A linear worst-case lower bound on the number of holes inside regions visible due to multiple diffuse reflections

Sudebkumar Prasant Pal,[1] Siddhartha Brahma
Department of Computer Science and Engineering,
Indian Institute of Technology, Kharagpur
Kharagpur 721302, India.
email: spp@cse.iitkgp.ernet.in
url:- http://www.angelfire.com/or/sudebkumar

Dilip Sarkar
Department of Computer Science
University of Miami, Coral Gables
Miami, FL 33124, USA
email: sarkar@cs.miami.edu

Abstract

It is known that the region $V(s)$ of a simple polygon P, directly visible (illuminable) from an internal point s, is simply connected. Aronov et al. [**?**] established that the region $V_1(s)$ of a simple polygon visible from an internal point s due to at most one diffuse reflection on the boundary of the polygon P, is also simply connected. In this paper we establish that the region $V_2(s)$, visible from s due to at most two diffuse reflections

[1]A part of this work was done when this author was visiting the University of Miami, Coral Gables, Florida, USA.

may be multiply connected; we demonstrate the construction of an n-sided simple polygon with a point s inside it so that the region of P visible from s after at most two diffuse reflections is multiply connected. We also show that $V_3(s)$, the region of P visible from s after at most three diffuse reflections, can have $\Omega(n)$ holes.

Mathematics Subject Classification (2000): 52C45, 68U05.
Keywords: multiple connectedness, diffuse reflection, visibility, simple polygon.

1 Introduction

In recent years, visibility problems have been studied extensively (see [**?**, **?**]). Visibility computations abound in computer graphics, motion planning, robotics and computer vision. Two points inside a simple polygon are mutually *visible* if the line segment joining them is not obstructed by any edge of the polygon. Several algorithms exist for computing the region visible from a point light source inside a simple polygon [**?**, **?**, **?**]. The problems of computing the region of a simple polygon which is *weakly visible* from an internal segment [**?**, **?**], or a convex set [**?**], are also well studied. A point p inside a simple polygon P is said to be *weakly visible* from an edge e of the polygon if p is visible from some point in the *interior* of e. Certain portions of the polygon that are not directly illuminated from the source may become visible due to one or more reflections on the bounding edges of the polygon. Reflection at a point is called *specular* if the reflected ray is directed as per the standard law of reflection; the angle of incidence is the same as the angle of reflection. Most reflecting surfaces cause another type of reflection, called *diffuse* reflection, where light incident at a point is reflected in all possible interior directions. We assume that light incident at a vertex is absorbed and not reflected. Throughout this paper we consider the region visible from a point source s due to multiple reflections inside a simple polygon P. Such visible regions are also called *visibility polygons.*

1.1 Preliminaries and notation

Let P be a simple polygon with no three collinear vertices. Let $int(X)$ and $bd(X)$ denote the relative interior and boundary of a region X of P, respectively. If p and q are two points on $bd(P)$, let $bd(p, q)$ denote the part of $bd(P)$ traversed from p to q in counterclockwise order, keeping $int(P)$ to the left of the direction of traversal. Two points in P are *(mutually) visible* if the interior of the line segment joining them lies in $int(P)$. A point y is *visible under diffuse reflection* from a point x, if there exists a point p lying in the interior of an edge of P, such that p is visible from both x and y.

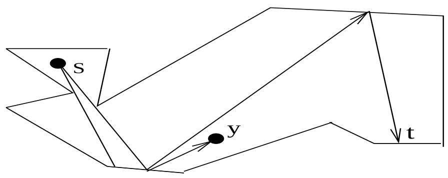

Figure 1.1: Visibility with multiple diffuse reflections.

Let s denote a point light source inside P. A point y is 1-visible from s if y and s are mutually visible. For $k > 1$, a point $y(= p_k)$ is *k-visible* from a point $s(= p_0)$, if there exists points $p_1, p_2, ..., p_{k-1}$ in the interiors of edges of P, such that p_i and p_{i+1} are mutually visible for $0 \leq i \leq k - 1$. In Figure 1.1, y is *2-visible* from s; y is visible from s after one diffuse reflection. In Figure 1.1, t is *3-visible* from s; t is visible from s after two diffuse reflections. We assume that the light incident at a vertex is absorbed and not reflected further. We represent a directed line segment passing through two points a and b and directed from a to b by $ray(a, b)$.

Let $V(s)$ denote the portion of P visible from s. We know that $V(s)$ is simply connected, has $O(n)$ edges, and at most one edge of $V(s)$ lies on each edge of P [**?**, **?**]. For a point $s \in P$, let $V(s)$ denote the polygonal region consisting of points in P that are directly visible from s. For $k \geq 0$, let $V_k(s)$ denote the polygonal region consisting of points that are l-visible from s, for some $1 \leq l \leq k+1$. $V_k(s)$ is the set of points that receive light from s after at most k diffuse reflections. We have $V_0(s) = V(s)$.

1.2 Combinatorial complexity and multiple connectedness of regions visible due to diffuse reflections

Regions visible due to reflections inside simple polygons were first studied by Aronov *et al.* [**?**]; a tight $\Theta(n^2)$ bound was established on the number of edges of visibility polygons when at most one diffuse reflection is permitted. The authors showed that such visibility polygons were simply connected. In sharp contrast, regions visible due to at most one specular reflection were shown to be multiply connected; the combinatorial complexity remained $\Theta(n^2)$ however [**?**]. (The combinatorial complexity of a visibility polygon is the number of edges defining its boundary.) The number of *holes* (also called *blind spots*) in such a visibility polygon (with at most one specular reflection) is also $\Theta(n^2)$ [**?**].

The case where at most k specular reflections are permitted, is dealt with in a subsequent paper [**?**]; an $O(n^{2k})$ upper bound and a lower bound of $\Omega((n/k - \Theta(1))^{2k})$ on the combinatorial complexity of the visibility polygons was established when at most a constant number k of specular reflections are permitted. The number of holes too obeys these bounds. The upper bound on the combinatorial complexity of visibility polygons improves to $O(n^{2\lceil (k+1)/2\rceil+1})$, when specular reflections are replaced by diffuse reflections [**?**]. Note that the improvement is by an asymptotic square root factor for large k. This is not surprizing because diffuse reflection spreads light much more widely than restricted pencils do in the case of specular reflections.

The best lower bound on the combinatorial complexity for visibility polygons $V_k(s)$ (with at most k diffuse reflections), for any $k \geq 1$, remains just $\Omega(n^2)$ to date (see [**?**, **?**]). One interesting and important problem concerning multiple diffuse reflections is that of bridging this gap between the very high upper bound of [**?**] and the $\Omega(n^2)$ lower bound of [**?**, **?**]. One way to approach this open problem is to study the structure of $V_k(s)$ and bound the total combinatorial complexity of holes in it. The presence of holes in $V_2(s)$ (and in $V_k(s)$ for $k \geq 1$, in general) was not known until we demonstrated the existence of holes in $V_k(s)$, $k \geq 1$, in [**?**]. In this paper we present the construction in [**?**] of an n-sided simple polygon P with a point s located inside P so that $V_2(s)$ has a hole. We construct a simple polygon P and place a point s inside it so that the portion of the polygon P visible from s due to at most two diffuse reflections does not include the interior of a triangle tqr (see Figures 2.2). Here, the triangle tqr is totally contained inside P and the boundary of triangle tqr as well as its exterior neighbourhood are visible from s after at most two diffuse reflections. The natural question is whether we can have more holes in $V_2(s)$. It is possible to construct an n-sided simple polygon with an interior point s so that $V_2(s)$ has two holes [**?**]. The construction demonstrating two holes in $V_2(s)$ is an extension of the construction above for a single hole as shown in Figures 2.2. However, a general construction demonstrating $\Omega(n)$ holes is not easy for $V_2(s)$. In this paper we establish a linear worst-case lower bound on the number of holes in the region $V_3(s)$ visible from a point source s due to at most three diffuse reflections inside an n-sided simple polygon P. We achieve this lower bound by extending the construction in section 2; we replicate the polygonal chain containing one hole in Figure 2.2 H times, where H is an integer, thereby realizing H holes inside the regions visible due to three diffuse reflections for the constructed polygon in Figure 3.3 with $O(H)$ edges.

The existence of holes in $V_k(s), k > 1$ is a bit surprising. Diffuse reflection, as the term suggests, results in bouncing back of incident light in all possible directions from the point of reflection. Clearly, a lot more region gets illuminated due to such reflections when compared with specular reflection. Multiple diffuse reflections would lead to even larger spreads and it was not intuitively apparent as to how holes could be created in such widely spread visible regions. In fact, it seems unlikely that too many such holes can be created; we suspect that the number of holes in $V_k(s)$ is $O(n)$. An interesting problem is to find asympototic upper bounds on the number of holes in $V_k(s)$. A linear upper bound on the number of holes or the total combinatorial complexity of holes in $V_k(s)$ for any $k \geq 1$ may improve the upper bound on the combinatorial complexity of $V_k(s)$ over that in [**?**].

2 The construction of a hole for two diffuse reflections

Now we show that $V_2(s)$ may have a hole. In what follows, we state this result in detail as in [**?**]. We show the existence of an island or hole by constructing a simple polygon P containing a point s such that $V_2(s)$ has a triangular hole with vertices t, q and r (see Figure 2.2). The interior of the triangle tqr is invisible from s, from all points of $bd(V_0(s))$ on $bd(P)$, and, from all points of $bd(V_1(s))$ on $bd(P)$. Note that the first reflection takes place on points of $bd(V_0(s))$ on $bd(P)$ and the second reflection takes place on points of $bd(V_1(s))$ on $bd(P)$.

We need a definition. We say that a point p *blocks* a region R of P from another region S if the shortest path from every point in R to every point in S passes through p. Clearly, no ray can reach the hole directly from s. This is because b blocks s from the (proposed) hole. We first show that no point in the interior of the triangle tqr is visible after the first reflection.

LEMMA 2.1 The interior of the triangle tqr is not visible from s after the first reflection.

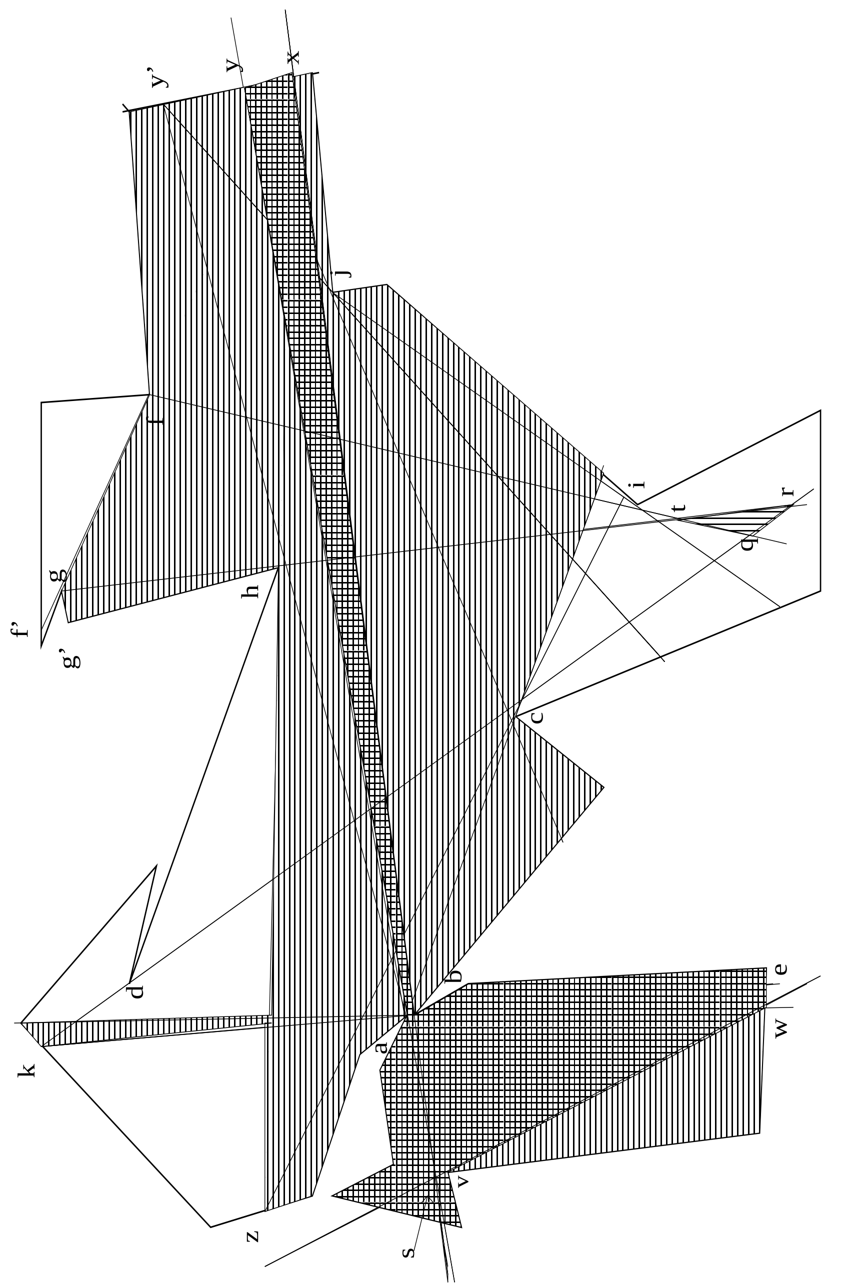

Figure 2.2: $V_0(s) = V(s)$ is deeply shaded and $V_1(s)$ is lightly shaded. Triangle tqr is a hole of $V_2(s)$.

Proof.
Consider $bd(V_0(s))$. We start by classifying portions of $bd(P)$ that form $bd(V_0(s))$. There are three portions- $bd(x, y)$, $bd(w, b)$ and $bd(a, v)$. These constitute all points on $bd(V_0(s))$ on $bd(P)$. The sequences of points, $< s, a, y >$, $< s, b, x >$ and $< s, v, w >$ are collinear. How do we guard the hole from the points of $bd(V_0(s))$ on $bd(P)$? The shortest path from any point in $bd(x, y)$ to any point in the hole has to pass through j. In other words, j *blocks* the hole from $bd(x, y)$. So, no point in $bd(x, y)$ can see the hole. Similarly, b blocks $bd(w, b)$ from the holes, and, c blocks $bd(a, v)$ from the hole. We conclude that the hole is not visible from any point of $bd(V_0(s))$ on $bd(P)$.

$\square$

All we need to do now is to show that no point of $bd(V_1(s))$ on $bd(P)$ can see any point inside the hole tqr. We start by classifying portions of $bd(P)$ that lie on $bd(V_1(s))$. These are

1. Points of $bd(a, b)$ visible from $bd(a, v)$, $bd(w, b)$ or $bd(x, y)$. (Note that $bd(x, y)$ is the only portion of $bd(P)$ visible from s outside $bd(a, b)$.)

2. Points of $bd(i, f)$ and $bd(b, c)$, visible from $bd(a, v)$. (Points of $bd(a, v)$ can see points within $bd(i, f)$ because (i) the vertex i is to the right of the ray starting at a and passing through c, and, (ii) the vertex f is to the left of the ray starting at v and passing through a. Observe that $bd(a, v)$ can also see points on $bd(b, c)$. These are the only portions of $bd(P)$, visible from $bd(a, v)$.)

3. Points of $bd(h, a)$ visible from $bd(w, b)$. (Points in $bd(w, b)$ can see points of $bd(P)$ only within $bd(d, a)$, or more precisely, points within $bd(d, k)$, where e, b, a and k are collinear.)

4. Points of $bd(b, c)$ visible from $bd(x, y)$. (The point c is to the left of $ray(y, j)$. So, all points of $bd(V_1(s))$ on $bd(b, j)$ are to the right of $ray(y, j)$).

5. Points of $bd(h, a)$ visible from $bd(x, y)$.

6. Points of $bd(i, h)$ visible from $bd(x, y)$.

The remaining points on $bd(P)$ can not be on $bd(V_1(s))$.

Now we show that none of these six possible portions of $bd(V_1(s))$ can illuminate the triangular hole tqr of $V_2(s)$.

1. The second reflection at $bd(a, b)$ causes rays to miss the hole due to the blocker c. This is due to the fact that the shortest path from any point in $bd(a, b)$ to any point in the hole tqr, has to cross segment ac. In particular, points in $bd(v, b)$ are blocked from the hole by b.

2. Consider points of $bd(V_1(s))$ on $bd(i, f)$. It is sufficient to consider the extreme points y' and c' of $bd(i, f)$ in $bd(V_1(s))$. The point y' is collinear to v and a. The point j blocks y' and the whole of $bd(j, y')$ from the hole. The point c' is collinear with a and c. The point i blocks the whole of $bd(i, j)$ from the hole tqr.

   Now consider points of $bd(b, c)$ that belong to $bd(V_1(s))$ due to first reflection at $bd(a, v)$. These points are blocked from the hole by c.

3. Only points within $bd(d, k)$ in $bd(h, a)$ can cause the second reflection from rays undergoing the first reflection on $bd(w, b)$. All these points in $bd(d, k)$ are blocked by d from the hole tqr. Indeed, k, d, q and r are collinear, defining the edge qr of tqr. Note that q is the intersection of $ray(k, d)$ and $ray(f, i)$ and, r is the intersection of $ray(k, d)$ and $ray(g, h)$.

4. Consider points of $bd(b, c)$ visible from xy. Points of $bd(b, c)$ cannot see tqr because they are blocked by c.

5. Consider $bd(d, a)$. Points in $bd(d, z)$ are blocked by h from points on xy. Since z, h and x are collinear and tqr is to the right of $ray(z, c)$, all points of $bd(z, a)$ are blocked from tqr by c.

6. Consider $bd(i,h)$ and xy. Points on $bd(j,y)$ are blocked from tqr by j. So we need to consider only $bd(y,h)$. Any point on $bd(y,f)$ may reflect into the region to the right of $ray(f,i)$ but not to the left of $ray(f,i)$. In the construction, f, i, t and q are collinear and tq defines an edge of the hole tqr. Note that $ray(g,h)$ and $ray(f,i)$ intersect at vertex t.

   Now consider point f' on $bd(f,h)$ where f' is the point where $ray(x,f)$ meets $bd(P)$. We construct $bd(f,g)$ so that the shortest path from f' to h passes through g and g lies in the segment joining f and f'. Points on xy cannot see any point in the interior of $bd(f,f')$. Since g, h, t and r are collinear, (i) the shortest path from any point in $bd(f',g)$ to any point inside tqr has to pass through blockers h and g, and, (ii) the shortest path from any point in $bd(g,h)$ to any point in tqr has to pass through blocker h. Indeed, $ray(g,h)$ defines the edge tr of hole tqr.

   Note that reflections on edge $g'h$ will not reach the interior of triangle tqr because triangle tqr is to the right of $ray(g',h)$.

In summary, we have established that tqr is a hole in $V_2(s)$. We state the result as a theorem.

THEOREM 2.2 It is possible to construct a simple polygon P with internal points s, t, q and r such that (i) tqr is a triangle totally contained in $int(P)$ (ii) all points inside the triangle tqr belong to $P \setminus V_2(s)$ and (iii) tq, tr and rq are on $bd(V_2(s))$.

3 Constructing $\Omega(n)$ holes using three diffuse reflections

We show that the number of holes in $V_3(s)$ is $\Omega(n)$. We demonstrate this using a construction of a simple polygon P with $n = D * H + D'$ vertices with a location of a point source s inside P so that $V_3(s)$ has H triangular holes. Here, D and D' are small integers. This construction

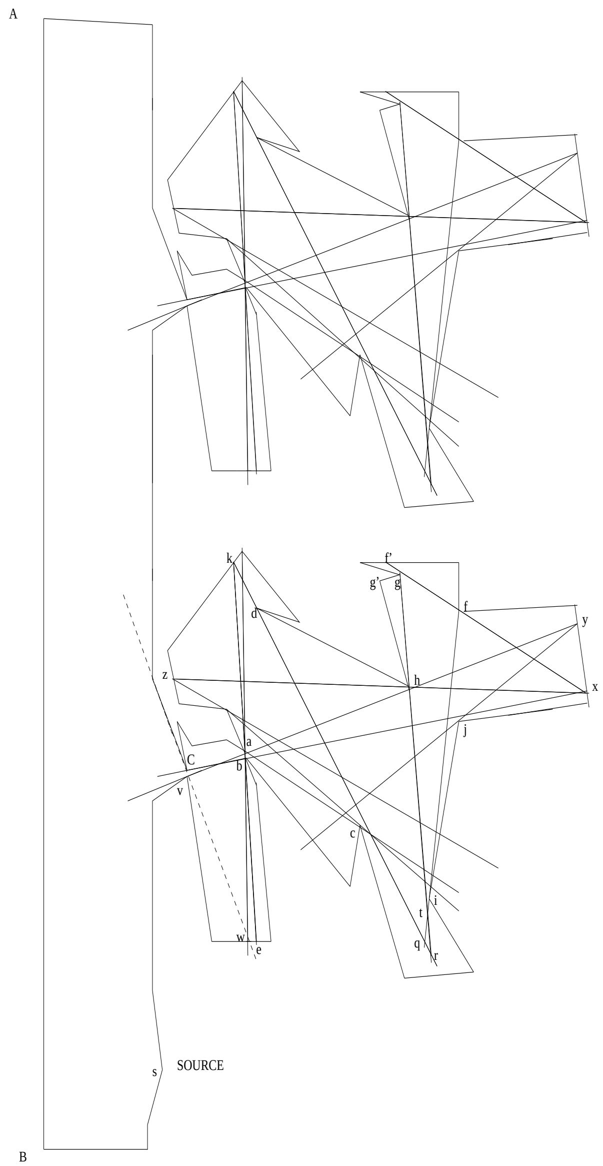

Figure 3.3: Existence of $\Omega(n)$ holes in $V_3(s)$

derives from that in section 2; we arrange multiple polygonal chains (precisely H chains) like $bd(v,a)$ in Figure 2.2 , in a linear fashion (see Figure 3.3). The chain $bd(v,a)$ of Figure 2.2 appears as the chain $bd(v,C)$ in Figure 3.3. The source s is dropped and instead diffuse light is projected into the regions bounded by these H chains from a long edge AB of P. This edge AB in turn gets illuminated from the lone new source s in P as shown in Figure 3.3. The source s does not directly illuminate any portion inside any of the H chains.

Consider the first such chain $bd(v,C)$ as in Figure 3.3. We have the same arguments for all the chains of P. For the sake of convenience we have only two chains in Figure 3.3. We show that after three reflections, each chain like $bd(v,C)$ has one hole in $V_3(s)$. We essentially simulate the effect of the single source as in Figure 2.2; the main objective is to ensure that only the portion xy of $bd(a,b) \subset bd(v,C)$ receives light after the first diffuse reflection on AB and, furthermore (as in section 2), only the portion $bd(w,b)$ of $bd(v,b)$ and $bd(a,C)$ receive light from AB. Note that although we do not have a single point source as in section 2, we still can create the appropriate illumination on xy where the second reflection takes place to illuminate critical vertices like f, g, k and z. We place v and C such that the slit defined by v and C and the slit defined by a and b, together define a bundle of rays that reach only the portion xy of $bd(b,a)$ from AB. This is ensured by the collinearity of x, b and C on the one hand and the collinearity of y, a and v on the other hand. These collinearities limit the extreme rays reaching $bd(b,a)$ from the entire AB.

We have already observed that the only portion of $bd(v,C)$ receiving light after the first reflection is the segment xy and certain parts of $bd(a,C)$ and $bd(w,b)$. Now we characterize the portions of $bd(v,C)$ that receive light after the second reflection and thereafter cause the third reflection inside $bd(v,C)$. These are parts of $bd(d,k)$, $bd(j,f)$, $bd(f',h)$ and $bd(z,c)$. It is easy to observe that by virtue of the construction in section 2, the portions of $bd(P)$ in Figure 3.3 causing the second and third reflections cannot illuminate any point inside the triangular hole tqr. Reflections (second reflections) from xy, $bd(a,C)$ and $bd(w,b)$ miss the hole tqr due to blockers j, c and b, respectively. Third reflections at

$bd(d,k)$, $bd(j,f)$, $bd(f',h)$ and $bd(z,c)$ miss the hole tqr due to blockers d, i, h and c, respectively. Indeed, as in Figure 2.2, the lines kd, fi and gh define the three edges qr, tq and tr of the hole in Figure 3.3. We summarize the result in this section as a theorem.

THEOREM 3.1 Given any integer $H \geq 1$, it is possible to construct a simple polygon P with an interior point s and $D * H + D'$ vertices, where D and D' are small integers, such that $V_3(s)$, the region of P visible due to at most three diffuse reflections, has H holes.

4 Concluding remarks

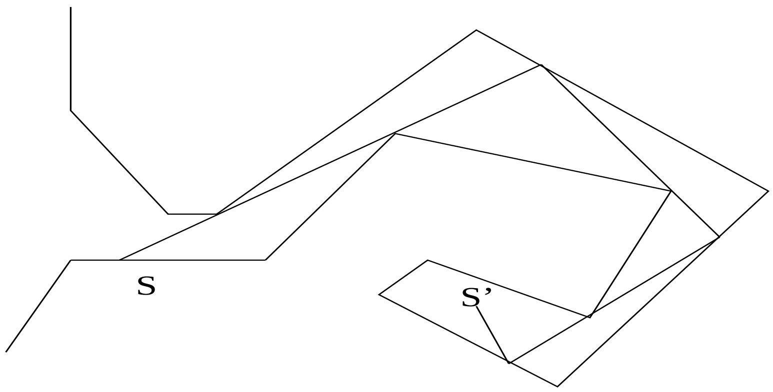

Figure 4.4: Forcing light from the source s' to take a zig-zag path gives a construction for the general case of $V_k(s)$.

We have shown that the region of a simple polygon visible from a point source due to at most two diffuse reflections can have a hole. We have also shown that in the worst case the number of holes can be linear in the number of vertices of the simple polygon when at most three diffuse reflections are considered. These results hold for single point light sources as well as single extended light sources like a line segment or a circle. The details are simple and are omitted here. For four or more diffuse reflections too, such multiply connected visible regions can be created by extending our constructions; the source s in our construction can be created as an illuminating line segment on an edge of the polygon after a predermined number $k-2, k \geq 4$ of reflections from a point s', where light from s' must undergo $k-2$ reflections (inside say, a winding spiral) and hit s before spreading to the rest of the polygon as shown in Figure 4.4. The details are straighforward and are omitted here.

An important research direction is that of establishing upper bounds on the number of holes in diffuse visibility regions. We believe that the upper bound is as low as linear in the number of vertices of the input polygon, matching the worst-case linear lower bound esatablished in this paper.

Acknowledgements: S. P. Pal thanks Alan Davis and Boris Aronov for helpful suggestions and comments on an earlier version of this paper.