Available online at www.sciencedirect.com
SCIENCE DIRECT
Chaos, Solitons and Fractals 29 (2006) 707–722
CHAOS SOLITONS & FRACTALS
www.elsevier.com/locate/chaos

Oligopoly games with nonlinear demand and cost functions: Two boundedly rational adjustment processes

Ahmad K. Naimzada [a], Lucia Sbragia [b,*]

[a] *Department of Economics, University of Milan, Bicocca, Italy*
[b] *Department of Economics, University of Urbino, Italy*

Abstract

We consider a Cournot oligopoly game, where firms produce an homogenous good and the demand and cost functions are nonlinear. These features make the classical best reply solution difficult to be obtained, even if players have full information about their environment. We propose two different kinds of repeated games based on a lower degree of rationality of the firms, on a reduced information set and reduced computational capabilities. The first adjustment mechanism is called "Local Monopolistic Approximation" (LMA). First firms get the correct local estimate of the demand function and then they use such estimate in a linear approximation of the demand function where the effects of the competitors' outputs are ignored. On the basis of this subjective demand function they solve their profit maximization problem. By using the second adjustment process, that belongs to a class of adaptive mechanisms known in the literature as "Gradient Dynamics" (GD), firms do not solve any optimization problem, but they adjust their production in the direction indicated by their (correct) estimate of the marginal profit. Both these repeated games may converge to a Cournot–Nash equilibrium, i.e. to the equilibrium of the best reply dynamics. We compare the properties of the two different dynamical systems that describe the time evolution of the oligopoly games under the two adjustment mechanisms, and we analyze the conditions that lead to non-convergence and complex dynamic behaviors. The paper extends the results of other authors that consider similar adjustment processes assuming linear cost functions or linear demand functions.

1. Introduction

The classical oligopoly games, and the associated notion of Nash equilibrium, are based on quite demanding notion of rationality that includes assumptions on the available information set, on the firms' capability of extracting correct estimates from it and on the computational skills required to solve the optimization problems through which firms make their decisions. Many authors claim that such assumptions are too strong and that real producers are not so rational when they make their decisions. Moreover, as clearly stated by [3], the more refined the decision-making

* Corresponding author.
E-mail addresses: ahmad.naimzada@unimib.it (A.K. Naimzada), l.sbragia@uniurb.it (L. Sbragia).

0960-0779/$ - see front matter
doi:10.1016/j.chaos.2005.08.103

process, the more expensive it is likely to be. Therefore, especially when a (single) decision is not of crucial importance, no more than an approximate solution may be justified. They call "optimally imperfect decisions" the decisions such that "The calculation of the appropriate decision is simple, inexpensive, and well suited for frequent repetition". This point of view is also shared by other authors, see [30,11,12].

In this paper we consider an industry where n firms, indexed by $i = 1,\dots,n$, produce an homogeneous good with production levels q_i, $i = 1,\dots,n$, respectively. Strategic interactions arise because the price of the good depends on the total output of the industry according to a given inverse demand function

$$p = f(Q), \tag{1}$$

where $Q = \sum_{i=1}^{n} q_i$ is the total output of the oligopoly market. If $C_i(q_i)$ denotes the cost function of producer i, then his profit at time period t is

$$\pi_i(t) = p(t)q_i(t) - C_i(q_i(t)). \tag{2}$$

We assume that the market is characterized by the following nonlinear inverse demand function

$$p = a - b\sqrt{Q}, \tag{3}$$

which has been used in other oligopoly models and in the experimental economics dealing with learning and expectations formation (see e.g. [26,2]). The cost functions are also nonlinear, that is

$$C_i = c_{i0} + c_{i1}q_i + c_{i2}q_i^2, \quad c_{ik} \geqslant 0, \quad k = 0,1,2. \tag{4}$$

Indeed, quadratic cost functions are often met in applications. For example, in the modeling of renewable resources exploitation, such as fisheries, cost functions without the linear term (i.e. functions of the form (4) with $c_{i2} > 0$ and $c_{i1} = 0$) are usually considered.[1] Negishi [25] considers cost functions of the form (4) and investigates the role of convexity, i.e. the role of the coefficient of the term of second degree.

With these demand and cost functions the profit of firm i becomes

$$\pi_i(t) = q_i(t)(a - b\sqrt{Q}) - (c_{i0} + c_{i1}q_i(t) + c_{i2}q_i^2(t)). \tag{5}$$

In a classical Cournot oligopoly game, producers are assumed to be price takers and, at each time t, they decide their production levels by maximizing the expected profit

$$q_i(t+1) = \arg\max_{q_i} \pi_i^e(t+1) = \arg\max_{q_i}[p^e(t+1)q_i - C_i(q_i)], \tag{6}$$

where the expected price $p^e(t+1) = f(q_i, q_{-i}^e(t+1))$ and $q_{-i}^e(t+1)$ represents the output decisions of the other players as expected by player i. Moreover, the inverse demand function is assumed to be known by all firms. Cournot [14] assumes *naive expectations*, i.e. $q_{-i}^e(t+1) = q_{-i}(t)$, that is, each firm expects that the production of the other firms will remain the same as in the current period.[2] This implies that the condition for solving the optimization problem (6) gives implicit relations between productions at time t and those at time $t+1$. In the simplest (and lucky) case one can uniquely express $q_i(t+1)$ as functions of $q_{-i}(t)$

$$q_i(t+1) = r_i(q_{-i}(t+1)), \tag{7}$$

where r_i are called *reaction functions*, and (7) gives rise to a discrete-time dynamical system (*Best Reply Dynamics*). Cournot [14] studied the properties of the best reply dynamics when both the demand and the cost functions are linear getting also linear reaction functions. Other examples of explicit computation of the best reply dynamics are given in the literature, see e.g. [27], where a nonlinear demand function is considered together with linear cost functions, [21,7], where nonlinear cost functions are considered together with a linear demand function. In all these cases, the Nash equilibria correspond to the fixed points of the map (7), i.e. they are located at the intersections of the reaction curves.

In our case, considering the demand function (3) and the cost functions (4), the first order conditions for the optimization problem (6) become

$$\frac{\partial \pi_i^e}{\partial q_i} = (a - b\sqrt{Q}) - \frac{bq_i}{2\sqrt{Q}} - c_{i1} - 2c_{i2}q_i = 0 \quad i = 1,\dots n.$$

[1] In fisheries the most frequently used cost function is given by $C(x) = \gamma\frac{x^2}{X}$ where x is the quantity of fish harvested (production) when a fish stock X is available. This cost function can be derived from a Cobb–Douglas-type "production function" with fishing effort (labor) and fish biomass (capital) as production inputs (see [13,32,10]).

[2] Other kinds of expectations mechanisms can be used, such as adaptive expectations, see e.g. [31,8].

In order to get the best reply solution, each of these irrational equation should be solved with respect to q_i, that means to solve equations of third degree after rationalization.

In other words, in this case, the best reply solution requires not only a very demanding information set, but also a considerable computational effort of the producers. In the spirit of [3], we will consider two adjustment mechanisms characterized by lower degrees of rationality and information, such that, in the long run (i.e. after several repetitions of the game) the obtained dynamic processes may converge to the same equilibria (i.e. Nash equilibria) of the best reply dynamics. These two mechanisms have already been proposed in the literature with different kinds of demand and cost functions. For example, [9] consider a Cournot oligopoly game with local monopolistic approximation, linear costs and a nonlinear demand function of the form $p = 1/Q$. Tuinstra [33] proposes a Bertrand oligopoly model,[3] under the assumption of local monopolistic approximation, with nonlinear demand functions and linear costs. Agiza et al. [2] study a Cournot oligopoly game based on "gradient dynamics" with the nonlinear demand function (3) and linear cost function. In [7] a nonlinear cost function is considered, associated with a linear demand function, so that the best reply dynamics can be explicitly written and compared with the "gradient dynamics". In our paper we consider nonlinear demand and cost functions, and we propose two different adjustment processes such that the computation of the reaction functions is not necessary. The aim of the paper is to investigate the dynamical properties that characterize these two boundedly rational decisional processes, and compare them to understand what they have in common and in what they differ.

The paper is organized as follows. In Section 2 we briefly describe the local monopolistic approximation mechanism and we introduce the corresponding dynamical system for the demand and cost functions introduced above. In Section 2.1 we focus on the simplest case of a duopoly. The local and the global stability analysis of the equilibrium of the two-dimensional map describing this decisional process are developed. In Section 3 we describe the second mechanism that bounded rational players may use to make their production decisions, that is the "gradient dynamics", and we will study the dynamical properties of the corresponding system. We finally conclude with considerations on the two mechanisms.

2. The model with local and monopolistic approximation

The best reply dynamics are obtained under the assumptions that, at each time period, firms have a global knowledge of the demand function, they know their current production and the current production of their competitors. They choose their next productions according to a profit maximization problem. In [9] firms are not required to know the complete demand function but just a point of it, and they try to get a linear approximation of the demand function on the basis of their local knowledge (see also [25,19,30,33]). In particular, at any time period t, each firm performs market experiments that permit it to compute the effects of small quantity variations on the price. For example, introducing at time t a small output variation Δq_i, firm i can compute

$$\frac{f(q_i(t) + \Delta q_i, q_{-i}(t)) - f(q_i(t), q_{-i}(t))}{\Delta q_i} \tag{8}$$

and we assumed that this variation corresponds to the correct estimate of the partial derivative

$$f_i(t) := \frac{\partial f(q_i(t), q_{-i}(t))}{\partial q_i}. \tag{9}$$

Each firm i uses this estimate to obtain the "rule of thumb" computation of the expected price

$$p_i^e(t+1) = p(t) + f_i(t)(q_i(t+1) - q_i(t)), \tag{10}$$

where $p(t) = f(q_1(t), \ldots, q_n(t))$ and $f_i(t)$ is defined in (9). The competition based on this kind of approximation is referred as "*monopolistic competition*", because the expected price depends only on the firm's production change. This approximation tries to catch the fact that, in the real world, firms are not able to observe their competitors' moves, so that they cannot anticipate their effects on the demand and, consequently, they take their rivals' actions as fixed.

The approximation (10) is easier to be obtained than a global knowledge of the demand function and it does not correspond to the linear approximation of $f(Q)$. In fact, as firm i cannot obtain the estimate of $f_j(t)$, with $j \neq i$, it simply

[3] Bertrand oligopoly means that the firms are assumed to be quantity takers and they decide their prices in order to maximize their profits. This differs from the assumption of Cournot oligopoly games, where players decide the quantities to be produced and take prices as determined by the market demand function.

neglects the influence of the competitors' production in the computation of the expected price. Of course, this is a very rough approximation. However, many authors claim that this is not far from reality, see e.g. [20] on this point. Moreover, as we shall see below, even if firms neglect the influence of the competitors' outputs in the computation of the expected price, the dynamic process generated by such a repeated game may converge to the same equilibria as the best reply dynamics do.

If the producer i uses the local monopolistic approximation (10) to compute the expected price, then the first order conditions for the optimization problem (6) become

$$\frac{\partial}{\partial q_i(t+1)}[q_i(t+1)(p(t)+f_i(t)(q_i(t+1)-q_i(t)))-C_i(q_i(t+1))]=0 \quad i=1,\ldots,n,$$

i.e.

$$p(t)+2f_i(t)q_i(t+1)-f_i(t)q_i(t)-C_i'(q_i(t+1))=0 \quad i=1,\ldots,n, \tag{11}$$

where $p(t)=f(Q(t))$ and C_i' denotes the derivative of the cost function. Notice that, in order to compute $q_i(t+1)$, at time t firm i needs the following *information set*: its current output $q_i(t)$, the current price of the good $p(t)$, the partial derivative $f_i(t)$ and its own cost function $C_i(q_i)$.

In [9] the following proposition is given, that states that even if players use a linear and monopolistic approximation of the demand function, the equilibria of this game are the same as in the oligopoly game with full information, that is the Nash equilibria.

Proposition 1. *The equilibria of the Cournot game (7) with best reply and perfect knowledge of the inverse demand function are steady states of the optimization problem with local monopolistic approximation (11).*

A study of the dynamic properties of the decisional process (11), based on the local monopolistic approximation of the demand function, is possible if the implicit equation (11) can be written in the form of an explicit discrete time dynamical system, i.e. if one can uniquely compute $q_i(t+1)$ from the knowledge of the state variables at time t. This can be obtained if we consider cost functions in the form (4). In fact, in this case we have $C_i'(q_i(t+1))=c_{i1}+2c_{i2}q_i(t+1)$, and (11) gives

$$q_i(t+1)=\frac{c_{i1}+q_i(t)f_i(Q(t))-f(Q(t))}{2[f_i(t)-c_{i2}]} \quad i=1,\ldots,n. \tag{12}$$

So, if we consider linear or quadratic cost functions and any kind of nonlinear differentiable demand function then we can get an explicit dynamical system that describes the time evolution of the oligopoly game. In particular, if we consider the demand function (3) we have $f_i(Q)=-b/(2\sqrt{Q})$, and the dynamic equations (12) become

$$q_i(t+1)=\frac{1}{2}\left(\frac{bq_i(t)+2(a-c_{i1})\sqrt{Q(t)}-2bQ(t)}{b+2c_{i2}\sqrt{Q(t)}}\right) \quad i=1,\ldots,n. \tag{13}$$

2.1. Duopoly Cournot game with LMA

The simplest kind of oligopoly is a market with only two producers, that is a duopoly. The discrete dynamical system (13), with $n=2$, becomes an iterated two-dimensional mapping $T:\mathbb{R}^2\to\mathbb{R}^2$ defined by

$$T:\begin{cases} q_1(t+1)=\dfrac{1}{2}\left(\dfrac{2(a-c_{11})\sqrt{q_1(t)+q_2(t)}-bq_1(t)-2bq_2(t)}{b+2c_{12}\sqrt{q_1(t)+q_2(t)}}\right), \\ q_2(t+1)=\dfrac{1}{2}\left(\dfrac{2(a-c_{21})\sqrt{q_1(t)+q_2(t)}-bq_2(t)-2bq_1(t)}{b+2c_{22}\sqrt{q_1(t)+q_2(t)}}\right). \end{cases} \tag{14}$$

Of course, only non-negative trajectories are feasible, that is they are economically meaningful. In the following we shall call *feasible set* the set of initial conditions that generate bounded and non-negative trajectories.

We already know, from Proposition 1, that the steady states of the dynamical system (14) are the equilibria of the best reply dynamics (7) (even if we do not have an explicit analytic representation of the reaction curves). An analytic computation of the steady states of (14) is not easy, because the equations to find the fixed points of the map T, obtained by setting $q_i(t+1)=q_i(t)$ in (14), give an irrational system that after rationalization gives rise to an algebraic system of sixth degree. Instead, under the assumption that the two firms have the same cost functions (homogenous costs)

$$c_{i1} = c_1 \quad \text{and} \quad c_{i2} = c_2 \quad \text{for each } i, \tag{15}$$

the steady states can be computed analytically because the equations for the fixed points become

$$\begin{aligned} &2(a - c_1 - 2c_2q_1)\sqrt{q_1 + q_2} - 3bq_1 - 2bq_2 = 0, \\ &2(a - c_1 - 2c_2q_2)\sqrt{q_1 + q_2} - 3bq_2 - 2bq_1 = 0. \end{aligned} \tag{16}$$

In this case the following result holds

Proposition 2. *If (15) holds and $a > c_1$ then a unique equilibrium of (14) exists which is locally asymptotically stable.*

Note that we do not consider the solution (0; 0) because it is the non-trade solution. A proof of Proposition 2 is given in Appendix A.

Proposition 2 only concerns the local stability of the Nash equilibrium, and it says nothing about the stability extent, that is, the extension of the feasible basin of the equilibrium. In Fig. 1 the feasible basin is shown for two different sets of parameters. In both the situations, the white region represents the feasible basin, i.e. the set of initial conditions generating economically feasible trajectories (i.e. bounded and non-negative) that converge to the stable Nash equilibrium, instead the grey region is the economically unfeasible set, i.e. the set of initial states generating trajectories going to infinity or to negative quantities. Fig. 1a is obtained with the set of parameters $a = 3$, $b = 1$, $c_1 = 1$, $c_2 = 0.3$, and Fig. 1b is obtained with the parameters $a = 2$, $b = 1.4$, $c_1 = 1.2$, $c_2 = 0.3$. In both cases the equilibrium is locally stable, with eigenvalues $\lambda_1 = -0.18$ and $\lambda_2 = 0.35$ in Fig. 1a, $\lambda_1 = -0.21$ and $\lambda_2 = 0.42$ in Fig. 1b respectively, but the extension of the feasible basin is very different and in Fig. 1b the stability of the Nash equilibrium has a poor practical meaning.

A delimitation of the feasible basin, and an estimate of how its extension is affected by the parameters of the model, are very important issues for practical purposes. This question requires a global analysis of the dynamical system represented by the iteration of the map (14). Indeed, the feasible basin is bounded by the coordinate axes and by the portions, in the first quadrant, of their rank-1 preimages, given by arcs of parabolas (their expressions are given in Appendix A). From the expression of the preimages of the axes we can easily deduce that the coefficients of the second degree term of the cost functions have no role in the delimitation of the boundary of the feasible basin. This means that an increase of the cost parameters c_{i2} cause a decrease in the Nash equilibrium quantities (as the production becomes more expensive) as well as on the associated eigenvalues, but they have no effects on the extension of the feasible basin. This has been confirmed by numerical simulations.

A more explicit discussion on the effects of the other parameters can be obtained as follows. The point where the preimages of the coordinate axes intersect each other is a rank-1 preimage of the point $O = (0, 0)$, denoted by O_{-1} in Fig. 1a, that can be computed by solving the algebraic system

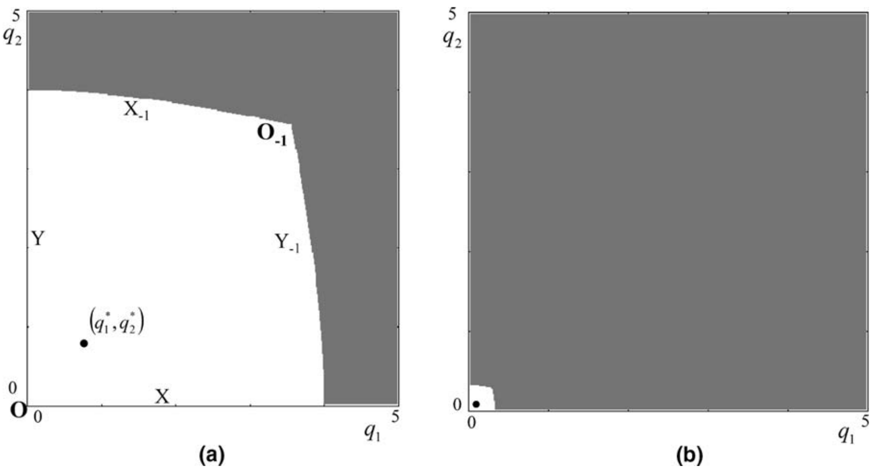

Fig. 1. Duopoly model (14) with LMA. The white region represents the feasible set, the grey region is the set of points that generate unbounded (and negative) trajectories. X_{-1} represents the rank-1 preimage of the horizontal axis X, Y_{-1} represents the rank-1 preimage of the vertical axis Y, O_{-1} is the inner rank-1 preimage of the origin O. (a) For $a = 3$, $b = 1$, $c_1 = 1$, $c_2 = 0.3$, the Nash equilibrium has coordinates $q_1^* = q_2^* = 0.2358$ and the eigenvalues in it are $\lambda_1 = -0.14$, $\lambda_2 = 0.28$. (b) For $a = 2$, $b = 1.4$, $c_1 = 1.2$, $c_2 = 0.3$ the Nash equilibrium has coordinates $q_1^* = q_2^* = 0.0908$, and the eigenvalues in it are $\lambda_1 = -0.21$, $\lambda_2 = 0.42$.

$$2(a-c_1)\sqrt{q_1+q_2}=bq_1+2bq_2,$$
$$2(a-c_1)\sqrt{q_1+q_2}=bq_2+2bq_1,$$

obtained from (14) with $q_1(t+1)=0$ and $q_2(t+1)=0$, and under the assumption of homogeneous costs. The preimage of the origin is

$$O_{-1}=\left(\frac{8(a-c_1)^2}{9b^2},\frac{8(a-c_1)^2}{9b^2}\right). \tag{17}$$

This gives an estimate of the extension of the feasible basin, as well as of the effects of the parameters on it. For example, an increase of the maximum price a, ceteris paribus, enlarges the feasible basin, instead an increases of the marginal cost c_1 or of the sensitivity of the demand b determine a contraction of the basin of attraction. These effects have been confirmed by numerical simulations run for different sets of parameters.

Summing up, a higher maximum price, a less sensitive demand to changes of the aggregate output and a cheaper production are all conditions that help firms to finally select the Nash production quantity, as they all increase the number of initial states starting from which the repeated game based on the local monopolistic approximation of the demand function converges to the equilibrium.

3. The adjustment process based on marginal profits

In this section we propose a repeated Cournot oligopoly game where the bounded rational players update their production strategies, at discrete time periods, by an adjustment mechanism based on a local estimate of the marginal profit $\partial\pi_i/\partial q_i$. At each time period t, a firm decides to increase (decrease) its production for the period $t+1$ if it perceives positive (negative) marginal profit on the basis of information held at time t, according to the following dynamic adjustment mechanism (see e.g. [5,2])

$$q_i(t+1)=q_i(t)+\alpha_i(q_i(t))\frac{\partial\pi_i}{\partial q_i}(q_1(t),\dots,q_n(t));\quad i=1,\dots,n, \tag{18}$$

where $\alpha_i(q_i)$ is a positive function which gives the extent of production variation of the ith firm following a given profit signal. With this kind of local adjustment mechanism, the two producers are not requested to have a complete knowledge of the demand and cost functions. They only need to infer how the market will respond to small production changes by an estimate of the marginal profit. This estimate may be obtained by brief experiments of small (or local) production variations performed at the beginning of period t (see e.g. [34]). We notice that, in this case, players do not decide their future production quantities by solving an optimization problem, but the selection occurs just following the direction of increasing profits. This adjustment mechanism, which is sometimes called *myopic* (see [15,16]), has been proposed by many authors, see e.g. [17,29], mainly with continuous time and constant α_i. However, following [5,2] we believe that a discrete time decision process is more realistic, since, in real economic systems, production decisions cannot be revised at every time instant. In the following, we also assume linear functions $\alpha_i(q_i)=v_iq_i$, $i=1,2$, since this assumption captures the fact that *relative* production variations are proportional to marginal profits, i.e.

$$\frac{q_i(t+1)-q_i(t)}{q_i(t)}=v_i\left(\frac{\partial\pi_i}{\partial q_i}\right),$$

where v_i is a positive *speed of adjustment*, which represents firm's i reaction to profit signals per unitary production. Given these assumptions, and considering the case of a duopoly, we obtain a discrete dynamical system of the form $(q_1(t+1),q_2(t+1))=T(q_1(t),q_2(t))$, with the map $T:\mathbb{R}^2\to\mathbb{R}^2$ given by

$$\begin{cases} q_1(t+1)=q_1(t)+v_1q_1(t)\left[a-c_{11}-b\sqrt{q_1(t)+q_2(t)}-\dfrac{bq_1(t)}{2\sqrt{q_1(t)+q_2(t)}}-2c_{12}q_1(t)\right],\\ q_2(t+1)=q_2(t)+v_2q_2(t)\left[a-c_{21}-b\sqrt{q_1(t)+q_2(t)}-\dfrac{bq_2(t)}{2\sqrt{q_1(t)+q_2(t)}}-2c_{22}q_2(t)\right]. \end{cases} \tag{19}$$

If $c_{12}=c_{22}=0$, i.e. the two cost functions are linear, then the dynamic game (19) reduces to the one studied by [2], which may be considered as a benchmark case in this context. As shown in [2], bounded trajectories that never converge to a steady state can be observed, as well as complex dynamics. Unfeasible trajectories are obtained if the initial condition is taken sufficiently far from the Nash equilibrium.

3.1. Steady states and basins

Also in this case we compute the steady states, under the assumption of homogenous costs (15), by solving the equations

$$v_1q_1[2(a-c_1-2c_2q_1)\sqrt{q_1+q_2}-3bq_1-2bq_2]=0,$$
$$v_2q_2[2(a-c_1-2c_2q_2)\sqrt{q_1+q_2}-3bq_2-2bq_2]=0.$$

Besides the Cournot–Nash equilibrium E_*, already computed in Section 2.1, the map (19) has two boundary equilibria located along the coordinate axes

$$E_1=(q_1^m,0),\qquad E_2=(0,q_2^m),\tag{20}$$

where q_1^m and q_2^m are solutions of the systems

$$\begin{cases}q_2=0,\\ v_1[2(a-c_1-2c_2q_1)\sqrt{q_1+q_2}-3bq_1-2bq_2]=0\end{cases}$$

and

$$\begin{cases}q_1=0,\\ v_2[2(a-c_1-2c_2q_2)\sqrt{q_1+q_2}-3bq_2-2bq_2]=0,\end{cases}$$

respectively. The fixed points E_1 and E_2 can be denoted as *monopoly equilibria*. In fact, the coordinate axes $q_i=0$, $i=1,2$, are invariant submanifold, i.e. if $q_i(t)=0$ then $q_i(t+1)=0$. This means that starting from an initial condition on a coordinate axis (*monopoly case*) the dynamics are trapped into the same axis for each t, thus giving *monopoly dynamics* governed by the restriction of the map T to that axis. Such a restriction is given by the following unimodal one-dimensional map, obtained from (19) with $q_i=0$

$$q_j(t+1)=q_j(t)\left(1+v_j(a-c_{j1})-\frac{3bv_jq_j(t)+4c_{j2}v_jq_j(t)\sqrt{q_j(t)}}{2\sqrt{q_j(t)}}\right)\quad j\neq i.\tag{21}$$

In order to get some insight into the global stability properties of the Nash equilibrium under this kind of adjustment process we performed some numerical simulations to understand weather or not the fixed point is stable, how it loses stability when we change the parameter values and what kinds of attractors appear. Before going through this analysis we can make some observations on the delimitation of the feasible region and on how its extension is affected by the different parameters. Two numerical computations of the feasible region (white area) are shown in Fig. 2. As it can be

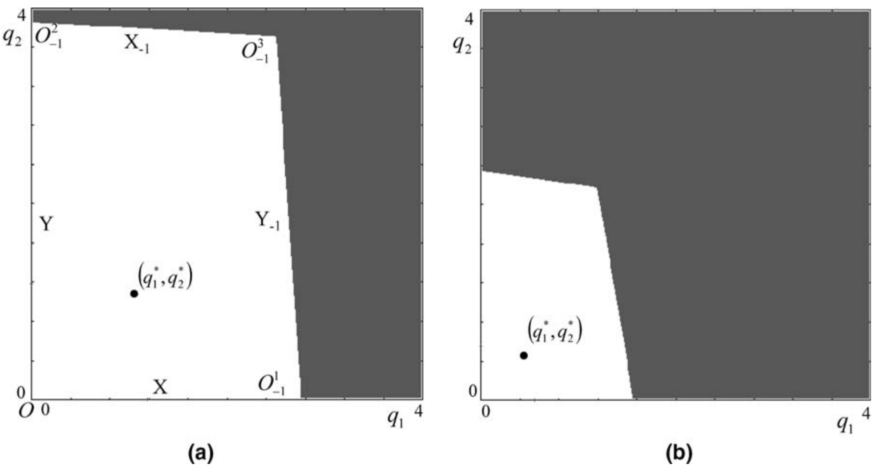

Fig. 2. Duopoly model (19) based on the marginal profit. The white area represents the feasible set, the grey region is the set of points that generate unbounded (and negative) trajectories. (a) $a=3$, $b=0.5$, $c_1=1$, $c_2=0.5$, $v_1=0.5$, $v_2=0.3$, the Nash quantities are given by $q_1^*=q_2^*=1.08$; (b) presents a higher sensitivity of the demand function, that is $b=1.3$, and the Nash quantities are given by $q_1^*=q_2^*=0.45$.

seen, the boundaries of the feasible region are very simple, being formed by four segments joining at four vertexes, given by the point $O = (0,0)$ and its rank-1 preimages, denoted by O^k_{-1}, $k = 1,2,3$ (Fig. 2a). In fact, like in the model studied in the previous section, two sides of the "quadrilateral" region are portions of the coordinate axes, and the other two sides are their rank-1 preimages. As the coordinate axes are invariant submanifold, we can compute the two preimages of O located along the coordinate axes, O^k_{-1}, $k = 1,2$, by computing the zeroes of the one-dimensional map (21), restriction of (19) to the axis $q_i = 0$ (see Appendix A). In Appendix A, the coordinates of O^k_{-1}, $k = 1,2$ are obtained as intersections between a parabola and a straight line, and a straightforward graphical analysis (see Fig. 7) allows us to say that an increase of the parameter b and/or an increase of a cost coefficients c_1 or c_2 reduce the value of the coordinates of the preimages O^1_{-1} and O^2_{-1}, hence the extension of two segments bounding the feasible region. This causes a negative impact on the stability extent, in the sense that the feasible region shrinks. On the contrary, an increase of the maximum price a moves the intercept of the straight line of Fig. 7 upwards, thus increasing the values of O^k_{-1}, $k = 1,2$ and having a positive effect on the extension of the feasible region. Analogously, an increase of the speeds of adjustment v_i, $i = 1,2$, determines a reduction of the extension of the feasible region. These conclusions are confirmed by computer simulations (compare Fig.2a and b).

4. Two kinds of complexity

In the situations shown in Fig. 2 the shape of the boundaries of the feasible region is very simple, being formed by simple curves joining at four vertexes. Moreover, the attractor to which the feasible trajectories converge is simple as well, being given by a stable steady state. As we shall see in the following, the process described by (19) may exhibit complex kinds of non-equilibrium asymptotic dynamics as well as complicated topological structures of the feasible set. In order to understand the transition from simple to complicated structures of the feasible set we have first to notice that the map (19) is a non-invertible map. This means that, given a point $(q_1(t+1), q_2(t+1))$, if we try to solve the algebraic system (19) with respect to the unknowns $(q_1(t), q_2(t))$, i.e. we try to compute the rank-1 preimages (or backward iterates), such solutions may be more than one. This can be expressed by saying that the map T is "many-to-one", or, equivalently, that the inverse relation T^{-1} is multivalued. The number of solutions of the system (19) and the number of the rank-one preimages may change, as pairs of real preimages appear or disappear as the point $(q_1(t+1), q_2(t+1))$ crosses the boundary separating regions whose points have a different number of preimages. Such boundaries are generally characterized by the presence of two coincident (merging) preimages. This leads to the definition of *critical curves*, one of the distinguishing features of non-invertible maps (see e.g. [18,24] for a deeper treatment, see also [6,8,28], for several applications in economic dynamic modeling). The critical curve of rank-1, denoted by LC, is defined as the locus of points having two, or more, coincident rank-1 preimages located on a set called LC_{-1}. LC is the two-dimensional generalization of the notion of critical value (local minimum or maximum value) of a one-dimensional map, LC_{-1} is the generalization of the notion of critical point (local extremum point). Arcs of LC separate the plane into regions characterized by a different number of real preimages.

As in the case of differentiable one-dimensional maps, where the derivative necessarily vanishes at the local extremum points, for a two-dimensional continuously differentiable map the set LC_{-1} belongs to the set of points in which the Jacobian determinant vanishes: $LC_{-1} \subseteq \{(x,y) \in \mathbb{R}^2 | \det DT = 0\}$. In fact, as LC_{-1} is defined as the locus of coincident rank-1 preimages of the points of LC, in any neighborhood of a point of LC_{-1} there are at least two distinct points mapped by T in the same point near LC. This means that the map T is not locally invertible in the points of LC_{-1} and, if the map T is continuously differentiable, it follows that $\det DT$ necessarily vanishes along LC_{-1}. If the set LC_{-1} is determined by the solutions where the Jacobian of the map (19) vanishes, then LC is simply obtained as the image of LC_{-1}, i.e. $LC = T(LC_{-1})$.

Consider Fig. 3 where the market parameters are $a = 3$, $b = 0.5$, the cost coefficients are $c_1 = 1$ and $c_2 = 0.5$ and the speeds of adjustment are $v_1 = 0.8$ and $v_2 = 0.1$ respectively. A numerical computation of the critical curves is shown for the map T defined in (19). The critical set LC is formed by two branches, that separate regions Z_k whose points have four, two or no preimages respectively. Following the terminology introduced in [24], the map (19) results to be of the kind $Z_0 - Z_2 - Z_4$. Indeed, as already noticed, the origin $O = (0,0)$ has four rank-1 preimages: one is O itself, the other ones are given by O^k_{-1}, $k = 1,2,3$ (see Appendix A), whereas points of the positive orthant that are too far from O have no preimages.

In Fig. 3, the quadrilateral $OO^1_{-1}O^3_{-1}O^2_{-1}$ constitutes the whole boundary of the feasible set because the rank-1 preimages of portions $X = OO^1_{-1}$ and $Y = OO^2_{-1}$ of the coordinate axes, denoted by X_{-1} and Y_{-1}, are entirely included inside the region Z_0 with no preimages. The situation is different when the values of the parameters are such that some portions of these curves belong to the regions Z_2 or Z_4 whose points have two or four preimages respectively. In this

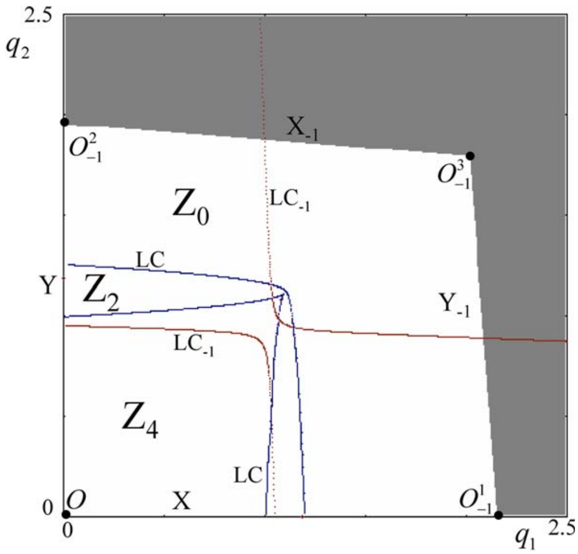

Fig. 3. Duopoly model (19) based on the marginal profit. The curves denoted by LC_{-1} represent the set of points for which the Jacobian of the map (19) vanishes. The critical curves $LC = T(LC_{-1})$ separate regions Z_k whose points have different numbers of preimages. The set of parameters used in the picture is $a = 3$, $b = 0.5$, $c_1 = 1$, $c_2 = 0.5$, $v_1 = 0.8$, $v_2 = 1$.

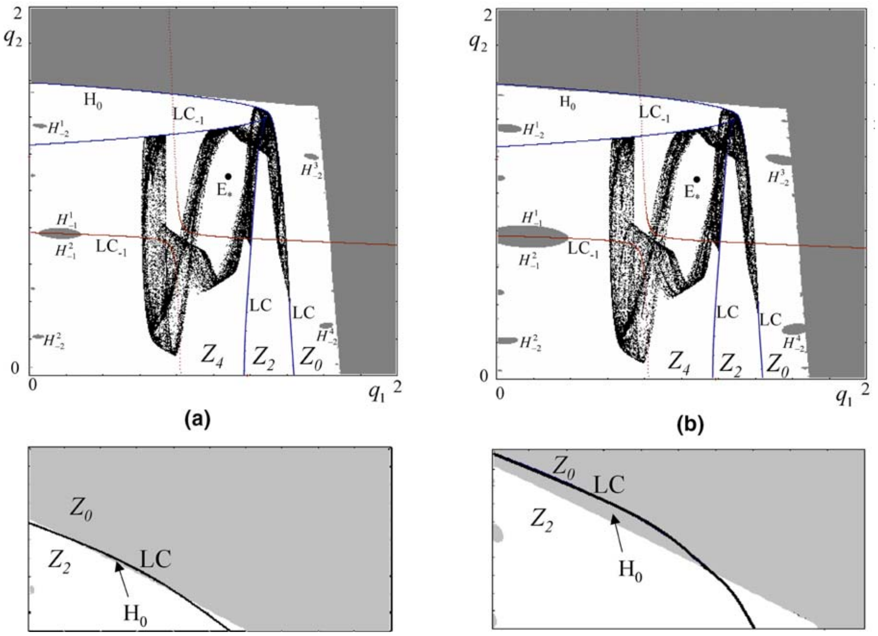

Fig. 4. Duopoly model (19) based on the marginal profit. In both the figures the Nash equilibrium is a repelling node, and a chaotic attractor exists around it. (a) For $a = 3$, $b = 0.5$, $c_1 = 1$, $c_2 = 0.5$, $v_1 = 1.5$, $v_2 = 1.86$ part of the unfeasible region, H_0, has crossed LC (see the enlargement below) and entered Z_2. (b) For $v_1 = 1.5$, $v_2 = 1.88$ the upper branch of LC is above the boundary preimage O^2_{-1}. Now main hole merges with the vertical axis and its preimages undergo to the same process.

case the preimages of higher order of X and Y exist, say X_{-k} and Y_{-k}, and they form new portions of the boundary, like in Fig. 4.

In fact, if some parameters are changed, for example the speeds of adjustment v_1 or v_2 are increased, a contact between the boundary of the feasible set and the critical curve LC that separates Z_0 from Z_2 occurs, and this constitutes a global bifurcation (well known after the work of [24]) that causes a qualitative change in the topological structure of the feasible region. In Fig. 4 a portion of the grey area, denoted by H_0, representing a portion of the region generating unfeasible economic trajectories, is inside region Z_2, thus suggesting that a contact bifurcation has occurred. In fact, changes in the parameter values move the critical lines, a contact between the basin boundary and LC occurs and the portion of the economic unfeasible region H_0, which was in region Z_0 before the bifurcation, enters inside Z_2. The points belonging to H_0 have now two distinct preimages, say H^1_{-1} and H^2_{-1}, located at opposite sides with respect to the line LC_{-1} and the points of the curve LC inside the economic unfeasible region have preimages merged on LC_{-1}. The rank-1 preimages of H_0 are formed by two areas joining along LC_{-1} and constitute a *hole* of the economic unfeasible region nested inside the economic feasible region (this hole is also called "lake" in [23]). This is the largest hole in Fig. 4a, and it is called the *main hole*. As it lies inside region Z_4 it has 4 preimages, which are smaller holes denoted by H^j_{-1}, $j = 1, \ldots, 4$ in Fig. 4. Two of them are inside Z_0, so that they do not give rise to further preimages; one of them is inside Z_2, so that it has two further preimages inside Z_2. These preimages constitute the rank-3 preimages of H_0, that is their points are mapped into H_0 after three iterations of the map T. These preimages have, on their turn, their own preimages so that we can repeat the same analysis. The global bifurcation just described transforms a *simply connected* basin into a *multiply connected* one, with a countable infinity of holes, called *arborescent sequence of holes*, inside it (see [23,24] for a rigorous treatment of this type of global bifurcation, or [1] for a simpler and charming exposition).

If we further increase the speeds of adjustment, then the upper branch of the critical line LC moves upwards including a wider portion of the unfeasible region. This causes an enlargement of the main hole and of its preimages as well as they all move towards the basin boundaries. When the boundary preimage O^2_{-1} of the origin along the vertical axis crosses the critical line LC (Fig. 4b with $v_1 = 1.5$ and $v_2 = 1.88$) part of the main hole merges with the vertical axis and also its preimages undergo to the same process with their closest basin boundary.

Our numerical explorations also show that this model, where firms make their production decisions following the direction of the marginal profit, can be characterized by attracting sets of increasing complexity, such as the chaotic regions shown in Fig. 4. Indeed, if we increase the speeds of adjustment starting from the values used in Fig. 2, we first notice that the stable fixed point loses stability via a flip (or period doubling) bifurcation, then a cascade of flip bifurcations leads to the creation of chaotic attractors. Some of these attractors are shown in Figs. 4 and 5. As usual when dealing with non-invertible maps, the boundaries of a chaotic attractor $\mathscr{A}$ can be easily obtained by portions of critical curves. Indeed, following [24] (see also [4,1,28]) a practical procedure can be outlined in order to obtain the boundary of an absorbing area (although it is difficult to give a general method). Starting from a portion of LC_{-1}, approximately taken in the region occupied by the chaotic attractor $\mathscr{A}$, its images by T of increasing rank are computed until a closed region is obtained. When such a region is mapped into itself, then it is a trapping region. The length of the initial segment is to be taken, in general, by a trial and error method, although several suggestions are given in the books referenced above. Once a trapping region (also called absorbing area) is found, in order to see if it is invariant or not the same procedure must be repeated by taking only the portion

$$\gamma = \mathscr{A} \cap LC_{-1} \tag{22}$$

as the starting segment. Then one of the following two cases occurs:

Case I: the union of m iterates of γ (for a suitable m) covers the whole boundary of $\mathscr{A}$; in which case $\mathscr{A}$ is an invariant absorbing area, and

$$\partial\mathscr{A} \subset \bigcup_{k=1}^{m} T^k(\gamma). \tag{23}$$

Case II: no natural m exists such that $\bigcup_{i=1}^{m} T^i(\gamma)$ covers the whole boundary of $\mathscr{A}$; in which case $\mathscr{A}$ is not invariant but strictly mapped into itself. An invariant absorbing area is obtained by $\cap_{n>0} T^n(\mathscr{A})$ (and may be obtained by a finite number of images of $\mathscr{A}$).

This method has been applied, as an example, to the chaotic attractors shown in Fig. 5. In Fig. 5a two segments of LC_{-1} were chosen inside the chaotic attractor, instead in Fig. 5b only one segment was enough to delimit the corresponding chaotic attractor. In Fig. 6a we can see that the delimitation of a closed absorbing area that includes the chaotic attractor of Fig. 5a is obtained after three iterations of the two segments of LC_{-1}, i.e. $LC = T(LC_{-1})$, $LC_i = T^i(LC)$, $i = 1, 2, 3$. Higher rank images of LC will give a more precise delimitation of the boundaries of the chaotic area. In Fig. 6b the boundaries of the two-cyclic chaotic attractor of Fig. 5b are obtained after five iterations of LC_{-1}, i.e. $LC = T(LC_{-1})$, $LC_i = T^i(LC)$, $i = 1, \ldots, 5$. The two-cyclic chaotic attractor is now well delimited and if we consider further iterations of the segment of LC_{-1} we get segments of critical curves LC_i inside the chaotic area.

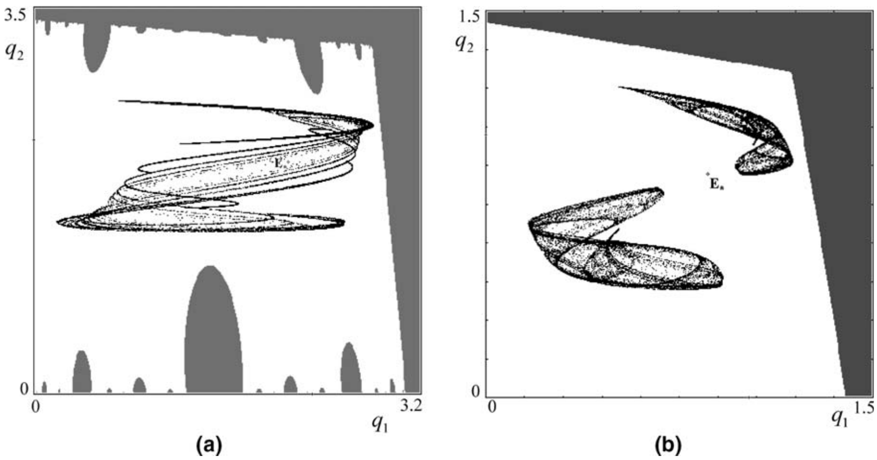

Fig. 5. Some examples of chaotic attractors of the map (19). (a) $a = 6.5$, $b = 1$, $c_1 = 1$, $c_2 = 0.7$, $v_1 = 0.7$, $v_2 = 0.5$. (b) $a = 5$, $b = 1.7$, $c_1 = 1$, $c_2 = 0.7$, $v_1 = 1.04$, $v_2 = 0.9$.

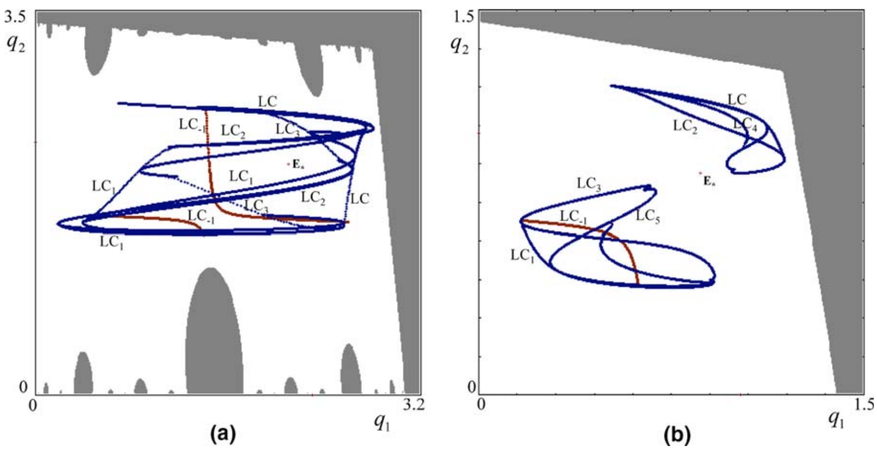

Fig. 6. Procedure for obtaining the boundary of an absorbing area. (a) Delimitation of an absorbing area that includes the chaotic attractor of Fig. 5a; (b) delimitation of the chaotic attractor of Fig. 5b.

We finally note that, as shown in Figs. 4 and 5a, both kinds of complexity, the one related to chaotic attractors and the one related to multiply connected basins, can coexist. However, these two kinds of complexity are not necessarily related, in the sense that simple attracting sets (such as steady states) may be associated with complex structures of the basins (see e.g. [8]) or complex attractors may have simple basins, like in Fig. 5b.

5. Conclusions

In this paper we have proposed two different kinds of repeated oligopoly games based on the idea that firms are not fully rational. Weaker assumptions have been made on the available information set, on the firms' capability of extracting correct estimates from it and on the computational skills required to solve the optimization problems through which firms make their decisions. In particular, we developed two models where n firms produce an homogeneous good and face nonlinear demand and cost functions. They are assumed to use two different rules of thumb, or adjustment mechanisms, denoted by Local Monopolistic Approximation (LMA) and Gradient Dynamics (GD) respectively, to make their production decisions. The main conceptual difference between these two mechanisms is that even if firms use a LMA of the demand function they still have to solve their profit maximization problem to decide their production levels, instead no maximization problem is considered when the GD is used.

Both these models give rise to discrete dynamical systems and, for both, we have considered the simplest case of the duopoly, whose dynamics are expressed by the iteration of a two-dimensional map. The study of the dynamical properties of the map has shown that they both share the same Nash equilibrium, which is also the equilibrium of the so called Best Reply dynamics characterized by a complete knowledge of the demand function.

For these two maps of the plane we have compared the stability properties, at the local and global level. First, the Nash equilibrium under LMA is always locally asymptotically stable, instead under GD it can lose stability via flip (period doubling) bifurcation. We have then obtained an analytical estimate of the extension of the basins of attraction. This result is very important in practical applications but it is often neglected in the literature on dynamical systems of dimension greater than one because it requires a global analysis of the iterated map. A study of the stability only based on a local analysis may be misleading if it is not associated to a global analysis of the extension of the basin of attraction. In fact we have shown that for the model with LMA a locally sympatrically stable equilibrium may have an extremely small basin of attraction, so that its stability may have no practical meaning. For both the adjustment mechanisms the boundaries of the feasible region are given by portions of the coordinate axes and by their rank-1 preimages. The study on how the different parameters affect its dimension has shown that, for both models, an increase of the maximum price and/or a reduction of the sensitivity of the demand function and/or a reduction of the cost coefficient of the term of first degree have a positive impact determining an enlargement of the basin of attraction. One difference in the global dynamical properties between the two models is that the cost coefficient of the term of second degree has non-effects on the feasible region under LMA, instead its increase determines a contraction of the feasible region under GD. A second difference is given by the presence of the speed of adjustment in the model with GD. This parameter has an important role in the local and global dynamical properties of the map of the model, creating two different kinds of complexity absent in the model with LMA. One complexity is related to the creation of basins with complex topological structures, as a consequence of global bifurcations. A second complexity is related to the creation of complex attractors, as a consequence of the lost of stability of the equilibrium. From the numerical simulations we have seen that complex boundaries of the basins are more likely to be observed in presence of a high speed of adjustment and chaotic attractors are more likely to be observed in presence of a high maximum price and/or a high speed of adjustment to the marginal profit.

We conclude noting that the comparison we have made between the two adjustment processes is not directed to say which one of the two is better than the other, because the adjustment process based on the marginal profit presents two more parameters than the one based on the local monopolistic approximation. These are given by the sensitivity of each firm to its marginal profit when it decides its production level and it resulted to have a strong influence on the dynamics of the system. Our idea was to build decisional models based on weaker assumptions than the complete rationality, but this rises the question that there are several ways to be bounded rational which share the same equilibrium but they can generate different dynamics.

Acknowledgements

The authors thanks Gian Italo Bischi and the participants of the workshop MDEF-2004, Urbino (Italy). We express our thanks to the two anonymous referees for their interesting remarks and very useful suggestions. The usual disclaimer applies. This work has been performed within the activities of the national research project "Nonlinear models in economics and finance: complex dynamics, disequilibrium, strategic interactions", MIUR, Italy, under the Joint Research Grant (0382): "Reconsideration of economic dynamics from a new perspective of nonlinear theory", Chuo University, Japan.

Appendix A. Proof of Proposition 2

If Eq. (16) to find the fixed points of (14) are subtracted then the following equation is obtained

$$(q_1 - q_2)(b(a - c_1) + 4bc_2(q_1 + q_2)) = 0. \tag{24}$$

If we assume $a > c_1$, a quite natural assumption as it states that the maximum price cannot be less that the unitary cost (otherwise any production will give negative profits), then the second factor is always positive. This implies that the system (16) is equivalent to

$$\begin{cases} q_2 = q_1, \\ 2\sqrt{2q_1}(a - c_1 - 2c_2q_1) = 5bq_1. \end{cases} \tag{25}$$

After rationalization, the second equation is equivalent to

$$q_1[32c_2^2q_1^2 - (32(a-c_1)c_2 + 25b^2)q_1 + 8(a-c_1)^2] = 0 \tag{26}$$

with the condition

$$q_1 \leqslant \frac{a-c_1}{2c_2}. \tag{27}$$

Eq. (26) has always two real and positive solutions, but only the smaller one satisfies the condition (27), which is given by

$$q^* = \frac{a-c_1}{2c_2} + \frac{5b(5b - \sqrt{25b^2 + 64(a-c_1)c_2})}{64c_2^2}.$$

This proves that a unique positive steady state $E_* = (q^*, q^*)$ exists.

We now prove that the positive equilibrium $E_* = (q^*, q^*)$ is locally asymptotically stable. Let us consider the Jacobian matrix of (14)

$$DT(q_1,q_2) = \frac{1}{2}\begin{bmatrix} \dfrac{b(a-c_1-b\sqrt{q_1(t)+q_2(t)}-c_2q_1(t))}{(b+2c_2\sqrt{q_1(t)+q_2(t)})^2\sqrt{q_1(t)+q_2(t)}} & \dfrac{b(a-c_1-2b\sqrt{q_1(t)+q_2(t)}-3c_2q_1(t)-2c_2q_2(t))}{(b+2c_2\sqrt{q_1(t)+q_2(t)})^2\sqrt{q_1(t)+q_2(t)}} \\ \dfrac{b(a-c_1-2b\sqrt{q_1(t)+q_2(t)}-3c_2q_2(t)-2c_2q_1(t))}{(b+2c_2\sqrt{q_1(t)+q_2(t)})^2\sqrt{q_1(t)+q_2(t)}} & \dfrac{b(a-c_1-b\sqrt{q_1(t)+q_2(t)}-c_2q_2(t))}{(b+2c_2\sqrt{q_1(t)+q_2(t)})^2\sqrt{q_1(t)+q_2(t)}} \end{bmatrix} \tag{28}$$

that, computed at the fixed point E^*, assumes the structure

$$DT(q^*, q^*) = \frac{1}{2}b\begin{bmatrix} A & B \\ B & A \end{bmatrix}, \tag{29}$$

where

$$A = \frac{a-c_1-b\sqrt{2q^*}-c_2q^*}{(b+2c_2\sqrt{2q^*})^2\sqrt{2q^*}}, \qquad B = \frac{a-c_1-2b\sqrt{2q^*}-5c_2q^*}{(b+2c_2\sqrt{2q^*})^2\sqrt{2q^*}}.$$

A set of sufficient conditions for the local asymptotic stability of E_*, i.e. for the eigenvalues to be inside the unit circle of the complex plane, is given by

$$1 - \mathrm{Tr} + \mathrm{Det} > 0,$$
$$1 + \mathrm{Tr} + \mathrm{Det} > 0,$$
$$1 - \mathrm{Det} > 0$$

(see e.g. [22, p. 52]) which, in this case, become

$$(b(A-B)-2)(b(A+B)-2) > 0, \tag{30}$$
$$(b(A-B)-2)(b(A+B)+2) > 0, \tag{31}$$
$$1 - \frac{1}{4}b^2(A-B)(A+B) > 0. \tag{32}$$

We first prove that (32) is always satisfied, because $A - B > 0$ and $A + B < 0$. In fact, the first inequality is immediately proved, being

$$\frac{a-c_1-b\sqrt{2q^*}-c_2q^*}{(b+2c_2\sqrt{2q^*})^2\sqrt{2q^*}} - \frac{a-c_1-2b\sqrt{2q^*}-5c_2q^*}{(b+2c_2\sqrt{2q^*})^2\sqrt{2q^*}} = \frac{b\sqrt{2q^*}+4c_2q^*}{(b+2c_2\sqrt{2q^*})^2\sqrt{2q^*}} > 0.$$

The second inequality is equivalent to

$$2(a-c_1) - 3b\sqrt{2q^*} - 6c_2q^* < 0. \tag{33}$$

From the condition for the equilibrium (25) we have

$$a - c_1 - 2c_2q^* = \frac{5bq^*}{2\sqrt{2q^*}},$$

that is

$$2(a-c_1)=4c_2q^*+\frac{5b\sqrt{2q^*}}{2}.$$

Substituting the above expression in (33) we obtain

$$4c_2q^*+\frac{5b\sqrt{2q^*}}{2}-3b\sqrt{2q^*}-6c_2q^*=-2c_2q^*-\frac{b\sqrt{2q^*}}{2}<0,$$

so that $A+B<0$.

We then prove that also condition (30) always holds. In fact condition (30) holds if

$$b(A-B)-2<0,$$

which is equivalent to

$$b\frac{b\sqrt{2q^*}+4c_2q^*}{(b+2c_2\sqrt{2q^*})^2\sqrt{2q^*}}-2<0,$$
$$-\frac{b^2\sqrt{2q^*}+12c_2q^*+16c_2^2q^*\sqrt{2q^*}}{(b+2c_2\sqrt{2q^*})^2\sqrt{2q^*}}<0,$$

that is always true. Finally, we have that

$$\frac{1}{4}(bA-2)^2-\frac{1}{4}b^2B^2<\frac{1}{4}(bA+2)^2-\frac{1}{4}b^2B^2,$$

so when (30) holds, then (31) is also true.

Preimages of the coordinate axes for the model with LMA

The rank-1 preimages of the points of the q_1 coordinate axis are the real solutions (q_1, q_2) of the algebraic system obtained from (14) with $(q_1(t+1), q_2(t+1))=(x,0)$, i.e.

$$\begin{cases}2(a-c_{11}-2c_{12}x)\sqrt{q_1+q_2}=2xb+bq_1+2bq_2,\\ 2(a-c_{21})\sqrt{q_1+q_2}=bq_2+2bq_1.\end{cases} \tag{34}$$

From the second equation it is easy to see that the preimages of the points of the q_1 axis are located on the parabola of equation

$$b^2(2q_1+q_2)^2-4(a-c_{21})^2(q_1+q_2)=0, \tag{35}$$

that intersects the coordinate axes q_1 and q_2 in the points $((a-c_{21})^2/b^2, 0)$ and $(0, 4(a-c_{21})^2/b^2)$ respectively. Analogously, the rank-1 preimages of the points of the q_2 axis are located on the parabola of equation

$$b^2(2q_2+q_1)^2-4(a-c_{11})^2(q_1+q_2)=0, \tag{36}$$

that intersects the coordinate axes q_1 and q_2 in the points $(4(a-c_{11})^2/b^2, 0)$ and $(0, (a-c_{11})^2/b^2)$ respectively.

Under the assumption of homogenous costs (15), it is easy to realize that the two parabolas also intersect each other in a unique positive point along the diagonal $q_1=q_2$, given by

$$O_{-1}=\left(\frac{8(a-c_1)^2}{9b^2},\frac{8(a-c_1)^2}{9b^2}\right),$$

which is the rank-1 preimage of the origin.

Preimages of the origin for the model with GD

The rank-1 preimages of the origin located on the invariant coordinate axes are the real solutions of Eq. (21) by setting $q_j(t+1)=0$. For example, the rank-1 preimages along the axis $q_2=0$ are given by the solutions of the equation

$$q_1\left[1+v_1(a-c_1)-\frac{3bv_1q_1+4c_2v_1q_1\sqrt{q_1}}{2\sqrt{q_1}}\right]=0.$$

One is $q_1=0$, i.e. the origin itself, that cannot be considered as the map (19) is not defined in it, and the positive one is the solution of the equation

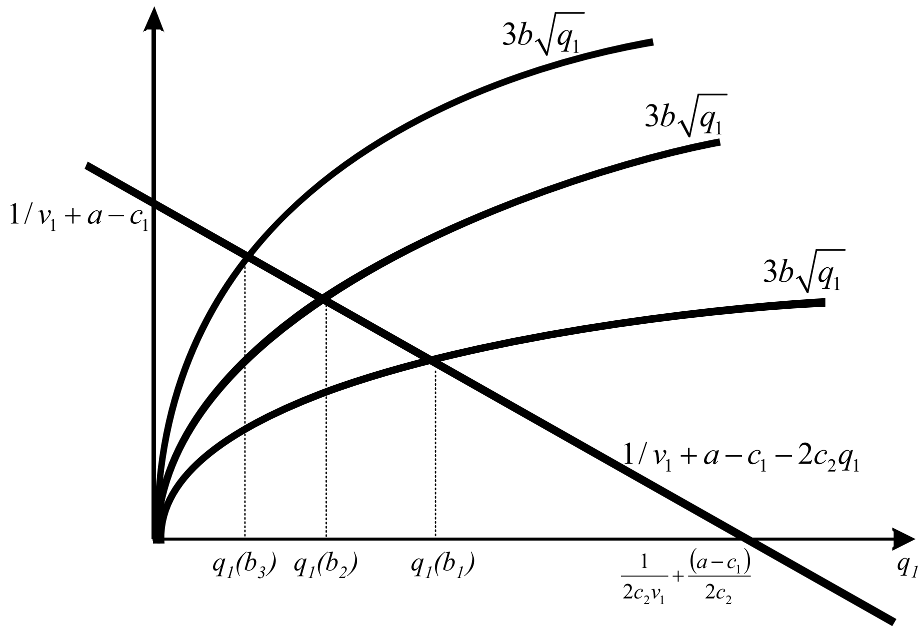

Fig. 7. Graphical representation of the parabola $3b\sqrt{q_i}$ and the straight line $1/v_i + a - c_1 - 2c_2q_i$.

$$2(a - c_1 + 1/v_1) - 4c_2q_1 = 3b\sqrt{q_1}. \tag{37}$$

This equation has always one positive solution, located at the intersection of a decreasing straight line (the left hand side of (37)) and the half-parabola of equation (the right hand side of (37)), see Fig. 7. This allows one to see the effects of the parameters' variations on the positions of the vertexes O^k_{-1}, $k = 1, 2$ of the feasible region. The inner preimage O^3_{-1} is given by the solution of the system

$$\begin{cases} q_1(t) + v_1q_1(t)\left[a - c_{11} - b\sqrt{q_1(t)+q_2(t)} - \dfrac{bq_1(t)}{2\sqrt{q_1(t)+q_2(t)}} - 2c_{12}q_1(t)\right] = 0, \\ q_2(t) + v_2q_2(t)\left[a - c_{21} - b\sqrt{q_1(t)+q_2(t)} - \dfrac{bq_2(t)}{2\sqrt{q_1(t)+q_2(t)}} - 2c_{22}q_2(t)\right] = 0. \end{cases} \tag{38}$$

References

[1] Abraham R, Gardini L, Mira C. Chaos in discrete dynamical systems. Springer-Verlag; 1997.

[2] Agiza HN, Hegazi AS, Elsadany AA. Complex dynamics and synchronization of a duopoly game with bounded rationality. Math Comput Simulat 2002;58:133–46.

[3] Baumol WJ, Quandt RE. Rules of Thumb and optimally imperfect decisions. Am Econom Rev 1964;54:23–46.

[4] Bischi GI, Gardini L. Role of invariant and minimal absorbing areas in chaos synchronization. Phys Rev E 1998;58(5):5710–9.

[5] Bischi GI, Naimzada A. Global analysis of a duopoly game with bounded rationality. Adv Dynam Games Appl 1999;5:361–85.

[6] Bischi GI, Gardini L, Kopel M. Analysis of global bifurcations in a market share attraction model. J Econom Dynam Control 2000;24:855–79.

[7] Bischi GI, Lamantia F. Nonlinear duopoly games with positive cost externalities due to spillover effects. Chaos, Solitons & Fractals 2002;13:805–22.

[8] Bischi GI, Kopel M. Equilibrium selection in a nonlinear duopoly game with adaptive expectations. J Econom Behav Organiz 2001;46(1):73–100.

[9] Bischi GI, Naimzada A., Sbragia L. Oligopoly games with local monopolistic approximation. J Econom Behav Organiz, forthcoming.

[10] Bischi GI, Lamantia F, Sbragia L. Competition and cooperation in natural resources exploitation: an evolutionary game approach. In: Carraro C, Fragnelli V, editors. Game practice and the environment. Edward Elgar Publishing; 2004. p. 187–211.

[11] Bonanno G. Oligopoly equilibria when firms have local knowledge of demand. Int Econom Rev 1988;29:45–55.

[12] Bonanno G, Zeeman C. Limited knowledge of demand and oligopoly equilibria. J Econom Theory 1985;35:276–83.
[13] Clark CW. Mathematical bioeconomics—the optimal management of renewable resources. New York: John Wiley & Sons; 1990.
[14] Cournot A. Recherches sur les principes matematiques de la theorie de la richesse. Paris: Hachette; 1838.
[15] Dixit A. Comparative statics for oligopoly. Int Econom Rev 1986;27:107–22.
[16] Flam SD. Nonlinear dynamics in economics and social sciences. In: Gori F, Geronazzo L, Galeotti M, editors. Lecture notes in economics and mathematical systems, vol. 399. Springer-Verlag; 1993. p. 232–7.
[17] Furth D. Stability and instability in oligopoly. J Econom Theory 1986;40:197–228.
[18] Gumowski I, Mira C. Dynamique chaotique. Toulose: Cepadues Editions; 1980.
[19] Jin JY. Monopolistic competition and bounded rationality. J Econom Behav Organiz 2001;45:175–84.
[20] Kirman AP. Learning by firms about demand conditions. In: Day RH, Groves T, editors. Adaptive economic models. Academic Press; 1975. p. 137–56.
[21] Kopel M. Simple and complex adjustment dynamics in Cournot duopoly models. Chaos, Solitons & Fractals 1996;7(12):2031–48.
[22] Medio A, Lines M. Nonlinear dynamics. Cambridge University Press; 2001.
[23] Mira D, Fournier-Prunaret, Gardini L, Kawakami H, Cathala JC. Basin bifurcations of two-dimensional noninvertible maps: fractalization of basins. Int J Bifurcat Chaos 1994;4:343–81.
[24] Mira C, Gardini L, Barugola A, Cathala JC. Chaotic dynamics in two-dimensional noninvertible maps. World Scientific; 1996.
[25] Negishi T. Monopolistic competition and general equilibrium. Rev Econom Stud 1961;27:136–9.
[26] Offerman T, Potters J, Sonnemans J. Imitation and belief learning in an oligopoly experiment. Rev Econom Stud 2002;69:973–97.
[27] Puu T. Chaos in duopoly pricing. Chaos, Solitons & Fractals 1991;1:573–81.
[28] Puu T. Attractors, bifurcations and Chaos. Berlin: Springer-Verlag; 2000.
[29] Sacco PL. Adaptive response and Cournotian behavior. Econom Notes 1991;20:474–96.
[30] Silvestre J. A model of general equilibrium with monopolistic behavior. J Econom Theory 1977;16:425–42.
[31] Szidarovszky F, Okuguchi K. A linear oligopoly model with adaptive expectations: stability reconsidered. J Econom 1988;48:79–82.
[32] Szidarovszky F, Okuguchi K. An oligopoly model of commercial fishing. Seoul J Econom 1998;11:321–30.
[33] Tuinstra J. A price adjustment process in a model of monopolistic competition. Int Game Theory Rev 2004;6(3):417–42.
[34] Varian HR. Microeconomic analysis. 3rd ed. W.W. Norton & Company; 1992.