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A dynamic model of patch differentiated exploitation of fisheries

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Abstract. We propose a discrete-time dynamic model for studying the time evolution of fish stocks in an aquatic environment divided into two patches with different harvesting policies. In one patch a central authority imposes a constant fishing effort, in the other profit maximizing fishermen engage in oligopolistic competition. The dynamics inside the patches are strongly interrelated by ecological factors, market interactions, and biomass diffusion. Particular cases are examined and we offer results on existence, stability, local bifurcations of the equilibria and numerical explorations of the global dynamical properties. The complete model with fishing activity in both regions is, mainly numerically, investigated.

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1 Introduction

In the last few decades, the mathematical modelling of managed natural populations has furnished encouraging results, as witnessed by the rapidly increasing amount of literature in the field of so called "Mathematical Bioeconomics" [after the book by Clark (1976 and 1990), see also Munro (1992), Conrad (1995)]. Even if these models are quite simple with respect to the complex systems they aim to mimic, and they generally give only qualitative and heuristic results, important lessons have been learned from the study of their properties. This has been particularly true in the study of renewable resources, especially for fisheries. Indeed, many of the world's marine and freshwater fisheries have been overexploited, and many fish stocks are depleted. This is a direct consequence of the so called "tragedy of the commons" after Hardin (1968) [see also Gordon (1954), Mesterton-Gibbons (1993)], that occurs in the unregulated exploitation of common property resources by competing individuals, societies, or countries. For these reasons many authors suggest that central institutions should impose severe forms of regulation by enforcing fishing restrictions.

Unfortunately, there are serious difficulties in implementing suitable regulation policies that are able to combine economically (and socially) efficient exploitation with issues of sustainable exploitation, due to nonlinear interactions

between ecologic, social, and economic components [see e.g. Rosser (2002)]. This makes the mathematical modelling of natural resources exploitation quite a challenging task. However, there is a general agreement that the use of mathematical models can help scientists and policy makers better understand some basic principles and to analyze the effects of fishery management policies, as well as making some qualitative and general forecasts about possible future scenarios.

In this paper we propose a discrete-time dynamic model for studying the time evolution of fish stocks in an aquatic environment divided into two adjacent zones, characterized by different fishing policies. Indeed, often legislation divides a fishery into regions such that different fishing rules are imposed in each of them. Often the division between neighboring regions is virtual, in the sense that no physical boundaries exist, and fish can move among regions so that the quantity of biomass in one region depends not only on harvesting and biological growth in that region, but also on the stock and catch in the neighboring regions [see e.g. Srinivasu (2005)]. The consequence is an interdependence which, at each time period, can increase the population of the zone where the fish density is smaller.

We will consider two harvesting methods which are in some way complementary: a) an imposed *constant fishing effort* for coastal waters, with a given number of fishing units, each of them employing a controlled technology and a constrained effort, and b) an *oligopolistic competition* for the deep sea, where a limited number of fishing units plan their harvesting on the basis of a free "rational" choice. In particular they try to maximize their profit in a classic Cournot competition framework [see Okuguchi (1998), Szidarovszky and Okuguchi (1998, 2000), Bischi, Kopel and Szidarovszky (2005), Bischi, Lamantia and Sbragia (2004)]. We recall that a subdivision of a fishing region into patches, with these two kinds of fishing policies, has recently been considered by Bischi and Lamantia (2005a). The division modelled in that paper is markedly different from the one considered here. In fact, Bischi and Lamantia (2005a) propose two independent growth functions, in the sense that the fish population of each region is assumed to interact only with individuals of the same region. Instead, following Srinivasu (2005) we assume that the growth function in each patch involves interaction with both subpopulations, so that if we consider the sum of the two dynamic equations, then the growth function of the whole undifferentiated population is obtained, without any influence of the artificial subdivision. Moreover, Bischi and Lamantia (2005a), as well as Srinivasu (2005) only consider the cases where one of the two patches is a no-take region, i.e. a marine reserve, whereas here we also consider the case of two regions where harvesting is allowed, even if with different fishing policies.

The aim of this paper is twofold. Firstly, we show how nonlinear dynamic modelling can be employed to represent the interaction between ecologic and economic issues involved in the problem of sustainable exploitation of renewable resources. Secondly, we describe how analytical and numerical methods can be combined to grasp some general information about the effects of different resource management policies.

The outline of the rest of the paper is as follows. In Section 2, we describe the model of natural growth and in Section 3 we derive the harvesting function for each patch. In Section 4, we analyze the bidimensional model, focusing on the two benchmark cases with the protected area before giving some insight on the complete model. Section 5 summarizes.

2 Logistic growth with patches

We now assume that the time evolution of a renewable natural resource (fish) follows a logistic growth function, with intrinsic growth rate α and carrying capacity K (with $\frac{1}{K} = \beta$). Hence, denoting by $z(t)$ the total quantity of biomass in the water basin, the unharvested resource grows according to the logistic equation

$$z(t+1) = z(t)(1 + \alpha - \beta z(t)). \tag{1}$$

Now we suppose that the water basin is virtually divided in two patches, labelled as patch 1 and patch 2, and we denote by X_1 and X_2 the quantity of biomass in the different regions. It is obviously $z = X_1 + X_2$. We can imagine that a central authority draws a virtual line on the water basin (sea or lake) so that it can impose a specific harvesting policy for agents operating in at least one of the two areas. Assuming that the resource can migrate between the different patches according to a linear diffusion mechanism, we can model the biological evolution of the resource in each patch by the following two dimensional dynamical system:

$$\begin{cases} X_1(t+1) = X_1(t) + X_1(t)\left[\alpha - \beta(X_1(t) + X_2(t))\right] - \sigma[X_1(t) - X_2(t)] \\ X_2(t+1) = X_2(t) + X_2(t)\left[\alpha - \beta(X_1(t) + X_2(t))\right] + \sigma[X_1(t) - X_2(t)] \end{cases} \tag{2}$$

$$= \begin{cases} X_1(t) + X_1(t) \cdot G(X_1(t), X_2(t)) - \sigma[X_1(t) - X_2(t)] \\ X_2(t) + X_2(t) \cdot G(X_1(t), X_2(t)) + \sigma[X_1(t) - X_2(t)] \end{cases}$$

having denoted by $G(X_1, X_2) = [\alpha - \beta(X_1 + X_2)]$. The main feature of this model is that it preserves the logistic growth of the total resource [see Srinivasu (2005)], i.e. we have

$$(X_1 + X_2)(t+1) = (X_1(t) + X_2(t))\left[1 + \alpha - \beta(X_1(t) + X_2(t))\right]$$

that is (1).

Let us now introduce the harvesting activity. The equation (1) describing the time evolution of the renewable resource becomes

$$z(t+1) = z(t)(1 + \alpha - \beta z(t)) - H(t), \tag{3}$$

where $H(t)$ can be any function specifying the harvesting exertion. Therefore, the model determining the time evolution of the natural resource with linear diffusion migration of resource between patches becomes

$$\begin{cases} X_1(t+1) = X_1(t) + X_1(t)\left[\alpha - \beta(X_1(t) + X_2(t))\right] - \sigma[X_1(t) - X_2(t)] - H_1(t) \\ X_2(t+1) = X_2(t) + X_2(t)\left[\alpha - \beta(X_1(t) + X_2(t))\right] + \sigma[X_1(t) - X_2(t)] - H_2(t) \end{cases} \tag{4}$$

where $H_1(t)$ and $H_2(t)$ are two non-negative valued functions such that $H_1(t) + H_2(t) = H(t)$, whose explicit specification is the main aim of the next section.

3 Different harvesting policies

In this section we analytically derive different harvesting functions for each patch. More precisely, $H_1(t)$ indicates the harvesting with an imposed constant fishing effort, that takes place at time t in patch 1, whereas $H_2(t)$ denotes the harvesting at time t from patch 2 by agents engaged in an oligopolistic competition.

3.1 Patch one: constant fishing effort

The idea of a constant fishing effort in the literature of mathematical bioeconomic modelling of the fisheries is very common, and probably constitutes the most employed model of renewable resources exploitation [see Gordon (1954), Clark (1990) and references therein]. Following this approach, we assume that a central authority imposes a constant total effort in the fishing activity in the coastal waters (referred to as patch 1 in the following). This can be justified by thinking that the authority's control is easier to implement in this part of the sea.

We denote by E the individual effort, by q the catchability rate and by n_1 the number of fishermen acting in region 1 and adopting this harvesting policy. Then, the harvesting policy of a *constant total effort* is represented by the linear function

$$H_1(X_1) = qEn_1X_1.$$

The time evolution of the fish stock X_1, when a constant total effort harvesting is applied, is therefore given by the one-dimensional map

$$X_1(t+1) = X_1(t)(1 + \alpha - qEn_1 - \beta X_1(t)) \tag{5}$$

which is conjugated to the standard logistic map [see Devaney (1987), Clark (1990)]. The assumption that only patch 1 exists, leading to (5), will be relaxed later on in the paper.

3.2 Patch two: oligopolistic competition

We also assume that there is a second area, for instance the deep sea (referred to as zone 2 in the following), where fishing activity is not regulated by a central authority. The idea is that the implementation of control in this area is either too expensive or impossible, since this region could be shared between different countries. For this reason, fishermen are free to adopt the harvesting policy they want. For the well-known externalities on the market side and on the natural resource side, fishermen engage in a *Cournot oligopolistic competition*, i.e. they decide their harvesting quantity from time to time by solving an individual profit maximization problem.

Let us assume that there are $N = n_1 + n_2 + n_3$ fishermen, with n_1 of them harvesting in region 1, n_2 in region 2 and n_3 in both regions. Here we assume the existence of a unique market for the resource, so that all fishermen sell the harvesting at a price p determined by the linear inverse demand function

$$p(t) = a - bH(t) \quad \text{with} \quad a > 0, \quad b > 0, \tag{6}$$

where $H(t) = H_1(X_1(t), X_2(t)) + H_2(X_1(t), X_2(t))$ is the total fish which is harvested and supplied in the market, and a and b are constants representing, respectively, the maximum selling price and the (opposite of the) slope of the inverse demand function. Following Clark (1990) [(see also Szidarovszky and Okuguchi (1998)], the cost function of fisherman i that chooses to fish in zone 2 for harvesting a quantity x_i when the fish stock is X_2 is given by

$$C_i(x_i, X_2) = c_i + \gamma \frac{{x_i}^2}{X_2}, \tag{7}$$

where γ is the technological parameter and c_i is a fixed cost that includes the effort for solving the profit maximization problem and the expenditures for going far from the coast.

Assuming that fishermen are homogeneous and have perfect knowledge of the information of the game, we can write the profit of fisherman i working in zone 2 as

$$\pi_i(t) = x_i(t)\left(a - bH(t)\right) - \gamma \frac{x_i(t)^2}{X_2(t)} - c_i,$$

where $H(t) = qE(N - n_2)X_1(t) + (N - n_1)x_i(t)$. The optimization problem for agent i is then $max_{x_i}\pi_i$, and leads to the following first order condition, assuming an interior maximizer solution[1]:

$$\frac{\partial \pi_i}{\partial x_i} = a - bqE(N - n_2)X_1 - 2b(N - n_1)x_i - \frac{2\gamma x_i}{X_2} = 0.$$

The optimal harvesting at time t for a representative agent in zone 2 is then given by[2]

$$x^*(t) = \frac{X_2(t)(a - bqE(N - n_2)X_1(t))}{2(b(N - n_1)X_2(t) + \gamma)}$$

with total harvesting in patch 2 given by $H_2(X_1, X_2) = (N - n_1)x^*$.

All in all, the time evolution of the fish stock X_2 when only zone 2 exists so that $n_1 = 0$, according to the previous discussions, is given by the one-

[1] As is shown later in this paragraph, the profit at the maximum is strictly positive, provided that the fixed costs c_i are sufficiently small.

[2] From the point of view of the model it is important to stress that the total harvesting in region 2 is function of the total biomass in both patches, hence we here postulate perfect knowledge on total biomass for agents operating in the second region. We can think of c_i as including the extra cost for learning the level of overall resource.

dimensional map[3]

$$X_2(t+1) = X_2(t)(1+\alpha - \frac{Na}{2(bNX_2(t)+\gamma)} - \beta X_2(t)). \tag{8}$$

The total fish landed at time t, when harvesting in both regions occurs, is then

$$H(t) = qE(N-n_2)X_1(t) + \frac{(N-n_1)X_2(t)(a-bqE(N-n_2)X_1(t))}{2\left(b(N-n_1)X_2(t)+\gamma\right)}$$

whence the corresponding profit of a representative fisherman working in zone 1, denoted as π_1, is given by

$$\begin{aligned}\pi_1(t) &= qEX_1(t)(a-bH(t)) - \delta E \\ &= qEX_1(t)\frac{(a-bqE(N-n_2)X_1(t))(b(N-n_1)X_2(t)+2\gamma)}{2(b(N-n_1)X_2+\gamma)} - \delta E,\end{aligned}$$

where δ is the unitary cost of effort, that we assume linear.

Analogously, the profit of a representative fisherman working in zone 2, π_2, is given by

$$\begin{aligned}\pi_2(t) &= x^*(a-bH(t)) - c_i - \gamma\frac{x^*(t)^2}{X_2(t)} \\ &= \frac{X_2(a-bqE(N-n_2)X_1)^2}{4(b(N-n_1)X_2+\gamma)} - c_i\end{aligned}$$

whose positivity depends on the amount of fixed costs c_i. Since in the general case $n_1+n_3 = N-n_2$ and $n_2+n_3 = N-n_1$, without loss of generality we explicitly assume in the rest of the paper that no agent harvests in both regions, i.e. $n_3 = 0$.

4 The Model

The two dimensional map that describes the time evolution of the natural resource corresponding to (4) when a number n_1 of fishermen harvest in region 1 and a number n_2 harvest in region 2 is

$$\begin{cases} X_1(t+1) = \max[0,\ X_1(t) + X_1(t)\left[\alpha - qEn_1 - \beta(X_1(t)+X_2(t))\right] \\ \quad -\sigma\left[X_1(t) - X_2(t)\right]] \\ X_2(t+1) = \max[0,\ X_2(t) + X_2(t)\left[\alpha - \frac{n_2(a-bqEn_1X_1(t))}{2[bn_2X_2(t)+\gamma]} - \beta(X_1(t)+X_2(t))\right] \\ \quad +\sigma[X_1(t) - X_2(t)]] \end{cases} \tag{9}$$

[3]Differently from (1), the map (8) is unimodal only if condition $1+\alpha > \frac{aN}{2\gamma}$ is verified, i.e. if the slope at the origin of (8) is positive. It is useful to observe for subsequent discussions that the origin is stable iff $\alpha < \frac{aN}{2\gamma} < 2+\alpha$, i.e. a range of stability of the extinction equilibrium exists.

Note that, in defining the system (9) we set to zero negative biomass levels X_i, $i = 1, 2$. In the following discussion we usually omit the *max* operator, since we primarily state results that are valid in the open set of strictly positive biomasses. However, we explicitly consider the definition in (9) for the numerical simulations. Moreover, when one biomass level drops below zero in one patch, then the evolution of the fish stock in the other patch is obtained from an unidimensional map of the type (5) or (8) by subtracting a (linear) output term of biomass going to the depleted area. That is, through the diffusion mechanism one patch can be thought as a restocking area for the exhausted one.

4.1 First benchmark case

A particular benchmark case is obtained by assuming that no fishing activity takes place in region 2, i.e. setting $n_2 = 0$ in (9). This particular case is important and it corresponds to the presence of a no-take zone in that region. As well as the origin $(0, 0)$, which is clearly an equilibrium point of (9) for all parameters values, it is also possible to calculate analytically the steady states when we set $n_2 = 0$ in (9) by solving a third degree algebraic equation. However, the expressions of the solutions are quite complicated and do not provide useful information in practise. Therefore, we try to locate the position of the unique strictly positive steady state in the plane (X_1, X_2) as stated in the following

PROPOSITION 1 *In the case of constant effort in patch 1 ($qEn_1 \geq 0$) and protected area in patch 2 ($n_2 = 0$), a strictly positive equilibrium (X_1^*, X_2^*) exists for all parameters values. Moreover*

- *if $\alpha > 2\sigma + qEn_1$, then $X_1^* \in \left(\frac{\sigma}{\beta}, \frac{\alpha - qEn_1 - \sigma}{\beta}\right)$ and $X_2^* \in \left(\frac{\sigma}{\beta}, \frac{\alpha - \sigma}{\beta}\right)$;*
- *if $2\sigma + qEn_1 > \alpha > 2\sigma$, then $X_1^* \in \left(\frac{\alpha - qEn_1 - \sigma}{\beta}, \frac{\sigma}{\beta}\right)$ and $X_2^* \in \left(\frac{\sigma}{\beta}, \frac{\alpha - \sigma}{\beta}\right)$;*
- *if $\alpha < 2\sigma$, then $X_1^* \in \left(\frac{\alpha - qEn_1 - \sigma}{\beta}, \frac{\sigma}{\beta}\right)$ and $X_2^* \in \left(\frac{\alpha - \sigma}{\beta}, \frac{\sigma}{\beta}\right)$.*

Proof. Setting $X_1(t+1) = X_1(t) = X_1$ and $X_2(t+1) = X_2(t) = X_2$ in (9) we get the following nonlinear system of equations

$$\begin{cases} X_1 \left[\alpha - \beta(X_1 + X_2)\right] - \sigma[X_1 - X_2] - qEn_1 X_1 = 0 \\ X_2 \left[\alpha - \beta(X_1 + X_2)\right] + \sigma[X_1 - X_2] - \dfrac{n_2 X_2 (a - bqEn_1 X_1)}{2\left[bn_2 X_2 + \gamma\right]} = 0 \end{cases}$$

which are two hyperbolas when $n_2 = 0$. In the plane (X_1, X_2), the two equations can be written as

$$X_2 = f_2(X_1) = \frac{X_1(\beta X_1 + qEn_1 + \sigma - \alpha)}{\sigma - \beta X_1}$$

and

$$X_1 = f_1(X_2) = \frac{X_2\left(\beta X_2 + \sigma - \alpha\right)}{\sigma - \beta X_2}$$

again when $n_2 = 0$, i.e. when no fishing occurs in the second patch. An equilibrium point (X_1^*, X_2^*) in the plane (X_1, X_2) is provided by any intersection between $f_1(.)$ and $f_2(.)$. Clearly $f_1(0) = 0$, $\lim_{X_2 \to \frac{\sigma}{\beta}} f_1(X_2) = \infty$, $\lim_{X_2 \to \infty} f_1(X_2) = -\infty$, $f_1'(X_2) = \frac{-(X_2\beta)^2 + 2X_2\beta\sigma + \sigma(\sigma-\alpha)}{(\sigma - X_2\beta)^2}$, we have that at the origin $f_1(X_2)$ is strictly decreasing $f_1'(0) = 1 - \frac{\alpha}{\sigma} < 0$ so that $f_1(X_2)$ is strictly decreasing when $\alpha > 2\sigma$. Otherwise when $\alpha \leq 2\sigma$, $f_1(X_2)$ is strictly increasing in the interval $\left(\frac{\sigma - \sqrt{\sigma(2\sigma-\alpha)}}{\beta}, \frac{\sigma + \sqrt{\sigma(2\sigma-\alpha)}}{\beta}\right)$, with $f_1(\frac{\sigma + \sqrt{\sigma(2\sigma-\alpha)}}{\beta}) < 0$. A similar analysis can be applied to the function $f_2(.)$. The proposition then easily follows from the signs and intervals of monotonicity of $f_1(.)$ and $f_2(.)$. Note that it is possible to define smaller intervals containing the fixed point, e.g. when $\alpha > 2\sigma + qEn_1$ then $X_1^* \in \left(\frac{\sigma}{\beta}, f_2^{-1}\left(\frac{\sigma}{\beta}\right)\right)$ and $X_2^* \in \left(\frac{\sigma}{\beta}, f_1^{-1}\left(\frac{\sigma}{\beta}\right)\right)$, and similarly for the remaining cases. □

As we are interested in the time evolution of the resource, we will study the stability properties of the two equilibria.

The Jacobian matrix of (9) with $n_2 = 0$ is given by

$$J(X_1, X_2) = \begin{bmatrix} 1 + \alpha - \beta(2X_1 + X_2) - \sigma - qEn_1 & -\beta X_1 + \sigma \\ -\beta X_2 + \sigma & 1 + \alpha - \beta(2X_2 + X_1) - \sigma \end{bmatrix} \tag{10}$$

PROPOSITION 2 *In the case of constant effort in patch 1 ($qEn_1 \geq 0$) and protected area in patch 2 ($n_2 = 0$), the extinction equilibrium $(0, 0)$ is unstable for all parameter values.*

Proof. We recall that a fixed point is stable if the following conditions hold

$$\begin{cases} 1 - \text{Tr} + \text{Det} > 0 \\ 1 + \text{Tr} + \text{Det} > 0 \\ 1 - \text{Det} > 0 \end{cases}, \tag{11}$$

where Tr and Det denote, respectively, the trace and the determinant of the Jacobian (10) at the fixed point $(0, 0)$. Let us assume that there is a set of parameters such that the origin is stable, then, from (11), the three following conditions should be verified

$$\begin{cases} -qEn_1(\alpha - \sigma) + \alpha(\alpha - 2\sigma) > 0 \\ -qEn_1(2 + \alpha - \sigma) + (2 + \alpha)(2 + \alpha - 2\sigma) > 0 \\ qEn_1(1 + \alpha - \sigma) - 2(\alpha - \sigma) - \alpha(\alpha - 2\sigma) > 0 \end{cases}.$$

From the first and the third inequalities we obtain that

$$\frac{2(\alpha - \sigma) + \alpha(\alpha - 2\sigma)}{1 + \alpha - \sigma} < qEn_1 < \frac{\alpha(\alpha - 2\sigma)}{\alpha - \sigma}$$

which leads to a contradiction, as easily verifiable. □

Proposition 2 is a very important in terms of the coherence of the model. If the biomass in the exploited patch drops below a given level, we have a guarantee that there is a regeneration to a positive level of biomass by endogenous forces. Moreover, since a no-take zone is present, we always have a minimum level of resource for avoiding the extinction. Moreover, an important insight on the kind of stability of the positive equilibrium is provided in the following

PROPOSITION 3 *In the model* (9) *with* $n_2 = 0$*, the positive equilibrium* (X_1^*, X_2^*) *can loose stability only through flip bifurcations.*

Proof. When no-harvesting takes place ($qEn_1 = 0$), the positive equilibrium corresponds to the carrying capacity level α/β, as in the standard analysis of the logistic case (1), i.e. $(X_1^*, X_2^*) = \left(\frac{\alpha}{2\beta}, \frac{\alpha}{2\beta}\right)$. Since an eigenvalue is given by $\lambda_1 = 1 - \alpha$ the destabilization of the positive fixed point starts here at $\alpha = 2$, through the well known flip bifurcation scenario. The second eigenvalues plays no role in the stability, being $\lambda_2 = 1 - 2\sigma$. In fact a flip bifurcation occurs at $\sigma = 1$, but that should be considered an unrealistic assumption on this parameter so we rule out this case. In general, the stability of the positive equilibrium can be investigated in terms of standard local analysis, i.e. valuing the magnitude of the eigenvalues of the Jacobian matrix at the equilibrium. In this case the eigenvalues are given by

$$\lambda_1 = 1 - \frac{1}{2}\left[qEn_1 - 2(\alpha - \sigma) + 3A\beta + \sqrt{(qEn_1)^2 + 2qEn_1B\beta + (A\beta - 2\sigma)^2}\right]$$
$$\lambda_2 = 1 - \frac{1}{2}\left[qEn_1 - 2(\alpha - \sigma) + 3A\beta - \sqrt{(qEn_1)^2 + 2qEn_1B\beta + (A\beta - 2\sigma)^2}\right],$$

where $A = X_1^* + X_2^*$ and $B = X_1^* - X_2^*$. It is possible to show that they are always real and furthermore, by elementary manipulations, that $\lambda_1 < \lambda_2 < 1$. Hence Neimark–Sacker and/or saddle bifurcations are definitely ruled out when patch 2 is a no-take region. Moreover it is possible to explicitly calculate that, for $\sigma \in [0, 1]$, it is $\lambda_1 = -1$ for a level of effort $qEn_1 = 2\sqrt{1 - \sigma^2}$. Furthermore, provided that $\alpha \in (2, 2 + \sigma) \cup (2 + 2\sigma, +\infty)$, we again have a positive level of effort, namely $qEn_1 = \frac{(\alpha-2)(\alpha-2(1+\sigma))}{\alpha-2-\sigma}$ such that $\lambda_1 = -1$. □

As an illustration of Proposition 1, we depict the bifurcation diagram in Figure 1a, obtained with $n_1 = 10$, $q = 1$, $\alpha = 2.3, \beta = 1, \sigma = 0.4$, and with an effort $E \in (0, 0.2)$. This figure shows that the positive equilibrium becomes stable through a backward flip bifurcation at $E_1 = \frac{(\alpha-2)(\alpha-2(1+\sigma))}{(\alpha-2-\sigma)qn_1} = 0.15$ and it looses stability, again through a flip bifurcation, at $E_2 = \frac{2}{qn_1}\sqrt{1 - \sigma^2} \approx 0.183303$. Then the well-known period-doubling cascade occurs and chaotic dynamics are broken down by periodic windows.

It is interesting to observe that as the effort level increases, the quantity of biomass in the first patch decreases and this has also a negative effect on

the biomass in patch 2, since there is less diffusion from patch 1 to patch 2, regulated by the parameter σ. However, this effect could be outbalanced by the fact that there is less mortality in the population, due to a lower level of the biomass in the first patch, as regulated by the parameter β. Since the latter effect is quadratic it is well possible that a decrement (on average) of biomass in the first patch leads to an increment (on average) in the second area.

In order to better understand this feature, we show in Figure 1b the same bifurcation diagram as in figure 1a but with a higher value of the diffusion parameter, i.e. $\sigma = 0.65$. For both regions the diagram is shifted towards the left, that is the backward and forward period doubling bifurcations and the period-doubling cascade occur for lower values of the effort. Moreover, the first cycle 2 at the very beginning becomes shorter, that is the backward flip bifurcation occurs at $E_1 \approx 0.0857 < 0.15$ and the set of effort values for which a fixed point exists is bigger. In region 1 the resource stocks are all shifted to higher levels. In region 2, on the other hand, they are all moved to lower levels.

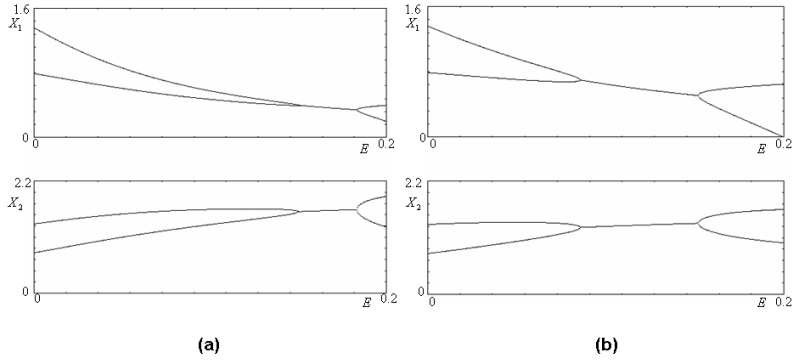

Figure 1: Benchmark case with constant fishing effort and reserve area. (a) Bifurcation diagram with biological parameters $\alpha = 2.3$, $\beta = 1$, $\sigma = 0.4$ and $q = 1$, $n_1 = 10$ and effort $E \in [0, 0.2]$. The positive equilibrium is stable after the backward flip bifurcation at $E_1 = 015$ and the flip bifurcation at $E_2 \approx 0.1833$. (b) Bifurcation diagram with an higher diffusion $\sigma = 0.65$. In this case, the positive equilibrium becomes stable at $E_1' \approx 0.08571 < E_1$ and looses stability at $E_2' \approx 0.15198 < E_2$.

4.2 Second benchmark and the complete case

The second benchmark case is given when region 1 is a protected area with no fishing activity, and in region 2 there is a free harvesting so that each fisherman decides his catch by maximizing his expected profit. In this case, the time evolution of the natural resource is obtained by (9) with $n_1 = 0$. Again, not only the extinction equilibrium $(0, 0)$ is an equilibrium point, but it is straightforward

to state that this equilibrium is unstable under all set of parameters, since the study is analogous to that of proposition 2. It is useful to remark that the presence of the reserve region in patch 1 is responsible for this result, since as we showed before, the unidimensional model without reserve can lead to situations where the extinction is a stable steady state, according to the parameters values.

To understand the dynamical properties of this benchmark case we start looking for the steady states. As an analytical computation of the steady states is quite involving, we will compute them numerically both in the second benchmark case and in the complete case. In Figure 2a there is the bifurcation diagram obtained with the same set of biological parameters as in Figure 1a. The market parameters are $a = 2.2$, $b = 0.45$ and the cost parameter is $\gamma = 2$. The number of fishermen n_2 is allowed to vary in the interval $[0, 7]$. The bifurcation diagram shows that the positive equilibrium becomes stable through a backward flip bifurcation as before, but it looses stability through a Neimark-Sacker bifurcation, with a quasi-periodic orbit that becomes wider as the number of fishermen increases. This fact constitutes a peculiar difference with respect to the first benchmark, where only flips occur. An example in the space of states is depicted in Figure 3, with $n_2 = 6.7$, where the quasiperiodic orbit surrounds the only positive unstable fixed point.

Like before, we observe that a decrement (on average) of biomass in the second patch leads to an increment (on average) in the first area. This time the more intensive fishing activity in the second patch is due to the increasing number of fishermen determining a reduction of the biomass level and the mortality of the resource. In general, we observed that an increase of the sensitivity b of the demand function, and/or a decrease of the maximum price a, and/or an increase of the coefficient cost γ make the positive steady state stable for a greater number of fishermen. If we increase the value of the diffusion parameter, as in Figure 2b where $\sigma = 0.65$, we observe that the bifurcation diagram of the first patch moves down to lower levels of the biomass, instead the bifurcation diagram of the patch with oligopolistic competition moves up to higher levels of the resource. Moreover, the positive equilibrium is stable for a greater number of fishermen, in the sense that the backward bifurcation occurs before and the Neimark-Sacker bifurcation occurs later than the scenario of Figure 2a.

We now move to the study of the dynamical properties of complete case, that is the one where we have different harvesting policies in the two patches. As mentioned before, we rely on numerical simulations to study the existence and the stability of the equilibria. In Figure 4a, a bifurcation diagram is depicted with parameters $\alpha = 2.3$, $\beta = 1$, $\sigma = 0.75$, $n_1 = 5$, $E = 0.33$, $q = 1$, $a = 2.2$, $b = 0.45$, $\gamma = 2$, and $n_2 \in [0, 5]$. Like before, as the number of fishermen in the second patch increases, a positive steady state becomes stable through a backward flip bifurcation even if it moves towards smaller biomass levels. In the complete case it is also possible to show numerically that the uniqueness of the positive stable state is not always guaranteed.

Furthermore, differently from the two benchmark cases, in the complete case the origin can become stable for certain values of the parameters. The precise result is summarized in the following

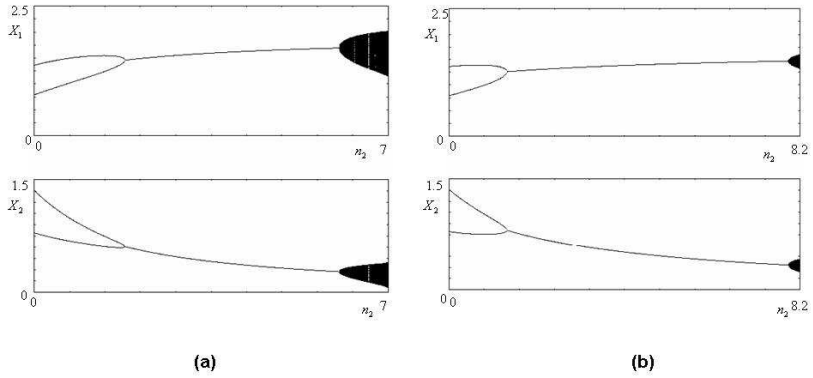

Figure 2: Benchmark case with fishermen engaged in an oligopolistic competition and reserve area. (a) Bifurcation diagram with biological parameters $\alpha = 2.3$, $\beta = 1$, $\sigma = 0.4$, market parameters $a = 2.2$, $b = 0.45$ and cost parameter $\gamma = 2$. The number of fishermen n_2 varies in the interval $[0, 7]$. In this case the positive equilibrium becomes stable through a backward flip bifurcation and it looses stability through a Neimark-Sacker bifurcation. (b) Bifurcation diagram for a more intensive diffusion $\sigma = 0.65$. In this case the positive equilibrium is stable for a higher number of fishermen.

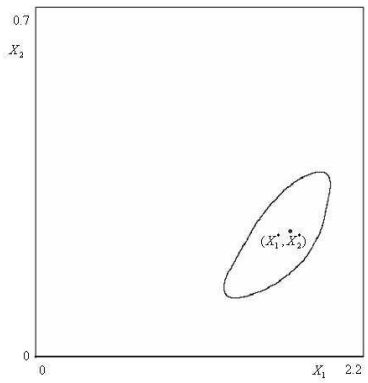

Figure 3: Benchmark case with fishermen engaged in an oligopolistic competition and reserve area. Numerical representation of the attractor and its basin of attraction. The set of parameters used is $\alpha = 2.3$, $\beta = 1$, $\sigma = 0.4$, $a = 2.2$, $b = 0.45$, $\gamma = 2$ and $n_2 = 6.7$. The positive steady state is unstable and the trajectories converge to a quasi-periodic orbit generated through a Neimark-Sacker bifurcation.

PROPOSITION 4 *In the model* (9) *with* n_1, $n_2 \neq 0$, *the extinction equilibrium* $(0, 0)$ *can loose stability through saddle or flip bifurcations.*

Proof. The assertion of the proposition follows from the standard conditions (11). In particular, due to the symmetry of the Jacobian matrix at the origin, its eigenvalues are always real and so it is $\mathrm{Tr}^2 > 4\,\mathrm{Det}$. Hence a sufficient condition for stability is that the first two conditions in (11) hold together with the condition that $-2 < \mathrm{Tr} < 2$. Summarizing, we get the following sufficient conditions for the stability of the origin:

$$2 + \alpha - qEn_1 + \frac{(qEn_1 - \alpha)^2}{qEn_1 - \alpha + \sigma} < \frac{an_2}{2\gamma} < \tag{12}$$
$$< \frac{qEn_1(2 + \alpha - \sigma) - (2 + \alpha)(2 + \alpha - 2\sigma)}{qEn_1 - 2 - \alpha + \sigma}$$
$$1 + \alpha - \sigma - \sqrt{1 - \sigma^2} < qEn_1 < 1 + \alpha - \sigma + \sqrt{1 - \sigma^2}.$$

Note that this result follows from a standard linearization of the map at the fixed point, so we did not deal with the max operator in the definition of (9). However, the result of the proposition is still valid in the intersection between the basin of the origin and the positive quadrant. □

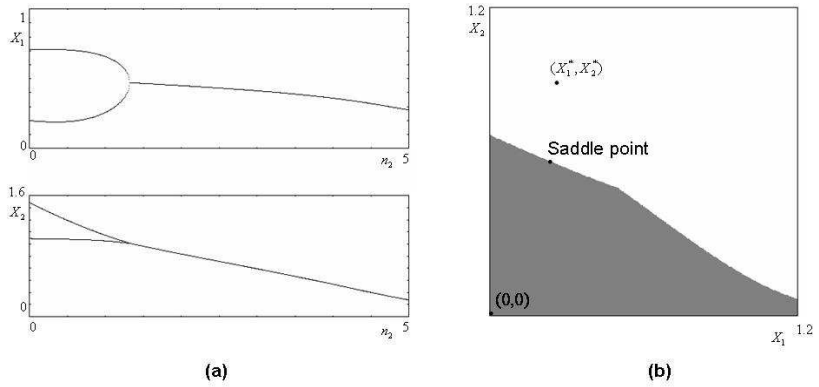

Figure 4: Model with two different fishing policies: in zone 1 fishermen harvest according to a constant fishing effort and in zone 2 fishermen are engaged in an oligopolistic competition. (a) Bifurcation diagram with parameters $\alpha = 2.3$, $\beta = 1$, $\sigma = 0.75$, $n_1 = 5$, $E = 0.33, q = 1$, $a = 2.2$, $b = 0.45$, $\gamma = 2$ and n_2 varying in the interval $[0, 5]$. The positive steady state becomes stable through a backward flip bifurcation. (b) Numerical representation of the attractors and their basins. The set of parameters used is $\alpha = 2.3$, $\beta = 1$, $\sigma = 0.45$, $n_1 = 10$, $E = 0.23$, $n_2 = 4$, $a = 3.5$, $b = 1.5$, $\gamma = 1$. Attractor are the extinction of the resource and the positive steady state $(X_1^*, X_2^*) = (0.25349, 0.90744)$. The white area is the basin of attraction of the extinction and the grey area is the basin of attraction of the positive steady state. The stable manifold of the saddle (X_1^s, X_2^s) separates the two regions.

The qualitative result given by proposition 4 differs strongly from the two benchmark cases, where the origin is always unstable. It is interesting to observe

that even if negative trajectories are ruled out by the map (9), exploitation in both patches can lead to extinction of the biomass in *infinite time* as stated by Clark (1990). This is confirmed by our numerical simulations, as with the parameter constellation $\alpha = 2.3$, $\beta = 1$, $\sigma = 0.45$, $n_1 = 10$, $E = 0.23, q = 1$, $a = 3.5$, $b = 1.5$, $\gamma = 1$, $n_2 = 2$, where the origin is the only stable fixed point of the map, as predicted by conditions (12). By increasing the number of agents in the second patch, we obtain the situation depicted in Figure 4b, where $n_2 = 4$. In this case, there is a stable steady state of coordinates $(X_1^*, X_2^*) \approx (0.25349, 0.90744)$, whose basin of attraction is represented by the grey area in Figure 4b with border given by the stable set of the saddle point $(X_1^s, X_2^s) \approx (0.21521, 0.60974)$. The white area is the basin of attraction of the extinction. In this case, conditions (12) are violated, but the generic trajectory with initial condition in the white area is prevented from diverging to minus infinity by the max operator in (9). In this case we could say that extinction of the biomass takes place in *finite time*. Even if we cast some light on the complete model, a thorough study of the global properties of the map is under investigation and will be developed in a further paper (see Mira et al. (1996) for the mathematical background). For instance, we are examining whether weak attractor in Milnor sense, such as riddled basins, can arise in our models (see Bischi and Lamantia (2005b) and references therein for an introductory survey with also applications in economics).

5 Conclusions

In this paper a discrete time nonlinear dynamic model has been proposed for the description of a commercial fishery where a central institution introduces an artificial subdivision of a common pool in two adjacent zones where different fishery management policies are adopted. Models of aquatic systems characterized by several compartments (or patches) have been recently proposed for the description of interactions between open access fisheries and marine protected areas, see e.g. Sumaila and Charles (2002), and references therein, and Bischi and Lamantia (2005a). Here we assumed that fish can freely move across the boundary separating the two patches, and we considered two particular kinds of fishing policies: the case of a regulated fishery according to constant effort, and the case where profit maximizing agents compete by engaging in an oligopoly game á la Cournot.

Some general results have been analytically proved. In particular, for different kinds of harvesting policies, results on the existence of the equilibria and their stability properties have been proved, as well as the different kinds of local bifurcations that may cause stability loss. Numerical simulations have also been given to show the different dynamic behaviors of the model for some parameters' constellations. In fact, a global analysis of the dynamic properties of the model, as well as a general overview of the influence of the parameters, requires an interplay between analytic and numerical methods. Indeed, the latter is the typical "modus operandi" in the study of both bifurcations and global dynam-

ics for nonlinear discrete dynamical systems of dimension greater (see Mira et al. (1996), Guckenheimer (1980), Brock and Hommes (1997), for interesting comments on this point). In fact, a direct numerical approach without any preliminary analytic study is generally hopeless task, however a study based only on formal analytic proofs very rarely can give a sufficiently clear understanding of the global properties and implications of the nonlinear dynamical system at hand.

The model proposed in this paper is only a first step towards a more complete modelling of interactions between different fishery regulations. In fact, many different assumptions can be made on specific growth functions, and other different harvesting policies may be introduced. An important extension may be obtained by considering an endogenous mechanism that allows fishermen to decide the region where they prefer to operate, according to some decision making method based on available information on profits or other indexes of performance. This means that the number of agents n_1 and n_2 that decide to harvest in region 1 and 2 respectively, are not given as parameters, but they are considered as dynamic variables governed by a dynamic equation that describes a switching criterion. For example, a replicator dynamics may be proposed to regulate the dynamics of n_1 and n_2 [see Taylor and Jonker (1978), Vega-Redondo (1996) and Bischi, Lamantia and Sbragia (2004)].

Another possible extension can be obtained by weakening the assumption on agents' perfect foresight on the available fish stock in the computation of the Nash equilibrium in the oligopoly competition, for example by introducing adaptive expectations as in Bischi, Kopel and Szidarovszky (2005). Of course, these extensions will cause an increase in the dimension of the dynamical system, because they require the addition of a dynamic variable (and consequently a dynamic equation) so that the difficulties of obtaining analytic results will become more evident. However, these extensions will be considered in a different paper.

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