Computing sharp and scalable bounds on the distance of an approximate zero from an exact zero of an analytic function

Siddhartha Brahma, Sudebkumar Prasant Pal
Hironmay Basu, Sudhir Kumar Singh*
Department of Computer Science and Engineering
Indian Institute of Technology
Kharagpur 721302, India.
email: spp@cse.iitkgp.ernet.in
http://www.angelfire.com/or/sudebkumar

Abstract

Numerical methods such as Newton's iterations can be used for computing approximate zeros of analytic functions in one complex variable. In this paper, we develop a simple and novel method for determining a sharp upper bound on the error in a given approximate zero of an analytic function using Taylor's theorem, Cauchy's integral formula and Rouché's theorem. Our bounds are scalable in the sense that we can compute sharper error bounds for better given approximations of zeros. We analyse the convergence of our method. We use requisite high precision computations for computing our bounds robustly and accurately.

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1 Introduction

The problem of solving the polynomial equation

$$g(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n = 0 \tag{1}$$

is a fundamental problem in algebra. Here, the coefficients a_i, $0 \leq i \leq n$ are complex numbers. Much attention has been paid to computing good approximations to zeros of such polynomials. Considerable research has been done on this problem and several methods are discussed in [8] and [17].

A natural generalization of univariate polynomials are univariate analytic functions. The problem of finding zeros of analytic functions is much more

*Department of Mathematics, Indian Institute of Technology, Kharagpur, 721302, India.

difficult than that of finding zeros of polynomials. An analytic function may have an infinite number of zeros. So, one may consider the problem of finding all zeros of an analytic function in a particular region γ of the complex plane. Many approaches have been proposed for solving this problem. One of the most popular approaches is the one proposed by Delves and Lyness [6]. In their pioneering work Delves and Lyness reduce the problem of computing zeros of analytic functions to that of computing the zeros of a polynomial. They develop methods based on numerical integration called *quadrature methods*. The works by Li [16] and Kravanja et al. [15] are variations of the above method. For a review of such methods see Iokimidis [9].

There are other approaches to the problem of computing zeros of analytic functions; Petković et al. [18] designed simultaneous iterative methods for computing zeros of analytic funtions where the zeros are known to be simple. His work was generalized to the case of multiple zeros with known mutiplicities by Atanassova [2]. Anastasselou and Ioakimidis [10, 11] have proposed methods for funtions that are analytic in the whole complex plane except for a finite number of dicontinuity arcs; these methods are based on the theory of Riemann-Hilbert boundary value problems. Kravanja et al. developed methods for solving systems of analytic functions [13] and for computing the poles and zeros of meromorphic functions [14].

Since all the above methods essentially compute only approximate zeros, we need to know how close these approximations are to the actual zeros. Although the zeros may be quite accurate, we still need to know how good they are. In other words, we need to calculate the distance of the approximate zero from an exact zero of the analytic function. In the case of polynomials this is a well studied problem. Smith computes upper bounds on errors in approximate zeros using a method based on Gerschgorin's theorems from linear algebra [25]. Several related approaches have been tried and there are works by Carstensen [4], Braess and Hadeler [3], Elsner [7] and Neumaier [17] which often give very good error bounds for good approximations of zeros. The present work is based on our previous work on bounding errors in approximate zeros of polynomials [20, 21, 22], in which we proposed a new and novel method for estimating upper bounds on errors based on Rouché's theorem. In this paper, we develop a method for determining a sharp upper bound on the error in a given approximate zero of an analytic function using Taylor's theorem, Cauchy's integral formula and Rouché's theorem. Our bounds are *scalable* in the sense that we can compute sharper error bounds for better given approximations of zeros. We use requisite high precision computations for computing our bounds robustly.

This paper is organized as follows. In section (2) we present our scheme for computing bounds on errors in approximate zeros of analytic functions in one variable. In section (3) we analyse methods to solve inequalities obtained in section (2). In section (4), we state examples where we apply our method of section (2) using **Mathematica 4.0**. We conclude the paper in section (5) with a few remarks and directions for future research.

2 A novel posteriori error bound for an approximate zero

We need the definition of an *analytic function* [1]. A *complex function* $f(z)$ is said to be *analytic* in a region if it is defined and differentiable at every point z in the region. Now we state Rouché's theorem.

Theorem 2.1 *[1, 8] Suppose the functions $f(z)$ and $g(z)$ are analytic inside and on a simple closed curve C. If f and g have no zeros on C and $|f(z) - g(z)| < |f(z)|$ for all z on C, then $f(z)$ and $g(z)$ have the same number of zeros inside C.*

The following theorem about analytic functions holds.

Theorem 2.2 *[1] If $t(z)$ is analytic in a region Ω, containing z_0, it is possible to write*

$$t(z) = t(z_0) + \frac{t'(z_0)}{1!}(z-z_0) + \frac{t''(z_0)}{2!}(z-z_0)^2 + \cdots + \frac{t^{(n)}(z_0)}{(n)!}(z-z_0)^n + t_{n+1}(z)(z-z_0)^{n+1}$$

where $t_{n+1}(z)$ is analytic in Ω.

Let $g(z)$ be the given analytic function with no singularities. Given an approximate zero z_0 of $g(z)$ we wish to determine positive real $r > 0$ such that $|z_0 - \alpha_0| < r$, where $g(\alpha_0) = 0$, that is, α_0 is an exact zero of $g(z)$. In other words, we wish to find an upper bound on the distance of the approximate zero z_0 from an exact zero α_0 of $g(z)$. Using **Theorem** 2.2 we have

$$g(z) = g(z_0) + \frac{g'(z_0)}{1!}(z-z_0) + \frac{g''(z_0)}{2!}(z-z_0)^2 + \cdots + \frac{g^{(n)}(z_0)}{(n)!}(z-z_0)^n + g_{n+1}(z)(z-z_0)^{n+1} \tag{2}$$

Taking $a_i = \frac{g^{(i)}(z_0)}{i!}$ for $0 \le i \le n$ and $E_{n+1}(z_0) = g_{n+1}(z)(z - z_0)^{n+1}$, we have

$$g(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + ... + a_n(z - z_0)^n + E_{n+1}(z_0)$$

We define $f(z)$ such that

$$g(z) = a_0 + (z - z_0)f(z) + E_{n+1}(z_0) \tag{3}$$

Note that $f(z)$ is an $(n - 1)$-degree polynomial.

The following **Lemma** follows directly from the definitions,

Lemma 2.3 *With the above definitions,*

$$|(z - z_0)f(z) - g(z)| \le |a_0| + |E_{n+1}(z_0)| \tag{4}$$

Proof Using (3),

$$(z - z_0)f(z) - g(z) = a_0 + E_{n+1}(z_0)$$

Therefore, by the triangle inequality,

$$|(z - z_0)f(z) - g(z)| = |a_0 + E_{n+1}(z_0)| \le |a_0| + |E_{n+1}(z_0)| \quad \blacksquare$$

Now, we define $f_1(z)$, a monic polynomial such that

$$(z-z_0)f(z) = (z-z_0)*b*(f(z)/b) = b(z-z_0)*f_1(z)$$

where b is the coefficient $\frac{g^{(n)}(z_0)}{n!}$ of z^{n-1} in $f(z)$.

Since $f_1(z)$ is a $(n-1)$-degree polynomial it must have $n-1$ zeros. Suppose we compute the $n-1$ zeros approximately. Let z_1,z_2,...,z_{n-1} be the approximate zeros. Let

$$L(z) = (z-z_1)(z-z_2)...(z-z_{n-1})$$

Since $z_1, z_2, ..., z_{n-1}$ are approximate zeros of $f_1(z)$, we account for the approximation of $f_1(z)$ by $L(z)$ as follows. We define $c(z)$ to be

$$c(z) = f_1(z) - L(z)$$

Since both $f_1(z)$ and $L(z)$ are monic polynomials of degree $n-1$,

$$c(z) = c_{n-2}z^{n-2} + c_{n-3}z^{n-3} + ... + c_0$$

$n \geq 2$, $c_i \in \mathbf{C}$. We have the follwing Lemma.

Lemma 2.4 *With the above definitions,*

$$|f(z)| \geq |b| \prod_{i=1}^{n-1}(||z-z_0| - |z_0 - z_i||) - \sum_{i=0}^{n-2}|c_i|(|z-z_0| + |z_0|)^i \tag{5}$$

Proof From the triangle inequality we have,

$$||f_1(z)| - |L(z)|| \leq |f_1(z) - L(z)| \tag{6}$$

Now,

$$\begin{aligned} |f_1(z) - L(z)| &= |c_{n-2}z^{n-2} + ... + c_0| \\ &\leq |c_{n-2}||z|^{n-2} + ... + |c_0| \end{aligned}$$

Therefore, from inequality (6)

$$|f_1(z)| \geq |L(z)| - (|c_{n-2}||z|^{n-2} + ... + |c_0|) \tag{7}$$

Again, we have the following implication using the triangle inequality.

$$L(z) = (z-z_1)(z-z_2)\cdots(z-z_{n-1})$$

$$\Rightarrow |L(z)| \geq (||z-z_0| - |z_0-z_1||)(||z-z_0| - |z_0-z_2||)\cdots(||z-z_0| - |z_0-z_{n-1}||)$$

Therefore from inequality (7) we have,

$$|f_1(z)| \geq \prod_{i=1}^{n-1}(||z-z_0| - |z_0-z_i||) - \sum_{i=0}^{n-2}|c_i||z|^i \tag{8}$$

Using the fact that $|z| \leq |z-z_0| + |z_0|$, we have

$$\begin{aligned} \sum_{i=0}^{n-2}|c_i||z|^i &= \sum_{i=0}^{n-2}|c_i||(z-z_0)+z_0|^i \\ &\leq \sum_{i=0}^{n-2}|c_i|(|z-z_0|+|z_0|)^i \end{aligned}$$

Since $f_1(z) = f(z)/b$, we have from inequality (8),

$$|f(z)| \geq |b| \prod_{i=1}^{n-1} (||z - z_0| - |z_0 - z_i||) - \sum_{i=0}^{n-2} |c_i|(|z - z_0| + |z_0|)^i \quad \blacksquare$$

We have two inequalities (4) and (5). We intend to apply Rouché's theorem on the functions $(z - z_0)f(z)$ and $g(z)$ over the circular closed curve C of radius r and centre z_0, represented as $|z - z_0| = r$. This is done by satisfying the inequality $|(z - z_0)f(z) - g(z)| < |(z - z_0)f(z)|$ over the circular closed curve C. If this premise is set then Rouché's theorem ensures that the functions $(z - z_0)f(z)$ and $g(z)$ have the same number of zeros in the interior of the circle C. Since z_0 is trivially a zero of $(z - z_0)f(z)$, $g(z)$ will have at least one zero (say α_0) in C. This implies that $|z_0 - \alpha_0| \leq r$. Thus r is an upper bound on the distance of z_0 from an exact zero of $g(z)$.

We state the main result of this paper in the following theorem.

Theorem 2.5 *If the inequality*

$$|E_{n+1}(z_0)| + |a_0| < |b|r \left(\prod_{i=1}^{n-1} (||z_0 - z_i| - r|) - \sum_{i=0}^{n-2} |c_i|(r + |z_0|)^i \right) \tag{9}$$

holds for every point z on the circle $|z - z_0| = r$, then $g(z)$ has at least one zero in the circular region of radius r centered at z_0.

Proof Consider the circular region bounded by the circle $|z - z_0| = r$. We wish to apply Rouché's theorem by ensuring that $|(z - z_0)f(z) - g(z)| < |(z - z_0)f(z)|$ for all z on the circle $|z - z_0| = r$. Since from **Lemma** (2.3)

$$|(z - z_0)f(z) - g(z)| \leq |a_0| + |E_{n+1}(z_0)|$$

and from **Lemma** (2.4)

$$|b|r \left(\prod_{i=1}^{n-1} (||z_0 - z_i| - r|) - \sum_{i=0}^{n-2} |c_i|(r + |z_0|)^i \right) \leq r|f(z)| = |(z - z_0)f(z)|$$

it is easy to see that if

$$|E_{n+1}(z_0)| + |a_0| < |b|r \left(\prod_{i=1}^{n-1} (||z_0 - z_i| - r|) - \sum_{i=0}^{n-2} |c_i|(r + |z_0|)^i \right),$$

by Rouché's theorem, $g(z)$ has at least one zero in the circular region of radius r centered at z_0. $\blacksquare$

We know that for any r satisfying inequality (9), Rouché's theorem ensures that $(z - z_0)f(z)$ and $g(z)$ must have the same number of zeros inside the region $|z - z_0| = r$. The inequality is automatically satisfied if it is satisfied for an upper bound on $|E_{n+1}(z_0)|$. The main challenge lies now in getting a good analytic form for an upper bound to $|E_{n+1}(z_0)|$. If the given approximate zero is a good approximation to an exact zero of $g(z)$, $|E_{n+1}(z_0)|$ is likely to be "small". Yet, for analytic rigor, it is important to get a good estimate of $|E_{n+1}(z_0)|$. In what follows, we demonstrate how such estimates can be derived.

We use **Theorem**(2.2) for accomplishing this task; we draw a circle C' about z_0 with radius R. Since $g(z)$ is analytic in the entire complex plane, the region Ω as required in **Theorem**(2.2) is the entire plane. So, there is no restriction on the radius R of C'. Recall that $E_{n+1}(z_0) = g_{n+1}(z)(z-z_0)^{n+1}$. The following fact can be proved about $g_{n+1}(z)$.

Lemma 2.6 *For C' defined above,*

$$g_{n+1}(z) = \frac{1}{2\pi i}\int_{C'} \frac{g(\zeta)d\zeta}{(\zeta - z_0)^{n+1}(\zeta - z)} \tag{10}$$

Proof (see [1]) By *Cauchy's Integral Formula*, we have

$$g_{n+1}(z) = \frac{1}{2\pi i}\int_{C'} \frac{g_{n+1}(\zeta)d\zeta}{\zeta - z}$$

Substituting the expression for $g_{n+1}(\zeta)$ from (2), we see that there will only be one main term containing $g(\zeta)$. The remaining terms are, except for constant factors, of the form

$$G_\nu(z_0) = \int_{C'} \frac{d\zeta}{(\zeta - z_0)^\nu(\zeta - z)} \qquad \nu \geq 1$$

But

$$G_1(z_0) = \frac{1}{z - z_0}\int_{C'} \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - z_0}\right) d\zeta = 0$$

identically for all z_0 inside C. It is easy to see that

$$G_{\nu+1}(z_0) = \frac{G_1^{(\nu)}(z_0)}{\nu!}$$

and thus $G_\nu(z_0) = 0$ for all $\nu \geq 1$. Hence the expression for $g_{n+1}(z)$ reduces to

$$g_{n+1}(z) = \frac{1}{2\pi i}\int_{C'} \frac{g(\zeta)d\zeta}{(\zeta - z_0)^{n+1}(\zeta - z)}$$

The representation is valid inside the circle C'. ∎

An estimate of $g_{n+1}(z)$ can be found out by using **Lemma**(2.6). Let the absolute value of $g(\zeta)$ be upper bounded by M on the circle C' with centre z_0. The radius of C' is R as defined above. Since $\int_{C'} \frac{d\zeta}{\zeta - z} = 2\pi i$ and $|\zeta - z_0| = R$, we have

$$|g_{n+1}(z)| \leq \frac{|g(\zeta)|}{2\pi R^{n+1}} \left|\int_{C'} \frac{d\zeta}{\zeta - z}\right| \leq \frac{M}{R^{n+1}}$$

for all z such that $|z - z_0| < R$.

Since $|z - z_0| = r$ for the circle C over which we are enforcing the premise of Rouché's theorem, we have by the definition of $E_{n+1}(z_0)$,

$$|E_{n+1}(z_0)| \leq \frac{Mr^{n+1}}{R^{n+1}}$$

for $r < R$.

Since we now have a bound on the error term $|E_{n+1}(z_0)|$ as a function of R, we may try to minimize this bound with respect to R, so that we have a tight upper bound. This will give us better values of r on solving inequality (9). This is not easy in general, but for a large number of common functions this can be achieved. In this paper we give two such examples.

2.1 $g(z) = e^z - 1$

In this case, on C',

$$|e^z - 1| \leq e^{|z_0|+R} + 1 = M$$

So, to get a good upper bound on the Taylor error $|E_{n+1}(z_0)|$, we need to minimize $\frac{Mr^{n+1}}{R^{n+1}}$. Let

$$\alpha = e^{|z_0|}, \;\; p = \frac{(\alpha e^R + 1)r^{n+1}}{R^{n+1}}$$

Differentiating p with respect to R and equating it to 0, we have

$$(\alpha e^R + 1)\left((n+1)R^n\right) = \alpha e^R R^{n+1}$$

or

$$\frac{\alpha e^R}{\alpha e^R + 1} = \frac{(n+1)}{R}$$

If we assume that $R >> 1 >> r$, for minimizing the Taylor error, we have

$$R \approx n + 1$$

Thus, we finally we have

$$|E_{n+1}(z_0)| \leq \frac{(e^{|z_0|}e^{n+1} + 1)r^{n+1}}{(n+1)^{n+1}}$$

and, from inequality (9), we get the upper bound r on the error in the approximate zero z_0 of $g(z)$ by finding feasible values for r in the inequality

$$\frac{(e^{|z_0|}e^{n+1} + 1)r^{n+1}}{(n+1)^{n+1}} + |a_0| < |b|r\left(\prod_{i=1}^{n-1}(|z_0 - z_i| - r) - \sum_{i=0}^{n-2}|c_i|(r + |z_0|)^i\right)$$

We try to compute as small a value of r satisfying this inequality as possible.

2.2 $g(z) = sin(z)$

In this case on C',

$$\begin{aligned} \left|sin(z)\right| = \left|sin(z_0 + Re^{i\theta})\right| &= \left|sin(z_0)cos(Re^i\theta) + cos(z_0)sin(Re^{i\theta})\right| \\ &\leq |sin(z_0)||cos(Re^{i\theta})| + |cos(z_0)||sin(Re^{i\theta})| \end{aligned}$$

It is easy to see that

$$|sin(z_0)| \leq \left(\frac{e^R + e^{-R}}{2}\right)$$

and

$$|cos(z_0)| \leq \left(\frac{e^R + e^{-R}}{2}\right)$$

Thus we can take

$$\begin{aligned} M &= \left(\frac{e^R + e^{-R}}{2}\right)|sin(z_0)| + \left(\frac{e^R + e^{-R}}{2}\right)|cos(z_0)| \\ &= \gamma\left(e^R + e^{-R}\right) \end{aligned}$$

where $\gamma = \left(\frac{|sin(z_0)|+|cos(z_0)|}{2}\right)$

Thus to get a good upper bound we have to minimize $\frac{Mr^{n+1}}{R^{n+1}}$.

Let $p = \frac{\gamma(e^R+e^{-R})r^{n+1}}{R^{n+1}}$.

Differentiating p with respect to R and equating it to 0, we have

$$(\gamma e^R + \gamma e^{-R})\left((n+1)R^n\right) = (\gamma e^R - \gamma e^{-R})R^{n+1}$$

i.e.,

$$\frac{\gamma + \gamma e^{-2R}}{\gamma - \gamma e^{-2R}} = \frac{R}{n+1}$$

Assuming $R >> 1 >> r$ due to reasons already stated above, the lhs of the above equation can be replaced by 1, whence

$$R \approx n+1$$

Thus, we finally have

$$|E_{n+1}| \leq \frac{\gamma(e^{n+1} + e^{-(n+1)})r^{n+1}}{(n+1)^{n+1}}$$

and we get the value of r by solving the inequality

$$\frac{\gamma(e^{n+1} + e^{-(n+1)})r^{n+1}}{(n+1)^{n+1}} + |a_0| < |b|r\left(\prod_{i=1}^{n-1}(|z_0 - z_i| - r) - \sum_{i=0}^{n-2}|c_i|(r + |z_0|)^i\right)$$

and taking the minimum value of r satisfying the inequality.

3 Solving the inequality and convergence criteria

The main **Theorem** 2.5 suggests a method for finding an upper bound on the distance of z_0 from an exact zero of $g(z)$. All we need to do is solve inequality (9) for a positive real value of r. To solve the inequality we need to have an estimate of $|E_{n+1}(z_0)|$. For a general analytic function it is not easy to give an explicit closed form description of the upper bound to $|E_{n+1}(z_0)|$. But as we have shown in the previous section, we can derive good upper bounds for specific cases. In what follows, we will analyse the inequalities for one of the special cases worked out in the previous section.

Let $g(z) = e^z - 1$. In this case inequality (9) takes the form

$$\frac{(e^{|z_0|}e^{n+1} + 1)r^{n+1}}{(n+1)^{n+1}} + |a_0| < |b|r\left(\prod_{i=1}^{n-1}(|z_0 - z_i| - r) - \sum_{i=0}^{n-2}|c_i|(r + |z_0|)^i\right) \quad (11)$$

Let

$$P(r) = |b|r\left(\prod_{i=1}^{n-1}(|z_0 - z_i| - r) - \sum_{i=0}^{n-2}|c_i|(r + |z_0|)^i\right) - \frac{(e^{|z_0|}e^{n+1} + 1)r^{n+1}}{(n+1)^{n+1}} - |a_0|$$

Computing a "small" value of r such that $P(r) > 0$ reduces to the non-linear optimization problem of solving inequality (11). There are several well known methods for solving polynomial inequalities such as (11). Typically, we could adopt a polynomial zero finding algorithm to solve polynomial inequalities. In this paper we illustrate the use of a safe variant of the Newton-Raphson method [19], that combines the bisection method and the Newton-Raphson method. The hybrid algorithm takes a bisection step whenever the Newton-Raphson method would take the solution out of bounds or whenever the Newton-Raphson method does not reduce the size of the brackets rapidly enough. In this algorithm, it is necessary to bracket the zero initially and it is ensured that the successive approximations to the zeros do not stray outside the brackets. Once we have the zero bracketed, we can apply the algorithm to get a very good approximation of the zero of $P(r)$. Then we can simply search in a small neighbourhood of the approximate zero to get a solution to the inequality. The above method converges always if we can bracket the root initially because the bisection method is guaranteed to converge and the Newton-Raphson method is not allowed to stray out of bounds.

In this paper, we analyse the solution of (11) for a special case when z_0 is a "good" approximation to a zero of $g(z)$ and the error in the zero is much smaller than $|z_i - z_0|, (1 \leq i \leq n-1)$. In what follows let

$$L = min\{|z_i - z_0|, 1 \leq i \leq (n-1)\}.$$

If the above assumption is valid, we can restict our attention to finding a solution of (11) in the interval (0,L). Since we are working in $(0, L)$, inequality (11) can be rephrased as

$$P(r) > 0 \tag{12}$$

The zero finding procedure typically works well if we can initially bracket the zero within a small interval. Since $P(0) < 0$, for bracketing a zero of $P(r)$ which lies in $(0, L)$, we have to find a positive real $r_0 \in (0, L)$ such that $P(r_0) > 0$. Let us rewrite (12) in the following equivalent form

$$r > Q(r) \tag{13}$$

where

$$Q(r) = \frac{|a_0| + \frac{(e^{|z_0|}e^{n+1}+1)r^{n+1}}{(n+1)^{n+1}} + |b|r\left(\sum_{i=0}^{n-2}|c_i|(|z_0|+r)^i\right)}{|b|\prod_{i=1}^{n-1}(|z_i - z_0| - r)} \tag{14}$$

Consider the equality corresponding to the inequality (13). If the above assumptions are valid, and

$$r \ll |z_0|\,,\, r \ll |z_i - z_0|(1 \leq i \leq n-1)\,,\, \frac{(e^{|z_0|}e^{n+1}+1)r^{n+1}}{(n+1)^{n+1}} \ll |a_0|$$

then

$$Q(r) \approx \frac{|a_0| + |b|r\left(\sum_{i=0}^{n-2}|c_i||z_0|^i\right)}{|b|\prod_{i=1}^{n-1}(|z_i - z_0|)}$$

Therefore, for $r_1 = \frac{|a_0|}{|b|\left(\prod_{i=1}^{n-1}(|z_i - z_0|) - \sum_{i=0}^{n-2}|c_i||z_0|^i\right)}$,we have

$$r_1 \approx Q(r_1)$$

Thus, under the assumptions mentioned above, if $r_1 < L$, then r_1 can serve as a good starting point for searching an r_0 such that $P(r_0) > 0$ and the method outlined above converges quickly. Note that for a "good" approximation $|z_0|$, the above assumptions are likely to hold and hence we are likely to get a good upper bound. This is illustrated by examples in the next section. Of course, if the assumptions made above do not hold, we will have to adopt a trial and error approach or use more sophisticated methods to bracket the root initially.

The sharpness of the upper bound obtained from the solution of inequality (11) depends on many factors. The closer z_0 is to an actual zero of $g(z)$, the smaller is the value of $|a_0|$. This results in better upper bounds and thus explains the scalability of our method. In this method we require the zeros of $f_1(z)$ to compute the bounds. This is not a serious drawback as there are several fast polynomial zero finding methods that give zeros with large values of precision. Clearly, the sharpness of the upper bound also depends on how close z_i's$(1 \leq i \leq n-1)$ are to the actual zeros of $f_1(z)$. The better the approximations are, the smaller will be the values of $|c_i|$'s. This will result in better upper bounds.

4 Results and observations

The method oulined in the paper has been extensively tested for analytic functions for which we could obtain good closed form expressions for an upper bound to $|E_{n+1}(z_0)|$. The results obtained were very positive. The calculations have been carried out using **Mathematica(ver.4.0)**. We show the results obtained for the two analytic functions considered in section(2).

1. $g(z) = e^z - 1$
2. $g(z) = sin(z)$

Ser.No.	z_0	Bound
1(a)	0.00000010000000	$1.00000000 \times 10^{-7}$
1(b)	0.0000000000001000	$1.00003872 \times 10^{-13}$
2(a)	6.2831853071795888 ($2\pi + 0.0000000000234454$)	$2.34522341 \times 10^{-11}$
2(b)	6.2831853072795864 ($2\pi + 0.0000000001000000$)	$1.00089311 \times 10^{-10}$

5 Conclusions

Intersections of algebraic curves can be determined using the standard method of resultants of multivariate polynomials representing algebraic curves [26]; the zeros of a univariate polynomial derived from the resultant are used in computing the intersections. Errors in the computed zeros of a polynomial can be estimated using the method in [20, 21, 22] and then applied to computing the errors in the intersections of algebraic curves. In similar vein, we feel that posteriori error bounds on approximate zeros of analytic functions may be used in computations of intersections of curves and surfaces defined using analytic functions in scientific, engineering and computational mathematics applications. This requires to be investigated.

Error bounds on zeros can be used in algorithms for computing approximate zeros of analytic functions to desired degrees of accuracy; for instance, a Newton-Raphson based algorithm for computing approximate zeros may repeatedly compute error bounds in each iteration using our method to decide whether approximate zeros of desired accuracy have already been computed.

Error computing methods are of central importance in evaluating scientific computing software that use various non-linear methods for solving non-linear equalities or for finding feasible solutions for non-linear inequalities, when the expressions involve analytic functions that are not polynomials. We have illustrated such an example in Section (4).

An interesting problem is the study of the behaviour and performance of our error computing method for analytic functions with close zeros or dense clusters of zeros. Generalizing it to multivariate analytic functions may be another direction of further research.

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