla \times \overrightarrow{J_i} + \nabla \times \sigma\overrightarrow{E} + \nabla \times (-\varepsilon j\omega\overrightarrow{E})$ which (due to the independence of $\overrightarrow{E}$ from time) can be rewritten as

$\nabla \times [\nabla \times \overrightarrow{H}] = \nabla \times \overrightarrow{J_i} + \sigma\nabla \times \overrightarrow{E} + \varepsilon j\omega\nabla \times \overrightarrow{E}$. Next we substitute the curl of $\overrightarrow{E}$ term in the previous equation using another of Maxwell's equations: $\nabla \times \overrightarrow{E} = -\overrightarrow{M_i} - \mu j\omega\overrightarrow{H}$; we get

$\nabla \times [\nabla \times \overrightarrow{H}] = \nabla \times \overrightarrow{J_i} + \sigma\left[-\overrightarrow{M_i} - \mu j\omega\overrightarrow{H}\right] + \varepsilon j\omega\left[-\overrightarrow{M_i} - \mu j\omega\overrightarrow{H}\right]$. Next we invoke the vector identity $\nabla \times \nabla \times \overrightarrow{A} = \nabla(\nabla \cdot \overrightarrow{A}) - \nabla^2\overrightarrow{A}$, to change the left hand side of the previous equation, which results in

$\nabla(\nabla \cdot \overrightarrow{H}) - \nabla^2\overrightarrow{H} = \nabla \times \overrightarrow{J_i} - \sigma\overrightarrow{M_i} - \sigma\mu j\omega\overrightarrow{H} - \varepsilon j\omega\overrightarrow{M_i} - \varepsilon j\omega\mu j\omega\overrightarrow{H}$. Here we recognize that $j^2 = -1$ so we have

$\nabla(\nabla \cdot \overrightarrow{H}) - \nabla^2\overrightarrow{H} = \nabla \times \overrightarrow{J_i} + \nabla \times \overrightarrow{J_c} - \varepsilon j\omega\overrightarrow{M_i} + \varepsilon\mu\omega^2\overrightarrow{H}$.Next, we use the Maxwell equation $\nabla \cdot \overrightarrow{H} = \frac{q_{mv}}{\mu}$ to get

$\nabla(\frac{q_{mv}}{\mu}) - \nabla^2\overrightarrow{H} = \nabla \times \overrightarrow{J_i} + \nabla \times \overrightarrow{J_c} - \varepsilon j\omega\overrightarrow{M_i} + \varepsilon\mu\omega^2\overrightarrow{H}$. If we assume a homogeneous material, we can pull the μ out of the first term of this equation

$\frac{1}{\mu}\nabla q_{mv} - \nabla^2\overrightarrow{H} = \nabla \times \overrightarrow{J_i} - \sigma\overrightarrow{M_i} - \sigma\mu j\omega\overrightarrow{H} - \varepsilon j\omega\overrightarrow{M_i} + \varepsilon\mu\omega^2\overrightarrow{H}$. Finally this reduces to

$\nabla^2\overrightarrow{H} = -\nabla \times \overrightarrow{J_i} + \sigma\overrightarrow{M_i} + j\omega\varepsilon\overrightarrow{M_i} + \frac{1}{\mu}\nabla q_{mv} + j\omega\mu\sigma\overrightarrow{H} - \omega^2\mu\varepsilon\overrightarrow{H}$.

6 Problem 2.3 (Balanis 3.2)

6.1 Statement

Verify that (3-28a) and (3-28b) are solutions to (3-26a).

Equations 3-26a from page 109 and 3-28a and 3-28b on page 110 are:

$\frac{1}{f}\frac{\partial^2 f}{\partial x^2} = -\beta_x^2 \Rightarrow \frac{\partial^2 f}{\partial x^2} = -\beta_x^2 f$	(3-26a)
$f_1(x) = A_1 e^{-j\beta_x x} + B_1 e^{+j\beta_x x}$	(3-28a)
$f_2(x) = C_1 \cos(\beta_x x) + D_1 \sin(\beta_x x)$	(3-28b)

6.2 Solution

3-28a To verify that 3-28a is a solution to 3-26a, we simply "do the math".

$$\begin{aligned}
f_1(x) &= A_1 e^{-j\beta_x x} + B_1 e^{+j\beta_x x} \\
f_1^{'}(x) &= A_1(-j\beta_x)e^{-j\beta_x x} + B_1(j\beta_x)e^{+j\beta_x x} \\
f_1^{''}(x) &= A_1(-j\beta_x)(-j\beta_x)e^{-j\beta_x x} + B_1(j\beta_x)(j\beta_x)e^{+j\beta_x x} \\
&= -A_1\beta_x^2 e^{-j\beta_x x} - B_1\beta_x^2 e^{+j\beta_x x} \\
&= -\beta_x^2(A_1 e^{-j\beta_x x} + B_1 e^{+j\beta_x x})
\end{aligned}$$

In this case, f is only a function of x, so $\frac{\partial^2 f}{\partial x^2} = f^{''}$ and we can see that in fact:

$$\begin{aligned}
\frac{\partial^2 f_1}{\partial x^2} &= -\beta_x^2 f_1 \\
-\beta_x^2(A_1 e^{-j\beta_x x} + B_1 e^{+j\beta_x x}) &= -\beta_x^2\left[A_1 e^{-j\beta_x x} + B_1 e^{+j\beta_x x}\right]
\end{aligned}$$

3-28b To verify that 3-28b is a solution to 3-26a, we simply "do the math".

$$\begin{aligned}
f_2(x) &= C_1\cos(\beta_x x) + D_1\sin(\beta_x x) \\
f_2^{'}(x) &= -C_1\beta_x\sin(\beta_x x) + D_1\beta_x\cos(\beta_x x) \\
f_2^{''}(x) &= -C_1\beta_x\beta_x\cos(\beta_x x) - D_1\beta_x\beta_x\sin(\beta_x x) \\
&= -C_1\beta_x^2\cos(\beta_x x) - D_1\beta_x^2\sin(\beta_x x) \\
&= -\beta_x^2(C_1\cos(\beta_x x) + D_1\sin(\beta_x x))
\end{aligned}$$

In this case, f is only a function of x, so $\frac{\partial^2 f}{\partial x^2} = f^{''}$ and we can see that in fact:

$$\begin{aligned}
\frac{\partial^2 f_2}{\partial x^2} &= -\beta_x^2 f_2 \\
-\beta_x^2(C_1\cos(\beta_x x) + D_1\sin(\beta_x x)) &= -\beta_x^2\left[C_1\cos(\beta_x x) + D_1\sin(\beta_x x)\right]
\end{aligned}$$

7 Problem 2.4 (Balanis 3.5)

7.1 Statement

Show that the vector wave equation (3-53) reduces, when $\overrightarrow{E}$ has a solution of the form (3-47), to the three scalar wave equations 3-54a through 3-54c.

Equation 3-47 from page 116 and equations 3-53, 3-54a, 3-54b, and 3-54c from page 117 are:

$\overrightarrow{E}(\rho,\phi,z) = \widehat{a}_\rho E_\rho(\rho,\phi,z) + \widehat{a_\phi} E_\phi(\rho,\phi,z) + \widehat{a}_z E_z(\rho,\phi,z)$	(3-47)
$\nabla(\nabla\cdot\overrightarrow{E}) - \nabla\times\nabla\times\overrightarrow{E} = -\beta^2\overrightarrow{E}$	(3-53)
$\nabla^2 E_\rho + \left(-\frac{E_\rho}{\rho^2} - \frac{2}{\rho^2}\frac{\partial E_\phi}{\partial\phi}\right) = -\beta^2 E_\rho$	(3-54a)
$\nabla^2 E_\phi + \left(-\frac{E_\phi}{\rho^2} + \frac{2}{\rho^2}\frac{\partial E_\rho}{\partial\phi}\right) = -\beta^2 E_\phi$	(3-54b)
$\nabla^2 E_z = -\beta^2 E_z$	(3-54c)

7.2 Solution

The solution to this problem is relatively straight forward, however, there are a lot of terms that need to be expanded before the final cancellations can take place. In order to accomplish this, the following solution splits equation 3-53 into pieces. First we solve for the gradient of the divergence of $\overrightarrow{E}$, then we solve for the curl of the curl of $\overrightarrow{E}$. The final step is to collect all the terms associated with each of the unit axis vectors $\widehat{a}_\rho$, $\widehat{a_\phi}$, and $\widehat{a}_z$.

Divergence of E in Cylindrical Coordinates

$$\begin{aligned}
\nabla\cdot\overrightarrow{E} &= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho E_\rho) + \frac{1}{\rho}\frac{\partial E_\phi}{\partial\phi} + \frac{\partial E_z}{\partial z} \\
&= \frac{1}{\rho}E_\rho + \frac{\partial}{\partial\rho}E_\rho + \frac{1}{\rho}\frac{\partial E_\phi}{\partial\phi} + \frac{\partial E_z}{\partial z}
\end{aligned}$$

Gradient of the Divergence of E in Cylindrical Coordinates

$$\begin{aligned}
\nabla(\nabla\cdot\overrightarrow{E}) &= \widehat{a}_\rho\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}E_\rho + \frac{\partial}{\partial\rho}E_\rho + \frac{1}{\rho}\frac{\partial E_\phi}{\partial\phi} + \frac{\partial E_z}{\partial z}\right) \\
&+ \widehat{a_\phi}\frac{1}{\rho}\frac{\partial}{\partial\phi}\left(\frac{1}{\rho}E_\rho + \frac{\partial}{\partial\rho}E_\rho + \frac{1}{\rho}\frac{\partial E_\phi}{\partial\phi} + \frac{\partial E_z}{\partial z}\right) \\
&+ \widehat{a}_z\frac{\partial}{\partial z}\left(\frac{1}{\rho}E_\rho + \frac{\partial}{\partial\rho}E_\rho + \frac{1}{\rho}\frac{\partial E_\phi}{\partial\phi} + \frac{\partial E_z}{\partial z}\right) \\
&= \widehat{a}_\rho\left(\frac{\partial}{\partial\rho}\frac{1}{\rho}E_\rho + \frac{\partial}{\partial\rho}\frac{\partial}{\partial\rho}E_\rho + \frac{\partial}{\partial\rho}\frac{1}{\rho}\frac{\partial E_\phi}{\partial\phi} + \frac{\partial}{\partial\rho}\frac{\partial}{\partial z}E_z\right) \\
&+ \widehat{a_\phi}\left(\frac{1}{\rho}\frac{\partial}{\partial\phi}\frac{1}{\rho}E_\rho + \frac{1}{\rho}\frac{\partial}{\partial\phi}\frac{\partial}{\partial\rho}E_\rho + \frac{1}{\rho}\frac{\partial}{\partial\phi}\frac{1}{\rho}\frac{\partial E_\phi}{\partial\phi} + \frac{1}{\rho}\frac{\partial}{\partial\phi}\frac{\partial}{\partial z}E_z\right) \\
&+ \widehat{a}_z\left(\frac{\partial}{\partial z}\frac{1}{\rho}E_\rho + \frac{\partial}{\partial z}\frac{\partial}{\partial\rho}E_\rho + \frac{\partial}{\partial z}\frac{1}{\rho}\frac{\partial E_\phi}{\partial\phi} + \frac{\partial}{\partial z}\frac{\partial E_z}{\partial z}\right) \\
&= \widehat{a}_\rho\left(-\frac{1}{\rho^2}E_\rho + \frac{1}{\rho}\frac{\partial}{\partial\rho}E_\rho + \frac{\partial^2}{\partial\rho^2}E_\rho - \frac{1}{\rho^2}\frac{\partial}{\partial\phi}E_\phi + \frac{1}{\rho}\frac{\partial}{\partial\rho}\frac{\partial}{\partial\phi}E_\phi + \frac{\partial}{\partial\rho}\frac{\partial}{\partial z}E_z\right) \\
&+ \widehat{a_\phi}\left(\frac{1}{\rho^2}\frac{\partial}{\partial\phi}E_\rho + \frac{1}{\rho}\frac{\partial}{\partial\phi}\frac{\partial}{\partial\rho}E_\rho + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}E_\phi + \frac{1}{\rho}\frac{\partial}{\partial\phi}\frac{\partial}{\partial z}E_z\right) \\
&+ \widehat{a}_z\left(\frac{1}{\rho}\frac{\partial}{\partial z}E_\rho + \frac{\partial}{\partial z}\frac{\partial}{\partial\rho}E_\rho + \frac{1}{\rho}\frac{\partial}{\partial z}\frac{\partial}{\partial\phi}E_\phi + \frac{\partial^2}{\partial z^2}E_z\right)
\end{aligned}$$

Curl of E in Cylindrical Coordinates

$$
\begin{aligned}
\nabla \times \overrightarrow{E} &= \frac{1}{\rho}\begin{bmatrix} \widehat{a_\rho} & \rho\widehat{a_\phi} & \widehat{a_z} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ E_\rho & \rho E_\phi & E_z \end{bmatrix} \\
&= \frac{1}{\rho}\left[\widehat{a_\rho}\frac{\partial}{\partial\phi}E_z + \rho\widehat{a_\phi}\frac{\partial}{\partial z}E_\rho + \widehat{a_z}\frac{\partial}{\partial\rho}\rho E_\phi - \widehat{a_z}\frac{\partial}{\partial\phi}E_\rho - \rho\widehat{a_\phi}\frac{\partial}{\partial\rho}E_z - \widehat{a_\rho}\frac{\partial}{\partial z}\rho E_\phi\right] \\
&= \widehat{a_\rho}\frac{1}{\rho}\left(\frac{\partial}{\partial\phi}E_z - \frac{\partial}{\partial z}\rho E_\phi\right) + \rho\widehat{a_\phi}\frac{1}{\rho}\left(\frac{\partial}{\partial z}E_\rho - \frac{\partial}{\partial\rho}E_z\right) + \widehat{a_z}\frac{1}{\rho}\left(\frac{\partial}{\partial\rho}\rho E_\phi - \frac{\partial}{\partial\phi}E_\rho\right) \\
&= \widehat{a_\rho}\left(\frac{1}{\rho}\frac{\partial}{\partial\phi}E_z - \frac{\partial}{\partial z}E_\phi\right) + \widehat{a_\phi}\left(\frac{\partial}{\partial z}E_\rho - \frac{\partial}{\partial\rho}E_z\right) + \widehat{a_z}\left(\frac{1}{\rho}E_\phi + \frac{\partial}{\partial\rho}E_\phi - \frac{1}{\rho}\frac{\partial}{\partial\phi}E_\rho\right)
\end{aligned}
$$

Curl of the curl of E in Cylindrical Coordinates Let:

$$
\begin{aligned}
A &= \left(\frac{1}{\rho}\frac{\partial}{\partial\phi}E_z - \frac{\partial}{\partial z}E_\phi\right) \\
B &= \left(\frac{\partial}{\partial z}E_\rho - \frac{\partial}{\partial\rho}E_z\right) \\
C &= \left(\frac{1}{\rho}E_\phi + \frac{\partial}{\partial\rho}E_\phi - \frac{1}{\rho}\frac{\partial}{\partial\phi}E_\rho\right)
\end{aligned}
$$

$$
\begin{aligned}
\nabla \times \nabla \times \overrightarrow{E} &= \frac{1}{\rho}\begin{bmatrix} \widehat{a_\rho} & \rho\widehat{a_\phi} & \widehat{a_z} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ A & \rho B & C \end{bmatrix} \\
&= \frac{1}{\rho}\left[\widehat{a_\rho}\frac{\partial}{\partial\phi}C + \rho\widehat{a_\phi}\frac{\partial}{\partial z}A + \widehat{a_z}\frac{\partial}{\partial\rho}\rho B - \widehat{a_z}\frac{\partial}{\partial\phi}A - \rho\widehat{a_\phi}\frac{\partial}{\partial\rho}C - \widehat{a_\rho}\frac{\partial}{\partial z}\rho B\right] \\
&= \widehat{a_\rho}\frac{1}{\rho}\left(\frac{\partial}{\partial\phi}C - \frac{\partial}{\partial z}\rho B\right) + \rho\widehat{a_\phi}\frac{1}{\rho}\left(\frac{\partial}{\partial z}A - \frac{\partial}{\partial\rho}C\right) + \widehat{a_z}\frac{1}{\rho}\left(\frac{\partial}{\partial\rho}\rho B - \frac{\partial}{\partial\phi}A\right) \\
&= \widehat{a_\rho}\left(\frac{1}{\rho}\frac{\partial}{\partial\phi}C - \frac{\partial}{\partial z}B\right) + \widehat{a_\phi}\frac{1}{\rho}\left(\frac{\partial}{\partial z}A - \frac{\partial}{\partial\rho}C\right) + \widehat{a_z}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho B - \frac{1}{\rho}\frac{\partial}{\partial\phi}A\right)
\end{aligned}
$$

Now we can expand A, B, and C to get:

$$
\begin{aligned}
\nabla \times \nabla \times \overrightarrow{E} &= \widehat{a_\rho}\left(\frac{1}{\rho}\frac{\partial}{\partial\phi}\left(\frac{1}{\rho}E_\phi + \frac{\partial}{\partial\rho}E_\phi - \frac{1}{\rho}\frac{\partial}{\partial\phi}E_\rho\right) - \frac{\partial}{\partial z}\left(\frac{\partial}{\partial z}E_\rho - \frac{\partial}{\partial\rho}E_z\right)\right) \\
&+ \widehat{a_\phi}\left(\frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial}{\partial\phi}E_z - \frac{\partial}{\partial z}E_\phi\right) - \frac{\partial}{\partial\rho}\left(\frac{1}{\rho}E_\phi + \frac{\partial}{\partial\rho}E_\phi - \frac{1}{\rho}\frac{\partial}{\partial\phi}E_\rho\right)\right) \\
&+ \widehat{a_z}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\left(\frac{\partial}{\partial z}E_\rho - \frac{\partial}{\partial\rho}E_z\right) - \frac{1}{\rho}\frac{\partial}{\partial\phi}\left(\frac{1}{\rho}\frac{\partial}{\partial\phi}E_z - \frac{\partial}{\partial z}E_\phi\right)\right) \\
&= \widehat{a_\rho}\left(\frac{1}{\rho}\frac{\partial}{\partial\phi}\frac{1}{\rho}E_\phi + \frac{1}{\rho}\frac{\partial}{\partial\phi}\frac{\partial}{\partial\rho}E_\phi - \frac{1}{\rho}\frac{\partial}{\partial\phi}\frac{1}{\rho}\frac{\partial}{\partial\phi}E_\rho - \frac{\partial}{\partial z}\frac{\partial}{\partial z}E_\rho + \frac{\partial}{\partial z}\frac{\partial}{\partial\rho}E_z\right) \\
&+ \widehat{a_\phi}\left(\frac{\partial}{\partial z}\frac{1}{\rho}\frac{\partial}{\partial\phi}E_z - \frac{\partial}{\partial z}\frac{\partial}{\partial z}E_\phi - \frac{\partial}{\partial\rho}\frac{1}{\rho}E_\phi - \frac{\partial}{\partial\rho}\frac{\partial}{\partial\rho}E_\phi + \frac{\partial}{\partial\rho}\frac{1}{\rho}\frac{\partial}{\partial\phi}E_\rho\right) \\
&+ \widehat{a_z}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial z}E_\rho - \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}E_z - \frac{1}{\rho}\frac{\partial}{\partial\phi}\frac{1}{\rho}\frac{\partial}{\partial\phi}E_z + \frac{1}{\rho}\frac{\partial}{\partial\phi}\frac{\partial}{\partial z}E_\phi\right) \\
&= \widehat{a_\rho}\left(\frac{1}{\rho^2}\frac{\partial}{\partial\phi}E_\phi + \frac{1}{\rho}\frac{\partial}{\partial\phi}\frac{\partial}{\partial\rho}E_\phi - \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}E_\rho - \frac{\partial^2}{\partial z^2}E_\rho + \frac{\partial}{\partial z}\frac{\partial}{\partial\rho}E_z\right) \\
&+ \widehat{a_\phi}\left(\frac{1}{\rho}\frac{\partial}{\partial z}\frac{\partial}{\partial\phi}E_z - \frac{\partial^2}{\partial z^2}E_\phi + \frac{1}{\rho^2}E_\phi - \frac{1}{\rho}\frac{\partial}{\partial\rho}E_\phi - \frac{\partial^2}{\partial\rho^2}E_\phi - \frac{1}{\rho^2}\frac{\partial}{\partial\phi}E_\rho + \frac{1}{\rho}\frac{\partial}{\partial\rho}\frac{\partial}{\partial\phi}E_\rho\right)
\end{aligned}
$$

$$+ \quad \widehat{a_z}\left(\frac{1}{\rho}\frac{\partial}{\partial z}E_\rho + \frac{\partial}{\partial \rho}\frac{\partial}{\partial z}E_\rho - \frac{1}{\rho}\frac{\partial}{\partial \rho}E_z - \frac{\partial^2}{\partial \rho^2}E_z - \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2}E_z + \frac{1}{\rho}\frac{\partial}{\partial \phi}\frac{\partial}{\partial z}E_\phi\right)$$

Laplacian in Cylindrical Coordinates The next step is to collect all the terms associated with each of the unit axis vectors $\widehat{a_\rho}$, $\widehat{a_\phi}$, and $\widehat{a_z}$ to create three separate equations. Before proceeding on to this step, however, there is one final equation from the text that is useful. Specifically, equation 3-55, which says:

$$\begin{aligned}
\nabla^2\psi(\rho,\phi,z) &= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \psi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2} \\
&= \frac{\partial^2 \psi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \psi}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2}
\end{aligned}$$

3-54a Collecting the $\widehat{a_\rho}$ terms:

$$\begin{aligned}
\left(-\frac{1}{\rho^2}E_\rho + \frac{1}{\rho}\frac{\partial}{\partial \rho}E_\rho + \frac{\partial^2}{\partial \rho^2}E_\rho - \frac{1}{\rho^2}\frac{\partial}{\partial \phi}E_\phi + \frac{1}{\rho}\frac{\partial}{\partial \rho}\frac{\partial}{\partial \phi}E_\phi + \frac{\partial}{\partial \rho}\frac{\partial}{\partial z}E_z\right) &\quad - \\
\left(\frac{1}{\rho^2}\frac{\partial}{\partial \phi}E_\phi + \frac{1}{\rho}\frac{\partial}{\partial \phi}\frac{\partial}{\partial \rho}E_\phi - \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2}E_\rho - \frac{\partial^2}{\partial z^2}E_\rho + \frac{\partial}{\partial z}\frac{\partial}{\partial \rho}E_z\right) &= -\beta^2 E_\rho \\
\frac{\partial^2}{\partial \rho^2}E_\rho + \frac{1}{\rho}\frac{\partial}{\partial \rho}E_\rho + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2}E_\rho + \frac{\partial^2}{\partial z^2}E_\rho - \frac{1}{\rho^2}E_\rho - \frac{1}{\rho^2}\frac{\partial}{\partial \phi}E_\phi - \frac{1}{\rho^2}\frac{\partial}{\partial \phi}E_\phi & \\
+\frac{1}{\rho}\frac{\partial}{\partial \rho}\frac{\partial}{\partial \phi}E_\phi + \frac{\partial}{\partial \rho}\frac{\partial}{\partial z}E_z - \frac{1}{\rho}\frac{\partial}{\partial \phi}\frac{\partial}{\partial \rho}E_\phi - \frac{\partial}{\partial z}\frac{\partial}{\partial \rho}E_z &= -\beta^2 E_\rho
\end{aligned}$$

Here we note that the first four terms reduce to $\nabla^2 E_\rho$ according to equation 3-55. Furthermore, the last four terms cancel each other out. This results in the final form of equation 3-54a:

$$\nabla^2 E_\rho - \frac{1}{\rho^2}E_\rho - \frac{2}{\rho^2}\frac{\partial}{\partial \phi}E_\phi = -\beta^2 E_\rho$$

3-54b Collecting the $\widehat{a_\phi}$ terms:

$$\begin{aligned}
\left(\frac{1}{\rho^2}\frac{\partial}{\partial \phi}E_\rho + \frac{1}{\rho}\frac{\partial}{\partial \phi}\frac{\partial}{\partial \rho}E_\rho + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2}E_\phi + \frac{1}{\rho}\frac{\partial}{\partial \phi}\frac{\partial}{\partial z}E_z\right) & \\
-\left(\frac{1}{\rho}\frac{\partial}{\partial z}\frac{\partial}{\partial \phi}E_z - \frac{\partial^2}{\partial z^2}E_\phi + \frac{1}{\rho^2}E_\phi - \frac{1}{\rho}\frac{\partial}{\partial \rho}E_\phi - \frac{\partial^2}{\partial \rho^2}E_\phi - \frac{1}{\rho^2}\frac{\partial}{\partial \phi}E_\rho + \frac{1}{\rho}\frac{\partial}{\partial \rho}\frac{\partial}{\partial \phi}E_\rho\right) &= -\beta^2 E_\phi \\
\frac{\partial^2}{\partial \rho^2}E_\phi + \frac{1}{\rho}\frac{\partial}{\partial \rho}E_\phi + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2}E_\phi + \frac{\partial^2}{\partial z^2}E_\phi - \frac{1}{\rho^2}E_\phi + \frac{1}{\rho^2}\frac{\partial}{\partial \phi}E_\rho + \frac{1}{\rho^2}\frac{\partial}{\partial \phi}E_\rho & \\
+\frac{1}{\rho}\frac{\partial}{\partial \phi}\frac{\partial}{\partial \rho}E_\rho + \frac{1}{\rho}\frac{\partial}{\partial \phi}\frac{\partial}{\partial z}E_z - \frac{1}{\rho}\frac{\partial}{\partial z}\frac{\partial}{\partial \phi}E_z - \frac{1}{\rho}\frac{\partial}{\partial \rho}\frac{\partial}{\partial \phi}E_\rho &= -\beta^2 E_\phi
\end{aligned}$$

Here we note that the first four terms reduce to $\nabla^2 E_\phi$ according to equation 3-55. Furthermore, the last four terms cancel each other out. This results in the final form of equation 3-54b:

$$\nabla^2 E_\phi - \frac{1}{\rho^2}E_\phi + \frac{2}{\rho^2}\frac{\partial}{\partial \phi}E_\rho = -\beta^2 E_\phi$$

3-54c Collecting the $\widehat{a_z}$ terms:

$$\begin{aligned}
\left(\frac{1}{\rho}\frac{\partial}{\partial z}E_\rho + \frac{\partial}{\partial z}\frac{\partial}{\partial \rho}E_\rho + \frac{1}{\rho}\frac{\partial}{\partial z}\frac{\partial}{\partial \phi}E_\phi + \frac{\partial^2}{\partial z^2}E_z\right) & \\
-\left(\frac{1}{\rho}\frac{\partial}{\partial z}E_\rho + \frac{\partial}{\partial \rho}\frac{\partial}{\partial z}E_\rho - \frac{1}{\rho}\frac{\partial}{\partial \rho}E_z - \frac{\partial^2}{\partial \rho^2}E_z - \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2}E_z + \frac{1}{\rho}\frac{\partial}{\partial \phi}\frac{\partial}{\partial z}E_\phi\right) &= -\beta^2 E_z
\end{aligned}$$

$$\begin{aligned}\frac{\partial^2}{\partial\rho^2}E_z + \frac{1}{\rho}\frac{\partial}{\partial\rho}E_z + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}E_z + \frac{\partial^2}{\partial z^2}E_z + \frac{1}{\rho}\frac{\partial}{\partial z}E_\rho + \frac{\partial}{\partial z}\frac{\partial}{\partial\rho}E_\rho + \frac{1}{\rho}\frac{\partial}{\partial z}\frac{\partial}{\partial\phi}E_\phi & \\ -\frac{1}{\rho}\frac{\partial}{\partial z}E_\rho - \frac{\partial}{\partial\rho}\frac{\partial}{\partial z}E_\rho - \frac{1}{\rho}\frac{\partial}{\partial\phi}\frac{\partial}{\partial z}E_\phi &= -\beta^2 E_z\end{aligned}$$

Here we note that the first four terms reduce to $\nabla^2 E_z$ according to equation 3-55, and the last six terms cancel each other out. This results in the final form of equation 3-54c:

$$\nabla^2 E_z = -\beta^2 E_z$$

8 Problem 3.1 (Balanis 4.1)

8.1 Statement

A uniform plane wave having only an x component of the electric field is traveling in the $+z$ direction in an unbounded lossless, source free region. Using Maxwell's equations write expressions for the electric and corresponding magnetic field intensities. Compare your answers to those of (4-2b) and (4-3c).

Equation 4-2b on page 132 and equation 4-3c on page 133 are:

$E_x^+(z) = E_0^+ e^{-j\beta z}$	(4-2b)
$H_y^+ = \frac{1}{\sqrt{\mu/\varepsilon}} E_x^+$	(4-3c)

8.2 Solution

We know that the equation for the wave will be of the form: $\overrightarrow{E} = E_0 \cos(\omega t - z\phi)\widehat{a_x}$ in the time domeain and the form $\widetilde{E} = \widehat{a_x} E_0 e^{-j\phi z}$ in the frequency domain; where ϕ is some unknown phase shift.

First, we choose Maxwell's equation $\nabla \times \widetilde{E} = -j\omega\mu\widetilde{H}$. This allows us to find an expression of the magnetic field intensity:

$$\begin{aligned}\widetilde{H} &= -\frac{1}{j\omega\mu}\nabla\times\widetilde{E} \\ &= -\frac{1}{j\omega\mu}\left[0\widehat{a_x} + \widehat{a_y}\left(\frac{\partial E_x}{\partial z} - 0\right) + \widehat{a_z}\left(0 - \frac{\partial E_x}{\partial y}\right)\right]\end{aligned}$$

NOTE: The previous reduction uses equation II-16 on page 925 for the expanded form of the curl in rectangular coordinates. Since $\widetilde{E}$ is defined only in terms of E_x, only those terms with A_x in II-16 need to be considered. Furthermore, it can be observed directly that $\frac{\partial E_x}{\partial y} = 0$, so we have:

$$\begin{aligned}\widetilde{H} &= -\frac{1}{j\omega\mu}\widehat{a_y}\frac{\partial E_x}{\partial z} \\ &= -\frac{1}{j\omega\mu}\widehat{a_y}\frac{\partial}{\partial z}\left(E_0 e^{-j\phi z}\right) \\ &= -\frac{1}{j\omega\mu}\widehat{a_y}\left(-j\phi\right)E_0 e^{-j\phi z} \\ &= \frac{\phi}{\omega\mu}\widehat{a_y}E_0 e^{-j\phi z}\end{aligned}$$

Next, we use Maxwell's equation $\nabla \times \widetilde{H} = j\omega\varepsilon\widetilde{E}$. This allows us to find another expression of the electric field intensity:

$$\begin{aligned}\widetilde{E} &= \frac{1}{j\omega\varepsilon}\nabla\times\widetilde{H} \\ &= \frac{1}{j\omega\varepsilon}\left[\widehat{a_x}\left(0 - \frac{\partial H_y}{\partial z}\right) + 0\widehat{a_y} + \widehat{a_z}\left(\frac{\partial H_y}{\partial x} - 0\right)\right]\end{aligned}$$

NOTE: The previous reduction uses equation II-16 on page 925 for the expanded form of the curl in rectangular coordinates. Since $\widetilde{H}$ is defined only in terms of H_y, only those terms with A_y in II-16 need to be considered. Furthermore, it can be observed directly that $\frac{\partial H_y}{\partial x} = 0$, so we have:

$$\begin{aligned}
\widetilde{E} &= -\frac{1}{j\omega\varepsilon}\widehat{a_x}\frac{\partial H_y}{\partial z} \\
&= -\frac{1}{j\omega\varepsilon}\widehat{a_x}\frac{\partial}{\partial z}\left(\frac{\phi}{\omega\mu}E_0 e^{-j\phi z}\right) \\
&= -\frac{1}{j\omega\varepsilon}\widehat{a_x}\left(-\frac{j\phi^2}{\omega\mu}\right)E_0 e^{-j\phi z} \\
&= \frac{\phi^2}{\omega^2\varepsilon\mu}\widehat{a_x}E_0 e^{-j\phi z}
\end{aligned}$$

We can now use the two expressions for $\widetilde{E}$ to determine the value of ϕ.

$$\begin{aligned}
\widehat{a_x}E_0 e^{-j\phi z} &= \frac{\phi^2}{\omega^2\varepsilon\mu}\widehat{a_x}E_0 e^{-j\phi z} \\
1 &= \frac{\phi^2}{\omega^2\varepsilon\mu} \\
\phi^2 &= \omega^2\varepsilon\mu \\
\phi &= \omega\sqrt{\varepsilon\mu} \\
\phi &= \beta
\end{aligned}$$

Now we can write the electric field intensity in a form that exactly equals equation 4-2b:

$$\begin{aligned}
\widetilde{E} &= E_0 e^{-j\beta z} \\
E_x^+ &= E_0^+ e^{-j\beta z}
\end{aligned}$$

We can also write the magnetic field intensity in a form that exactly equals equation 4-3c:

$$\begin{aligned}
\widetilde{H} &= \frac{\omega\sqrt{\varepsilon\mu}}{\omega\mu}E_0 e^{-j\beta z} \\
H_y^+ &= \frac{\sqrt{\varepsilon\mu}}{\mu}E_x^+ \\
&= \frac{\sqrt{\varepsilon}\sqrt{\mu}\sqrt{\mu}}{\mu\sqrt{\mu}}E_x^+ \\
&= \sqrt{\frac{\varepsilon}{\mu}}E_x^+ \\
H_y^+ &= \frac{1}{\sqrt{\mu/\varepsilon}}E_x^+
\end{aligned}$$

9 Problem 3.2 (Balanis 4.3)

9.1 Statement

The complex $\overrightarrow{H}$ field of a uniform plane wave, traveling in an unbounded source free medium of free-space is given by

$$\overrightarrow{H} = \frac{1}{120\pi}\left(\widehat{a_x} - 2\widehat{a_y}\right) e^{-j\beta_0 z}$$

Find (a) the corresponding electric field, (b) the instantaneous power density vector, and (c) the time-average power density.

9.2 Solution

Part (a) By inspecting the exponent in the given equation, we see that the plane wave is moving along the Z axis. This means that the direction of $\widetilde{E}$ can be calculated based on the fact that it will be perpendicular to both $\widetilde{H}$and the Z axis:

$$\begin{aligned}
Direction\widetilde{E} &= \widetilde{H} \times \widehat{a_z} \\
&= \begin{vmatrix} \widehat{a_x} & \widehat{a_y} & \widehat{a_z} \\ 1 & -2 & 0 \\ 0 & 0 & 1 \end{vmatrix} \\
&= -2\widehat{a_x} - \widehat{a_y}
\end{aligned}$$

So

$$\widetilde{E} = \frac{\eta}{120\pi}\left(-2\widehat{a_x} - \widehat{a_y}\right) e^{-j\beta_0 z}$$

Part (b) The instantaneous power is found by $\overrightarrow{E} \times \overrightarrow{H}$

$$\begin{aligned}
\overrightarrow{E} &= RE\left[\frac{\eta}{120\pi}\left(-2\widehat{a_x} - \widehat{a_y}\right) e^{-j\beta_0 z} e^{j\omega t}\right] \\
&= RE\left[\frac{\eta}{120\pi}\left(-2\widehat{a_x} - \widehat{a_y}\right) e^{j(\omega t - \beta_0 z)}\right] \\
&= \frac{\eta}{120\pi}\left(-2\widehat{a_x} - \widehat{a_y}\right)\cos(\omega t - \beta_0 z) \\
&= \frac{-2\eta}{120\pi}\left(\widehat{a_x}\right)\cos(\omega t - \beta_0 z) - \frac{\eta}{120\pi}\left(\widehat{a_y}\right)\cos(\omega t - \beta_0 z)
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{H} &= RE\left[\frac{1}{120\pi}\left(\widehat{a_x} - 2\widehat{a_y}\right) e^{-j\beta_0 z} e^{j\omega t}\right] \\
&= RE\left[\frac{1}{120\pi}\left(\widehat{a_x} - 2\widehat{a_y}\right) e^{-j(\omega t - \beta_0 z)}\right] \\
&= \frac{1}{120\pi}\left(\widehat{a_x} - 2\widehat{a_y}\right)\cos(\omega t - \beta_0 z) \\
&= \frac{1}{120\pi}\left(\widehat{a_x}\right)\cos(\omega t - \beta_0 z) - \frac{2}{120\pi}\left(\widehat{a_y}\right)\cos(\omega t - \beta_0 z)
\end{aligned}$$

Let $C = \cos(\omega t - \beta_0 z)$ and we can write:

$$\overrightarrow{E} \times \overrightarrow{H} = \begin{vmatrix} \widehat{a_x} & \widehat{a_y} & \widehat{a_z} \\ \frac{-2\eta}{120\pi}C & \frac{-\eta}{120\pi}C & 0 \\ \frac{1}{120\pi}C & \frac{-2}{120\pi}C & 0 \end{vmatrix}$$

$$
\begin{aligned}
&= \widehat{a_z}\left(\frac{-2\eta}{120\pi}C\right)\left(\frac{-2}{120\pi}C\right) - \widehat{a_z}\left(\frac{-\eta}{120\pi}C\right)\left(\frac{1}{120\pi}C\right) \\
&= \widehat{a_z}\eta\left(\frac{-2}{120\pi}C\right)^2 + \widehat{a_z}\eta\left(\frac{1}{120\pi}C\right)^2 \\
&= \widehat{a_z}\eta 5\left(\frac{1}{120\pi}\cos(\omega t - \beta_0 z)\right)^2
\end{aligned}
$$

Since we are in free-space $\eta \cong 377\Omega$ and we get

$$P_i \cong 0.01326\cos^2(\omega t - \beta_0 z)\widehat{a_z}$$

Part (c) The time average power is found by $\frac{1}{2}\widetilde{E} \times \widetilde{H^*}$.

$$
\begin{aligned}
\frac{1}{2}\widetilde{E} \times \widetilde{H^*} &= \frac{1}{2}\begin{vmatrix} \widehat{a_x} & \widehat{a_y} & \widehat{a_z} \\ \frac{-2\eta}{120\pi}e^{-j\beta_0 z} & \frac{-\eta}{120\pi}e^{-j\beta_0 z} & 0 \\ \frac{1}{120\pi}e^{+j\beta_0 z} & \frac{-2}{120\pi}e^{+j\beta_0 z} & 0 \end{vmatrix} \\
&= \frac{1}{2}\left[\widehat{a_z}\left(\frac{-2\eta}{120\pi}e^{-j\beta_0 z}\right)\left(\frac{-2}{120\pi}e^{+j\beta_0 z}\right) - \widehat{a_z}\left(\frac{-\eta}{120\pi}e^{-j\beta_0 z}\right)\left(\frac{1}{120\pi}e^{+j\beta_0 z}\right)\right] \\
&= \frac{1}{2}\left[\widehat{a_z}\eta 4\left(\frac{1}{120\pi}\right)^2 + \widehat{a_z}\eta\left(\frac{1}{120\pi}\right)^2\right] \\
&= \frac{1}{2}\left[\widehat{a_z}\eta 5\left(\frac{1}{120\pi}\right)^2\right]
\end{aligned}
$$

Since we are in free-space $\eta \cong 377\Omega$ and we get

$$P_{avg} \cong .00663\widehat{a_z}$$

10 Problem 3.3 (Balanis 4.5)

10.1 Statement

The magnetic field of a uniform plane wave in a source-free region is given by

$$\widetilde{H} = 10^{-6}\left[-\widehat{a_x}\left(2+j\right)+\widehat{a_z}\left(1+j3\right)\right]e^{+j\beta y}$$

Assuming that the medium is free space, determine (a) the corresponding electric field and (b) the time-average power density.

10.2 Solution

Part (a) By inspecting the exponent in the given equation, we see that the plane wave is moving along the -Y axis. This means that the direction of $\widetilde{E}$ can be calculated based on the fact that it will be perpendicular to both $\widetilde{H}$and the -Y axis:

$$\begin{aligned} Direction\widetilde{E} &= \widetilde{H}\times\widehat{a_z} \\ &= \begin{vmatrix} \widehat{a_x} & \widehat{a_y} & \widehat{a_z} \\ -\left(2+j\right) & 0 & \left(1+j3\right) \\ 0 & -1 & 0 \end{vmatrix} \\ &= \widehat{a_z}\left(2+j\right)+\widehat{a_x}\left(1+j3\right) \end{aligned}$$

So

$$\widetilde{E} = \eta 10^{-6}\left[+\widehat{a_x}\left(1+j3\right)+\widehat{a_z}\left(2+j\right)\right]e^{+j\beta y}$$

Part (b) The time average power is found by $\frac{1}{2}\widetilde{E}\times\widetilde{H^*}$.

$$\begin{aligned} \frac{1}{2}\widetilde{E}\times\widetilde{H^*} &= \frac{1}{2}\begin{vmatrix} \widehat{a_x} & \widehat{a_y} & \widehat{a_z} \\ \eta 10^{-6}\left(1+j3\right)e^{+j\beta y} & \eta 10^{-6}\left(2+j\right)e^{+j\beta y} & 0 \\ -10^{-6}\left(2+j\right)e^{-j\beta y} & 10^{-6}\left(1+j3\right)e^{-j\beta y} & 0 \end{vmatrix} \\ &= \frac{1}{2}\left[\widehat{a_z}\left(\eta 10^{-6}\left(1+j3\right)e^{+j\beta y}\right)\left(10^{-6}\left(1+j3\right)e^{-j\beta y}\right)-\widehat{a_z}\left(\eta 10^{-6}\left(2+j\right)e^{+j\beta y}\right)\left(-10^{-6}\left(2+j\right)e^{-j\beta y}\right)\right] \\ &= \frac{1}{2}\left[\widehat{a_z}\eta\left(10^{-6}\left(1+j3\right)\right)^2+\widehat{a_z}\eta\left(10^{-6}\left(2+j\right)\right)^2\right] \\ &= \frac{1}{2}\left[\widehat{a_z}\eta 10^{-12}\left(\left(1+j3\right)^2+\left(2+j\right)^2\right)\right] \\ &= \frac{1}{2}\left[\widehat{a_z}\eta 10^{-12}\left(\left(1+j6-9\right)+\left(4+j4-1\right)\right)\right] \\ &= \frac{1}{2}\left[\widehat{a_z}\eta 10^{-12}\left(-5+j10\right)\right] \end{aligned}$$

Since we are in free-space $\eta \cong 377\Omega$ and we get

$$\begin{aligned} P_{avg} &\cong \widehat{a_z}(-942.5\times 10^{-12}+j1885\times 10^{-12}) \\ &\cong 2.107\times 10^{-9}\angle 2.034rad \end{aligned}$$

11 Problem 3.4 (Balanis 4.8)

11.1 Statement

A uniform plane wave is traveling in the $-z$ direction inside an unbounded source-free, free-space region. Assuming that the electric field has only an E_x component, its value at $z = 0$ is 4×10^{-3} V/m, and its frequency of operation is 300 MHz, write expressions for (a) the complex electric and magnetic fields, (b) the instantaneous electric and magnetic fields, (c) the time-average and instantaneous power densities, and (d) the time-average and instantaneous electric and magnetic energy densities.

11.2 Solution

Part (a) We are given that $E_0 = 4 \times 10^{-3}$ and we know that $\beta = \frac{\omega}{c} = \frac{2\pi 300\times 10^6}{3\times 10^8} = 2\pi$ so we can directly write:

$$\widetilde{E} = 4 \times 10^{-3} e^{+j2\pi z}\widehat{a_x}$$

and

$$\begin{aligned} \widetilde{H} &= -\frac{4 \times 10^{-3}}{377} e^{+j2\pi z}\widehat{a_y} \\ &= -10.61 \times 10^{-6} e^{+j2\pi z}\widehat{a_y} \end{aligned}$$

Part (b)

$$\begin{aligned} \overrightarrow{E} &= RE\left[4 \times 10^{-3} e^{+j2\pi z}\widehat{a_x} e^{j\omega t}\right] \\ &= RE\left[4 \times 10^{-3} e^{+j(\omega t + 2\pi z)}\widehat{a_x}\right] \\ &= 4 \times 10^{-3}\cos(\omega t + 2\pi z)\widehat{a_x} \end{aligned}$$

$$\begin{aligned} \overrightarrow{H} &= RE\left[-10.61 \times 10^{-6} e^{+j2\pi z}\widehat{a_y} e^{j\omega t}\right] \\ &= RE\left[-10.61 \times 10^{-6} e^{+j(\omega t + 2\pi z)}\widehat{a_y}\right] \\ &= -10.61 \times 10^{-6}\cos(\omega t + 2\pi z)\widehat{a_y} \end{aligned}$$

Part (c) Instantaneous Power Density

Let $C = \cos(\omega t + 2\pi z)$

$$\begin{aligned} \overrightarrow{E} \times \overrightarrow{H} &= \begin{vmatrix} \widehat{a_x} & \widehat{a_y} & \widehat{a}_z \\ 4 \times 10^{-3} C & 0 & 0 \\ 0 & -10.61 \times 10^{-6} C & 0 \end{vmatrix} \\ &= \widehat{a}_z \left(4 \times 10^{-3} C\right)\left(-10.61 \times 10^{-6} C\right) \\ &= -42.44 \times 10^{-9}\cos^2(\omega t - 2\pi z)\widehat{a}_z \end{aligned}$$

Time Average Power Density

$$\begin{aligned} \frac{1}{2}\widetilde{E} \times \widetilde{H^*} &= \frac{1}{2}\begin{vmatrix} \widehat{a_x} & \widehat{a_y} & \widehat{a}_z \\ 4 \times 10^{-3} e^{+j2\pi z} & 0 & 0 \\ 0 & -10.61 \times 10^{-6} e^{-j2\pi z} & 0 \end{vmatrix} \\ &= \frac{1}{2}\left[\widehat{a}_z \left(4 \times 10^{-3}\right)\left(-10.61 \times 10^{-6}\right)\right] \\ &= 21.22 \times 10^{-9} Watts \end{aligned}$$

Part (d) Instantaneous Electric Energy Density

$$\begin{aligned}
W_e &= \frac{1}{2}\varepsilon\left|\overrightarrow{E}\right|^2 \\
&= \frac{1}{2}\left(8.85\times10^{-12}\right)\left|4\times10^{-3}\cos(\omega t+2\pi z)\right|^2 \\
&= 70.8\times10^{-18}\cos^2(\omega t+2\pi z) Joules
\end{aligned}$$

Instantaneous Magnetic Energy Density

$$\begin{aligned}
W_m &= \frac{1}{2}\mu\left|\overrightarrow{H}\right|^2 \\
&= \frac{1}{2}\left(4\pi\times10^{-7}\right)\left|-10.61\times10^{-6}\cos(\omega t+2\pi z)\right|^2 \\
&= 707.3\times10^{-12}\cos^2(\omega t+2\pi z) Joules
\end{aligned}$$

Time Average Electric Energy Density

$$\begin{aligned}
W_e &= \frac{1}{4}\varepsilon\left|\widetilde{E}\right|^2 \\
&= \frac{1}{4}\left(8.85\times10^{-12}\right)\left|4\times10^{-3}e^{+j2\pi z}\right|^2 \\
&= 35.4\times10^{-18}e^{+j4\pi z} Joules
\end{aligned}$$

Time Average Magnetic Energy Density

$$\begin{aligned}
W_m &= \frac{1}{4}\mu\left|\overrightarrow{H}\right|^2 \\
&= \frac{1}{4}\left(4\pi\times10^{-7}\right)\left|-10.61\times10^{-6}e^{+j2\pi z}\right|^2 \\
&= 353.65\times10^{-12}e^{+j4\pi z} Joules
\end{aligned}$$

12 Problem 3.5 (Balanis 4.22)

12.1 Statement

Sea water is an important medium in communication between submerged submarines or between submerged submarines and receiving and transmitting stations located above the surface of the sea. Assuming the constitutive electrical parameters of the sea are $\sigma = 4$ S/m, $\varepsilon_r = 81$, $\mu_r = 1$, and $f = 10^4$ Hz, find:

(a) The complex propogation constant (per meter).
(b) The phase velocity (meters per second).
(c) The wavelength (meters).
(d) The attenuation constant (Nepers per meter).
(e) The skin depth (meters).

12.2 Solution

Part (a) Complex Propogation Constant

$$\gamma = \alpha + j\beta$$

$$\begin{aligned}
\alpha &= \omega\sqrt{\mu\varepsilon}\left\{\frac{1}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right]\right\}^{1/2} \\
&= 2\pi\times 10^4\sqrt{(4\pi\times 10^{-7})(81)(8.85\times 10^{-12})}\left\{\frac{1}{2}\left[\sqrt{1+\left(\frac{4}{(2\pi\times 10^4)(81)(8.85\times 10^{-12})}\right)^2}-1\right]\right\}^{1/2} \\
&= 1.886\times 10^{-3}\left\{\frac{1}{2}\left[\sqrt{1+7.8868\times 10^9}-1\right]\right\}^{1/2} \\
&= 1.886\times 10^{-3}\left\{44.4\times 10^3\right\}^{1/2} \\
&= 397.4\times 10^{-3} Np/m
\end{aligned}$$

$$\begin{aligned}
\beta &= \omega\sqrt{\mu\varepsilon}\left\{\frac{1}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]\right\}^{1/2} \\
&= 2\pi\times 10^4\sqrt{(4\pi\times 10^{-7})(81)(8.85\times 10^{-12})}\left\{\frac{1}{2}\left[\sqrt{1+\left(\frac{4}{(2\pi\times 10^4)(81)(8.85\times 10^{-12})}\right)^2}+1\right]\right\}^{1/2} \\
&= 1.886\times 10^{-3}\left\{\frac{1}{2}\left[\sqrt{1+7.8868\times 10^9}+1\right]\right\}^{1/2} \\
&= 1.886\times 10^{-3}\left\{44.4\times 10^3\right\}^{1/2} \\
&= 397.4\times 10^{-3} rad/m
\end{aligned}$$

$$\gamma = (397.4 + j397.4)\times 10^{-3}\frac{1}{m}$$

Part (b) Phase Velocity

$$v = \frac{\omega}{\beta}$$

$$\begin{aligned} &= \frac{2\pi \times 10^4}{397.4 \times 10^{-3}} \\ &= 158.1 \times 10^3 m/s \end{aligned}$$

Part (c) Wavelength

$$\begin{aligned} \lambda &= \frac{2\pi}{\beta} \\ &= \frac{2\pi}{397.4 \times 10^{-3}} \\ &= 15.81m \end{aligned}$$

Part (d) Attenuation Constant

α is the attenuation constant $= 397.4 \times 10^{-3} Np/m$

Part (e) Skin Depth

$$\begin{aligned} \delta &= \frac{1}{\alpha} \\ &= \frac{1}{397.4 \times 10^{-3}} \\ &= 2.52m \end{aligned}$$

13 Problem 3.6 (Balanis 4.23)

13.1 Statement

The electrical constitutive parameters of moist earth at a frequency of 1 MHz are $\sigma = 10^{-1}$ S/m, $\varepsilon_r = 4$, $\mu_r = 1$. Assuming that the electric field of a uniform plane wave at the interface (on the side of the earth) is 3×10^{-2} V/m, find:

(a) The distance through which the wave must travel before the magnitude of the electric field reduces to $1.104 \times 10^{-2} V/m$.

(b) The attenuation the electric field undergoes in part (a) (in decibels).

(c) The wavelength inside the earth (in meters).

(d) The phase velocity incide the earth (in meters per second).

(e) The intrinsic impedance of the earth.

13.2 Solution

Part (a)

$$\begin{aligned}
\alpha &= \omega\sqrt{\mu\varepsilon}\left\{\frac{1}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right]\right\}^{1/2} \\
&= 2\pi \times 10^6\sqrt{(4\pi \times 10^{-7})(4)(8.85 \times 10^{-12})}\left\{\frac{1}{2}\left[\sqrt{1+\left(\frac{10^{-1}}{(2\pi \times 10^6)(4)(8.85 \times 10^{-12})}\right)^2}-1\right]\right\}^{1/2} \\
&= 0.0419\left\{\frac{1}{2}\left[\sqrt{1+202.13 \times 10^3}-1\right]\right\}^{1/2} \\
&= 0.0419\left\{224.3\right\}^{1/2} \\
&= 627.5 \times 10^{-3} Np/m
\end{aligned}$$

Now we can find the distance from

$$\begin{aligned}
1.104 \times 10^{-2} V/m &= 3 \times 10^{-2} e^{-\alpha d} \\
1.104 \times 10^{-2} V/m &= 3 \times 10^{-2} e^{-.13325d} \\
\ln\left(\frac{1.104}{3}\right) &= -0.13325d \\
-0.999 &= -0.13325d \\
d &= 7.5m
\end{aligned}$$

Part (b) Voltage Drop

$$\begin{aligned}
dB &= 20\log_{10}\left(\frac{V_2}{V_1}\right) \\
&= 20\log_{10}\left(\frac{1.104}{3}\right) \\
&= -8.68dB
\end{aligned}$$

Part (c) Wavelength

$$\beta = \omega\sqrt{\mu\varepsilon}\left\{\frac{1}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]\right\}^{1/2}$$

$$
\begin{aligned}
&= 2\pi\times 10^{6}\sqrt{(4\pi\times 10^{-7})(4)(8.85\times 10^{-12})}\left\{\frac{1}{2}\left[\sqrt{1+\left(\frac{10^{-1}}{(2\pi\times 10^{6})(4)(8.85\times 10^{-12})}\right)^{2}}+1\right]\right\}^{1/2}\\
&= 41.9\times 10^{-3}\left\{\frac{1}{2}\left[\sqrt{1+202.13\times 10^{3}}+1\right]\right\}^{1/2}\\
&= 1.886\times 10^{-3}\left\{225.3\right\}^{1/2}\\
&= 629.0\times 10^{-3} rad/m
\end{aligned}
$$

$$
\begin{aligned}
\lambda &= \frac{2\pi}{\beta}\\
&= \frac{2\pi}{629.0\times 10^{-3}}\\
&= 10m
\end{aligned}
$$

Part (d) Velocity

$$
\begin{aligned}
v &= \frac{\omega}{\beta}\\
&= \frac{2\pi\times 10^{6}}{629.0\times 10^{-3}}\\
&= 1\times 10^{7} m/s
\end{aligned}
$$

Part (e) Intrinsic Impedance

$$
\begin{aligned}
\eta_c &\approx \sqrt{\frac{\mu}{\varepsilon}}\\
&\approx \sqrt{\frac{(4\pi\times 10^{-7})}{4\,(8.85\times 10^{-12})}}\\
&\approx 188.4\Omega
\end{aligned}
$$

14 Problem 3.7 (Balanis 4.24)

14.1 Statement

The complex electric field of a uniform plane wave is given by

$$\widetilde{E} = 10^{-2}\left[\widehat{a_x}\sqrt{2} + \widehat{a_z}\left(1+j\right)e^{j\pi/4}\right]e^{-j\beta y}$$

(a) Find the polarization of the wave (linear, circular, or elliptical).
(b) Determine the sense of rotation (clockwise or counterclockwise).
(c) Sketch the figure the electric field traces as a function of ωt.

14.2 Solution

Part (a)

$$\begin{aligned}\widetilde{E} &= 10^{-2}\left[\widehat{a_x}\sqrt{2} + \widehat{a_z}\sqrt{2}e^{j\pi/4}e^{j\pi/4}\right]e^{-j\beta y} \\ &= 10^{-2}\left[\widehat{a_x}\sqrt{2} + \widehat{a_z}\sqrt{2}e^{j\pi/2}\right]e^{-j\beta y} \\ &= 10^{-2}\sqrt{2}\left[\widehat{a_x} + j\widehat{a_z}\right]e^{-j\beta y}\end{aligned}$$

From this last form, we can see that the plane wave is circularly polarized in the XZ-plane and is moving along the positive Y-axis.

Part (b)

$$\begin{aligned}\widetilde{E} &= 10^{-2}\left[\widehat{a_x}\sqrt{2} + \widehat{a_z}\sqrt{2}e^{j\pi/2}\right]e^{-j\beta y} \\ &= 10^{-2}\sqrt{2}\widehat{a_x}e^{-j\beta y} + 10^{-2}\sqrt{2}\widehat{a_z}e^{j\pi/2}e^{-j\beta y} \\ &= 10^{-2}\sqrt{2}\widehat{a_x}e^{-j\beta y} + 10^{-2}\sqrt{2}\widehat{a_z}e^{-j(\beta y-\pi/2)}\end{aligned}$$

$$\begin{aligned}\overrightarrow{E} &= RE\left[10^{-2}\sqrt{2}\widehat{a_x}e^{-j\beta y}e^{j\omega t} + 10^{-2}\sqrt{2}\widehat{a_z}e^{-j(\beta y-\pi/2)}e^{j\omega t}\right] \\ &= 10^{-2}\sqrt{2}\widehat{a_x}\cos\left(\omega t-\beta y\right) + 10^{-2}\sqrt{2}\widehat{az}\cos\left(\omega t-\beta y+\pi/2\right)\end{aligned}$$

From this last form, we can see that $\widehat{a_z}$ leads $\widehat{a_x}$ by $\frac{\pi}{2}$. “The sense of rotation is determined by rotating the phase-leading component toward the phase-lagging component. The field rotation must be viewed as the wave travels away from the observer.” (Balanis, p.161). Looking along the positive Y-axis from the origin, a clockwise rotation is required to move Z onto X, so the wave is polarized clockwise.

Part (c) Sketch