Classification of Physical Problems and Partial Differential Equations

Gerald W. Recktenwald*

January 5, 2004

1 Overview

Partial differential equations (PDEs) are mathematical models of continuous physical phenomena in which a dependent variable, say u, is a function of more than one independent variable, say t and x. PDEs are often derived by applying a fundamental principle such as conservation of mass, momentum, or energy. The resulting equation is referred to as the governing differential equation since it expresses the mathematical rule which acceptable solutions, e.g. $u(x, t)$, must obey. Other conditions are required to fully define the solution. These are usually boundary conditions for field problems and initial conditions for transient problems.

This paper provides a survey of the three primary types of partial differential equations: parabolic, hyperbolic, and elliptic. The equations are described in terms of the types of solutions that they admit, and the speed with which information propagates along coordinate directions. The presentation is non-rigorous.

Example: 1D Heat Conduction in a Rod The heat flux in the x-direction is given by Fourier's Law

$$q_x = -k\frac{\partial T}{\partial x}$$

where k is the thermal conductivity, a material property. Applying an energy balance to a differential section of the rod gives

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) = 0$$

If there are significant temperature gradients in both the x and r directions within the rod then heat flows in both of these directions. The heat flux is then a vector, $\mathbf{q}$, and the component heat fluxes are given by Fourier's Law

$$q_x = -k\frac{\partial T}{\partial x} \qquad q_r = -k\frac{\partial r}{\partial x}$$

*Associate Professor, Mechanical Engineering Department Portland State University, Portland, Oregon, gerry@me.pdx.edu

An energy balance on an infinitessimal chunk of material gives

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right) = 0$$

This is the two-dimensional Laplace equation in axisymmetric coordinates.

2 Classification of Physical Problems

Ames [1, Chapter 1] identifies three basic types of physical problems that lead to partial differential equations

1. Equilibrium Problems
   - Steady state conservative systems–field problems.
   - Physical phenomena inside the domain are described by an elliptic partial differential equation
   - The steady state of the sytem cannot be determined unless boundary conditions are supplied
   - A solution is the function that satisfies both the governing PDE and the boundary conditions
   - Example: Steady state temperature distribution inside a solid object
2. Eigenvalue Problems
   - Determine the critical parameters that describe
   - Examples: resonance phenomena, stability analysis
3. Propagation Problems
   - Time dependent spatial distributions–initial value problems.
   - Physical phenomena inside the domain are described by a hyperbolic partial differential equation
   - The steady state of the sytem cannot be determined unless initial conditions and boundary conditions are supplied
   - A solution is the function that satisfies the governing PDE, the initial conditions and the boundary conditions
   - Examples: Wave propagation, transient and high speed fluid flow

3 Classification of Partial Differential Equations

Partial differential equations can be classified as either parabolic, hyperbolic, or elliptic. The physical interpretation of these terms leads to a description of the way that information can flow along characteristic directions in the physical domain. These distinctions are important because the choice of numerical method depends on the physical behavior of the system, which is reflected in the type of PDE that models the system.

Propagation problems (see preceding section) give rise to parabolic or hyperbolic PDEs. Equilibrium problems result in elliptic PDEs.

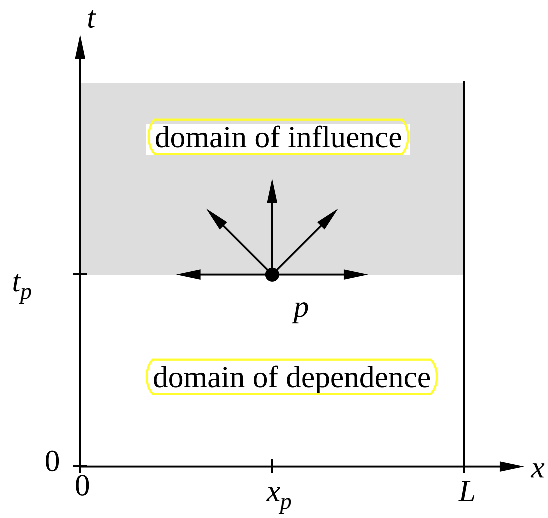


Figure 1: Information propagation along characteristics of the heat equation–a parabolic PDE.

3.1 Parabolic Partial Differential Equations

Example: The heat equation. Consider the one-dimensional (in space) heat equation for $u = u(x, t)$.

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \qquad 0 \leq x \leq L, \quad t \geq 0$$

with boundary and initial conditions

$$u(t, 0) = u_0 \qquad u(t, L) = u_L$$
$$u(0, x) = g(x)$$

where α is the diffusivity, a material property. The solution, $u(x, t)$ can be represented as a field in the semi-infinite strip shown in Figure 1. At time t_p the boundary conditions at $x = 0$ and $x = L$ both influence the solution at position x_p. This is true at all subsequent times. The solution at (t_p, x_p) also influences the solution at future times in the entire region above the line $t = t_p$. Thus, we can identify the semi-infinite strip above $t = t_p$ as a region in which the solution depends on the solution at (t_p, x_p). This is called the *domain of influence* of $u(t_p, x_p)$. This also means that the solution at (t_p, x_p) is affected by all points in the space-time where $t \leq t_p$. This region is called the *domain of dependence*.

3.2 Hyperbolic Partial Differential Equations

Example: The wave equation. Consider the one-dimensional (in space) wave equation for $u = u(x, t)$.

$$\frac{\partial^2 u}{\partial t^2} = c \frac{\partial^2 u}{\partial x^2} \qquad 0 \leq x \leq L, \quad t \geq 0$$

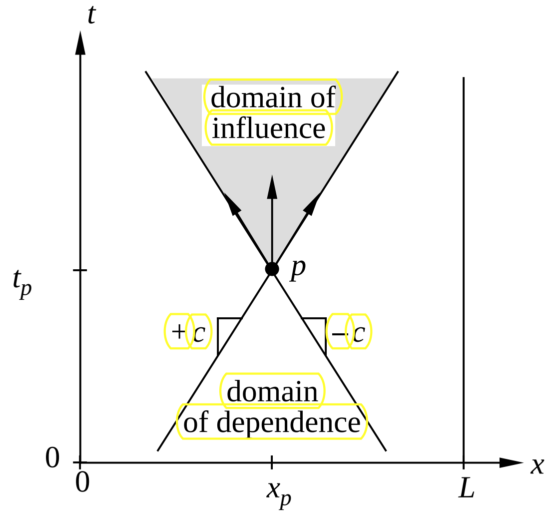


Figure 2: Information propagation along characteristics of the wave equation–a hyperbolic PDE.

with boundary and initial conditions

$$u(t,0) = u_0 \qquad u(t,L) = u_L$$
$$u(0,x) = f(x) \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$$

where $\sqrt{c}$ is the wave speed. The solution, $u(x,t)$ can be represented as a field in the semi-infinite strip shown in Figure 2. The state of the solution at time t_p and position x_p influences the solution at all other points in the wedge of space-time labelled domain of influence. This wedge is defined by two *characteristics* with slope $\pm c$.

3.3 Elliptic Partial Differential Equations

Example: The Laplace equation. Consider the Laplace equation, which is a model of steady heat conduction in a two-dimensional domain. Let $T = T(x,y)$ be the temperature field in a slab of some solid of width W and height H. The temperature field is governed by

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \qquad 0 \le x \le W, \quad 0 \le y \le H$$

with boundary conditions

$$T(x,0) = T_1 \qquad T(x,H) = T_3$$
$$T(0,y) = T_4 T(W,y) = T_2$$

The solution, $u(x,t)$ is a field in the rectangular domain shown in Figure 3. The state of the solution at time x_p and position y_p influences the solution at all

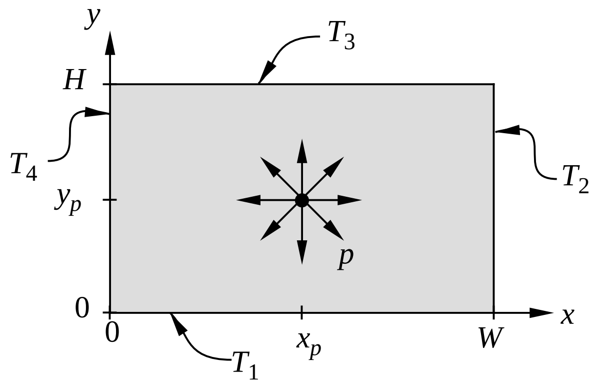


Figure 3: Information propagation along characteristics of the Laplace equation–an elliptic PDE.

other points in the domain. In other words the entire domain is both the domain of influence and the domain of dependence. One can think of temperature information as propagating with infinite speed in all coordinate directions. This may be confusing unless one realizes that there is no time dependence at all.

References

[1] William F. Ames. *Numerical Methods for Partial Differential Equations*. Academic Press, Inc., Boston, third edition, 1992.

Classification

1. A partial differential operator (PDO) L is linear if for any functions u and v and and scalars c,

$$L[u + cv] = Lu + cLv. \tag{1}$$

If L is a linear PDO, the equation

$$Lu = f, \tag{2}$$

is homogeneous if $f \equiv 0$, and inhomogeneous otherwise.

2. **Superposition:** If $u_1, \ldots, u_n$ satisfy the linear, homogeneous equation

$$Lv = 0, \tag{3}$$

then any linear combination $u = c_1 u_1 + \cdots + c_n u_n$ also satisfies it.

3. Extensions: If $\{u_k\}_{k=1}^{\infty}$ satisfy (3) and

$$u = \sum_{k=1}^{\infty} c_k u_k,$$

converges "well enough," then u also satisfies (3). If $u(x, \alpha)$ satisfy (3) for all α in some interval I and

$$u(x) = \int_I c(\alpha) u(x, \alpha)\, dx,$$

converges well enough, then u also satisfies (3).

4. Consider second-order, linear PDE in two independent variables. Associate with a t-derivative the symbol τ, with an x-derivative ξ and with a y-derivative η. The heat equation is

$$u_t - k u_{xx} = 0, \tag{4}$$

where $k > 0$. We thus associate with it the symbol $\tau - k\xi^2$, suggesting a parabola in the $\xi\tau$-plane. For this reason, the heat equation is called parabolic. Laplace's equation is

$$u_{xx} + u_{yy} = 0, \tag{5}$$

with the symbol $\xi^2 + \eta^2$. This suggests an ellipse in the $\xi\eta$-plane. Thus Laplace's equation is elliptic. Finally, the wave equation

$$u_{tt} - c^2 u_{xx} = 0, \tag{6}$$

has symbol $\tau^2 - c^2\xi^2$ and is thus a hyperbolic equation. We won't go into the details of classification, but there are a few general points to bear in mind:

a. Parabolic equations govern phenomena (e.g. diffusion) characterized by smoothing, spreading flow. The heat operator

$$H = \frac{\partial}{\partial t} - k\Delta$$

is the archetypal parabolic operator.

b. Elliptic equations govern equilibrium, energy-minimizing states. The Laplacian Δ is the archetypal elliptic operator.

c. Hyperbolic equations govern "distrubance preserving" phenomena such as travelling waves and shocks. The D'Alembertian $\Box$ is the archetypal hyperbolic operator.

d. Many PDO do not fall into one of the three categories. For example, the Tricomi operator (from gas dynamics)

$$L = y\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

is hyperbolic for $y < 0$ and elliptic for $y > 0$.

e. Classification is harder with more independent variables, with higher-order PDE and with systems of equations.

f. There are various refinements of the linear-nonlinear, homogeneous-inhomogeneous, parabolic-elliptic-hyperbolic classifications. You might, for example, have a "quasilinear elliptic" equation, or a system of equations that is "symmetric hyperbolic."

We restrict our attention to specialisations of the equation

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} = H, \qquad (1)$$

where $u = u(x,y)$ and A, B, C and H may depend on x, y, u, u_x and u_y. Such an equation is called a *quasi-linear* equation of *second-order* since the partial derivatives of *highest* order are involved in a linear way. The general *linear* equation of second-order is represented by the general form

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G, \qquad (2)$$

where $A, B, \ldots, G$ depend only on x and y. It can be shown that the quantity $B^2 - 4AC$ has a central role in the classification of equations of the form (1) or (2):

(i) if $B^2 - 4AC > 0$ at (x,y) the equation is said to be HYPERBOLIC at (x,y)
(ii) if $B^2 - 4AC < 0$ at (x,y) the equation is said to be ELLIPTIC at (x,y)
(iii) if $B^2 - 4AC = 0$ at (x,y) the equation is said to be PARABOLIC at (x,y)

The terminology arises from an associated conic section which occurs in the relevant theory.

e.g (i) $u_{xx} - u_{yy} = 0$ is everywhere HYPERBOLIC $(A = 1, B = 0, C = -1)$

(ii) $u_{xx} + u_{yy} = 0$ is everywhere ELLIPTIC $(A = 1, B = 0, C = 1)$

(iii) $u_{xx} = u_y$ is everywhere PARABOLIC $(A = 1, B = C = 0)$

(iv) $(1 - y)u_{xx} + 2xu_{xy} + (1 + y)u_{yy} = G$.

Here $B^2 - 4AC = 4x^2 - 4(1 - y^2) = 4(x^2 + y^2 - 1)$.

ELLIPTIC inside unit circle, HYPERBOLIC outside, PARABOLIC on unit circle.

(v) $u_{xx} + uu_{yy} = H$. Depends on sign of u at (x,y).

HYPERBOLIC equations usually arise in vibration problems (e.g. waves, vibrations of structures) and y represents time t. (i) is called the WAVE equation and $u(x,t)$ is the deflection of, for example, a string or beam.

ELLIPTIC equations usually apply to steady state problems. e.g. u can be temperature, deflection, electrostatic potential, velocity potential, stream function etc. (ii) is Laplace's equation.

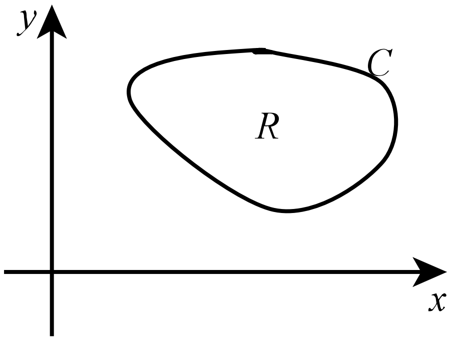
PARABOLIC equations normally arise in diffusion or heat conduction problems. u can be concentration, temperature etc. Here $y \equiv t$.

INITIAL CONDITIONS AND BOUNDARY CONDITIONS

The importance of the classification is that the *type* of the equation governs the number and nature of *initial* and/or *boundary conditions* which must be specified in order to obtain a *unique* solution.

(a) ELLIPTIC EQUATIONS: Boundary conditions

e.g. $u_{xx} + u_{yy} = G$ in a finite region R bounded by a closed curve C.



We must specify

(i) u on curve C.

or (ii) $\partial u / \partial n$ on C (n is outward normal to C).

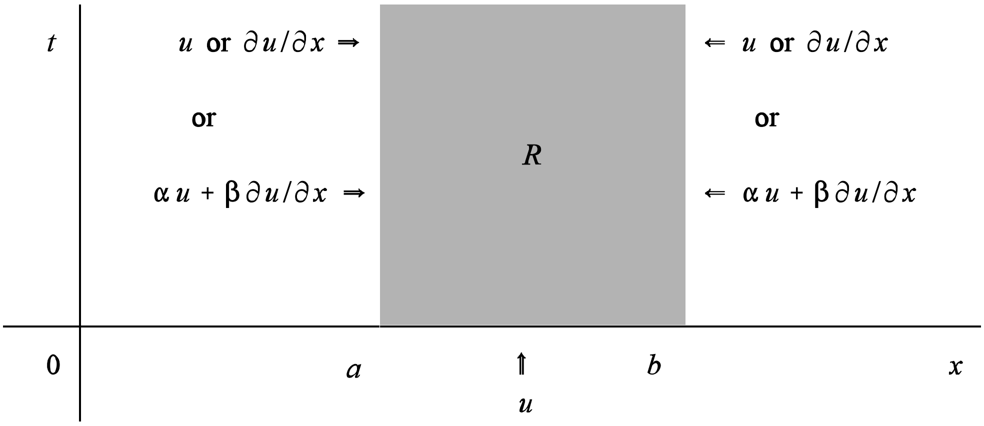
or (iii) $\alpha u + \beta \partial u / \partial n$ on C (α and β are given constants).

or (iv) a combination of (i), (ii) and (iii) on different parts of C.

In Cartesian coordinates the simplest case is if R is rectangular with boundary condition (i). R can extend to infinity, in which case we must specify how the solution behaves as x or y (or both x and y) tend to infinity.

(b) PARABOLIC EQUATIONS: Initial conditions and boundary conditions.

e.g. $u_{xx} = u_t$ in the *open* region R in the (x,t) plane. R is the region $a \le x \le b$, $0 \le t < \infty$.



We must specify u on $t = 0$ (i.e. $u(x,0)$) for $a \le x \le b$. This is an *initial* condition (e.g. an initial temperature distribution) and suitable *boundary* conditions on $x = a, b$ are as shown.

(c) HYPERBOLIC EQUATIONS: e.g. $u_{xx} = u_{tt}$ Initial conditions and boundary conditions as for (b) except that we must also specify u_t at $t = 0$ for $a \le x \le b$ (in addition to u) and R is the region $a \le x \le b$, $-\infty < t < \infty$.

pcl309.wpd

Proceedings of Institute of Mathematics of NAS of Ukraine 2002, Vol. 43, Part 1, 86–92

Group Classification of Nonlinear Partial Differential Equations: a New Approach to Resolving the Problem

P. BASARAB–HORWATH † *and V. LAHNO* ‡

† *Linköping University, S–581 83 Linköping, Sweden*
E-mail: *pehor@mai.liu.se*

‡ *Poltava State Pedagogical University, 2 Ostrogradskyj Str., Poltava 36000, Ukraine*
E-mail: *laggo@poltava.bank.gov.ua*

We describe a systematic procedure for classifying partial differential equations which are invariant with respect to low-dimensional Lie algerbas. This procedure is a synthesis of the infinitesimal Lie method, the technique of equivalence transformations and the theory of classification of abstract low-dimensional Lie algebras. By way of illustration, we consider three examples of group classification of partial differential equations in new approach.

1 Introduction

This article is based on two talks (one given by each of the authors) at the Fourth International Conference "Symmetry in Nonlinear Mathematical Physics" (9–14 July, 2001, Kyiv, Ukraine). More details can be found in [9] and [16], and a short description is given in [17].

The analysis and classification of differential equations using group theory goes back to Sophus Lie. The first systematic investigation of the problem of group classification was done by L.V. Ovsiannikov [1] in 1959 for nonlinear heat equation

$$u_t = [f(u)u_x]_x,$$

where $f(u)$ is an arbitrary nonlinearity. His approach is based on the concept of the equivalence group, which is the Lie transformation group (acting in the space whose local coordinates are independent variables, the functions and their derivatives) preserving the class of particular differential equations under study. It is possible to modify Lie's algorithm in order to make it applicable for the computation of this group (see, e.g., [2]). Having obtained the equivalence group one constructs the optimal system of subgroups of the equivalence group. The last step uses Lie's algorithm for obtaining specific partial differential equations that (a) belong to the class under study, and (b) are invariant with respect to these subgroups.

This approach has been applied to a number of equations of mathematical physics. Here we mention just a few of the papers in which the group classification of nonlinear heat equations has been studied:

Akhatov, Gazizov, Ibragimov (1987, [3])

$$u_t = G(u_x)u_{xx};$$

Dorodnitsyn (1982, [4])

$$u_t = G(u)u_{xx} + \frac{dG(u)}{du}u_x^2 + g(u);$$

Oron, Rosenau (1986, [5]), Edwards (1994, [6])

$$u_t = G(u)u_{xx} + \frac{dG(u)}{du}u_x^2 + f(u)u_x;$$

Cherniha and Serov (1998, [7])

$$u_t = G(u)u_{xx} + \frac{dG(u)}{du}u_x^2 + f(u)u_x + g(u);$$

Gandarias (1996, [8])

$$u_t = u^n u_{xx} + nu^{n-1}u_x^2 + g(x)u^m u_x + f(x)u^s.$$

However, the possibility of implementing Ovsiannikov's approach in its full generality presupposes that we are able to construct the optimal system of subgroups of the equivalence group. However, even for the case when the equivalence group is finite-parameter, there arise major algebraic difficulties, since the classification problem for all finite-parameter Lie groups has not yet been solved (to say nothing about infinite-parameter Lie groups, where this problem is completely open). Consequently, there is an evident need for Ovsiannikov's approach to be modified so as to be applicable to the case of infinite-parameter equivalence groups.

Here we turn out attention to a new approach, proposed by R. Zhdanov and V. Lahno in [9], that enables us to solve efficiently the symmetry classification problem for partial differential equations even for the case of infinite-dimensional equivalence groups. It is based mainly on the following facts:

- If the partial differential equation possesses non-trivial symmetry, then it is invariant under some finite-dimensional Lie algebra of differential operators. In the event that the maximal algebra of invariance is infinite-dimensional, then it contains, as a rule, some finite-dimensional Lie algebra.
- If there are local non-singular changes of variables which transform a given differential equation into another, then the finite-dimensional Lie algebras of invariance of these equations are isomorphic, and in the group-theoretic analysis of differential equations such equations are considered to be equivalent.
- Abstract Lie algebras of up to six dimensions have already been classified [10, 11, 12, 13].

What we have in [9] is a preliminary classification of inequivalent realizations of low-dimensional Lie algebras within some specific class of first-order linear differential operators. This class is determined by the structure of the equation under study. Its elements form a representation space for realizations of Lie algebras of symmetry groups admitted by the equations belonging to the class of partial differential equations under study. A natural equivalence relation is introduced on the set of all possible realizations. Namely, two realizations are called equivalent if they are transformed into each other by the action of the equivalence group. In other words, solving the problem of symmetry classification of partial differential equations having some prescribed form, is equivalent to constructing a representation theory of Lie transformation groups (or Lie algebras of first-order partial differential operators) realized as symmetry groups (algebras) of the equations in question.

2 Description of the method

The new approach to the classification of partial differential equations is a synthesis of Lie's infinitesimal method, the use of equivalence transformations and the theory of classification of abstract finite-dimensional Lie algebras. It provides a constructive solution of the problem of the group classification of partial differential equations possessing arbitrary elements and admitting *non-trivial finite-dimensional* invariance algebras.

The group classification in the approach described here is implementation of the following algorithm:

I. The first step involves finding the form of the infinitesimal operators which generate the symmetry group of the equation under consideration, and the construction of the equivalence group of this equation. To find the form of the infinitesimal operators one uses the usual Lie algorithm. In doing this we obtain a system of linear partial differential equations of first order which connect the coefficients of the infinitesimal operators with the arbitrary term of the equation. In the following we call this system *the characterizing system of the equation*. In order to construct the equivalence group $\mathcal{E}$ of the equation one can use the infinitesimal method as well as the direct method.

II. In the second step one carries out the group classification of those equations of the given form which admit finite-dimensional Lie algebras of invariance.

   For this, one carries out a step-by-step classification of finite-dimensional Lie algebras within the specified class of infinitesimal operators, up to equivalence under transformations of the group $\mathcal{E}$. In doing this, one has to check that each algebra obtained in this way can be an invariance algebra of the equation at hand before proceeding from the realization of Lie algebras of lower dimension to the realization of Lie algebras of higher dimension. This eliminates superfluous realizations of Lie algebras. Also, those realizations of Lie algebras which are invariance algebras of the equation will, as their dimension increases, correspond to greater fixing of the arbitrary term.

   This procedure is continued until the arbitrary term in the equation is completely determined or until it is no longer possible to extend the realization of Lie algebras beyond a given dimension within the specified class of infinitesimal operators.

III. The third step is then to exploit the characterizing system or the infinitesimal method of Lie in order to find, for each of the particular choices of the arbitrary term, the maximal invariance algebra of the equation under consideration. Moreover, the equivalence of the equations obtained in this manner is determined. We note that, in as much as equivalent equations have isomorphic invariance algebras, we may test the realizations of the invariance algebras for equivalence rather than test the equations themselves.

3 Examples of the group classification

Here we give some examples illustrating how the method works.

Example 1 ([14]). Group classification of

$$u_{tx} + A(t,x)u_t + B(t,x)u_x + C(t,x)u = 0. \qquad (1)$$

Ovsiannikov [15] gave a group classification of (1), using Laplace invariants

$$h = A_t + AB - C, \qquad k = B_x + AB - C.$$

His results can be formulated as follows:

Theorem 1. *Equation (1) admits a Lie symmetry algebra of dimension greater than 1 if and only the functions p, q given by*

$$p = \frac{k}{h}, \qquad q = \frac{1}{h}\partial_x\partial_y(\ln h)$$

are constant. In this case, equation (1) is equivalent either to the Euler–Poisson equation

$$u_{tx} - \frac{2u_t}{q(t+x)} - \frac{2pu_x}{q(t+x)} + \frac{4pu}{q^2(t+x)^2} = 0$$

when $q \neq 0$*, or to the equation*

$$u_{tx} + tu_t + pxu_x + ptxu = 0$$

when $q = 0$.

We have carried out the group classification of equation (1) using our method.

First, we find (by standard methods) that the infinitesimal generator of symmetries is given by

$$X = f(t)\partial_t + q(x)\partial_x + h(t,x)u\partial_u,$$

where the functions f, g, h satisfy

$$\begin{aligned} &h_t + B\dot{f} + fB_t + gB_x = 0,\\ &h_x + Ag' + gA_x + fA_t = 0,\\ &h_{tx} + C\dot{f} + fC_t + Cg' + gC_x + Ah_t + Bh_x = 0 \end{aligned} \qquad (2)$$

(we omit the trivial symmetry $X = \omega(t,x)\partial_u$, where ω is an arbitrary solution of (1)).

A direct analysis of (2) is not possible. The **equivalence group** of (1) is given by transformations of the two following types:

$$\begin{aligned} &(a)\quad r = \alpha(t), \quad \xi = \beta(x), \quad v = \theta(t,x)u + \rho(t,x);\\ &(b)\quad r = \alpha(x), \quad \xi = \beta(t), \quad v = \theta(t,x)u + \rho(t,x), \end{aligned}$$

where α, β are arbitrary smooth functions and θ, ρ satisfy

$$\theta_t\rho_x + \rho_t\theta_x - \theta\rho_{tx} + \rho\theta_{tx} - 2\frac{\rho}{\theta}\theta_t\theta_x + A[\theta_t\rho - \theta\rho_t] + B[\theta_x\rho - \theta\rho_x] - C\theta\rho = 0.$$

We note that equation (1) is invariant under the operator $u\partial_u$ and that $[X, u\partial_u] = 0$. So, X and $u\partial_u$ form a two-dimensional Lie algebra. There are only two canonical forms for a two-dimensional Lie algebra

$$\begin{aligned} &A_{21} = \langle e_1, e_2\rangle \quad \text{with} \quad [e_1, e_2] = 0,\\ &A_{22} = \langle e_1, e_2\rangle \quad \text{with} \quad [e_1, e_2] = e_2, \end{aligned}$$

and we clearly see that only A_{21} is suitable for our purposes.

We now need to find a canonical form for the operator X. We have the following result:

Proposition 1. *Let* A_{21} *be the invariance algebra of equation (1). There are two inequivalent canonical realizations of* $A_{21} = \langle u\partial_u, X\rangle$*:*

$$A^1_{21} = \langle u\partial_u, \partial_t\rangle, \qquad A^2_{21} = \langle u\partial_u, \partial_t + \partial_x\rangle,$$

and the corresponding canonical forms for equation (1) are

$$A^1_{21}:\ u_{tx} + B(x)u_x + u = 0, \qquad (3)$$

$$A^2_{21}:\ u_{tx} + B(z)u_x + C(z)u = 0 \qquad (4)$$

with $z = t - x$.

The system (2) for equation (3) then becomes

$$h_t + B\dot{f} + gB_x = 0, \qquad h_x = 0, \qquad \dot{f} + g' = 0, \tag{5}$$

where $B = B(x)$.

We easily integrate (5) and we find $B = mx$, where $m = \text{const} \neq 0$, and equation (3) takes on the form

$$u_{tx} + mxu_x + u = 0.$$

The invariance algebra of this equation is

$$\langle u\partial_u, \partial_t, t\partial_t - x\partial_x, \partial_x - mtu\partial_u\rangle.$$

For equation (4) we find (using the same procedure) that the corresponding canonical form for equation (1) is

$$u_{tx} + \frac{m}{z}u_x + \frac{k}{z^2}u = 0,$$

where m, k are constants with $k \neq 0$, and $z = t - x$.

The invariance algebra of this equation is

$$\langle u\partial_u, \partial_t + \partial_x, t\partial_t + x\partial_x + \frac{1}{2}mu\partial_u, t^2\partial_t + x^2\partial_x + mtu\partial_u\rangle.$$

These results are equivalent to the ones obtained by Ovsiannikov.

Example 2 ([9]). Group classification of nonlinear equation of the form

$$u_t = u_{xx} + F(t, x, u, u_x). \tag{6}$$

First, we find that the infinitesimal generator of symmetries is given by

$$X = 2a(t)\partial_t + (\dot{a}(t)x + b(t))\partial_x + f(t, x, u)\partial_u,$$

where functions a, b, f, F fulfil relation

$$\begin{aligned} f_t = u_x(\ddot{a}x + \dot{b}) + (f_u - 2\dot{a})F = f_{xx} + 2u_x f_{xu} + u_x^2 f_{uu} + 2aF_t \\ + (\dot{a}x + b)F_x + fF_u + f_x F_{u_x} + u_x(f_u - \dot{a})F_{u_x}. \end{aligned} \tag{7}$$

A direct analysis of (7) is not possible.

Using our approach we have established that there are three classes of equations (6) invariant with respect to one-parameter groups, seven classes of equations (6) invariant with respect to two-parameter groups, 28 classes of equations (6) invariant with respect to three-parameter groups and 11 classes of equation (6) invariant with respect to four-parameter groups.

Here we present all representatives of 11 classes of equations (6) invariant with respect to four-parameter groups only:

1. $u_t = u_{xx} + \dfrac{\lambda\epsilon u_x}{4\sqrt{|t|}} \ln|tu_x^2| + \dfrac{\beta u_x}{\sqrt{|t|}}$,
   $\epsilon = 1$ for $t > 0$, $\epsilon = -1$ for $t < 0$, $\beta \in \mathbb{R}$, $\lambda \neq 0$;
2. $u_t = u_{xx} - \lambda u_x(x + \ln|u_x|)$, $\lambda \neq 0$;
3. $u_t = u_{xx} + \lambda\exp(-u_x)$, $\lambda \neq 0$;
4. $u_t = u_{xx} + 2\ln|u_x|$;
5. $u_t = u_{xx} - u_x\ln|u_x| + \lambda u_x$, $\lambda \in \mathbb{R}$;

6. $u_t = u_{xx} + \lambda u_x^{\frac{2k-2}{2k-1}}, \quad \lambda \neq 0, \quad k = 0, \frac{1}{2}, 1;$
7. $u_t = u_{xx} + \frac{1}{4t} u_x^2;$
8. $u_t = u_{xx} - uu_{xx} + \lambda |u_x|^{\frac{3}{2}}, \quad \lambda \neq 0;$
9. $u_t = u_{xx} + \lambda^{-1} x + m\sqrt{|u_x|}, \quad \lambda > 0, \quad m \neq 0;$
10. $u_t = u_{xx} - \frac{1}{4}\lambda\epsilon(1-q)|t|^{-\frac{1}{2}(1+q)} u_x^2,$
 $\lambda \neq 0, \quad |q| \neq 1, \quad \epsilon = 1 \ \text{ for } \ t > 0, \quad \epsilon = -1 \ \text{ for } \ t < 0;$
11. $u_t = u_{xx} - \frac{1}{2}\dot{\alpha} u_x^2 (\lambda - \alpha)\left(1 + \alpha^2\right)^{-1}, \quad \lambda \in \mathbb{R}.$

Note that case 8) with $\lambda = 0$ gives rise to the Burgers equation

$$u_t = u_{xx} - uu_x,$$

which is invariant under a five-parameter group.

Example 3 ([16]). Group classification of nonlinear equations of the form

$$u_t = F(t, x, u, u_x) u_{xx} + G(t, x, u, u_x). \tag{8}$$

In [16] we find that the infinitesimal generator of symetries is given by

$$X = a(t)\partial_t + b(t, x, u)\partial_x + c(t, x, u)\partial_u,$$

where a, b, c are real-valued functions that satisfy the system of particular differential equations

$$\begin{aligned}
&(2b_x + 2u_x bu - \dot{a})F = aF_t + bF_x + cF_u + \left(c_x + u_x c_u - u_x b_x - u_x^2 bu\right) F_{u_x},\\
&c_t - u_x b_t + (c_u - \dot{a} - u_x b_u)G + (u_x b_{xx} - c_{xx} - 2u_x c_{xu} - u_x^2 c_{uu}\\
&\qquad + 2u_x^2 b_{xu} + u_x^3 b_{uu})F = aG_t + bG_x + cG_u + \left(c_x + u_x c_u - u_x b_x - u_x^2 b_u\right) G_{u_x}.
\end{aligned} \tag{9}$$

A direct analysis of system (9) is also not possible.

The principal result [16] of group classification of equations (8) is the following:

Proposition 2. *Equation (8) admits a Lie symmetry algebra of dimension greater than 4 if it is equivalent to one of the following equations:*

1. $u_t = u^{-4} u_{xx} - 2u^{-5} u_x^2;$
2. $u_t = u_{xx} + x^{-1} uu_x - x^{-2} u^2 - 2x^{-2} u;$
3. $u_t = \exp(u_x) u_{xx};$
4. $u_t = u_x^n u_{xx}, \quad n \geq -1, \quad n \neq 0;$
5. $u_t = \exp(n \arctan u_x)\left(1 + u_x^2\right)^{-1} u_{xx}, \quad n \geq 0.$

These equations are invariant under five-dimensional Lie algebras.

Note that equation 1) is equivalent to the equation obtained by Ovsiannikov [1], equation 2) is equivalent to Burgers equation, and equations 3)–5) was obtained by Akhatov, Gazizov and Ibragimov [3].

Acknowledgements

The authors gratefully acknowledge Prof. R.Z. Zhdanov's collaboration in this research.

[1] Ovsiannikov L.V., Group properties of nonlinear heat equation, *Dokl. AN SSSR*, 1959, V.125, N 3, 492–495 (in Russian).

[2] Ovsiannikov L.V., Group analysis of differential equations, New York, Academic Press, 1982.

[3] Akhatov I.S., Gazizov R.K. and Ibragimov N.K., Group classification of equation of nonlinear filtration, *Dokl. AN SSSR*, 1987, V.293, 1033–1035 (in Russian).

[4] Dorodnitsyn V.A., On invariant solutions of non-linear heat equation with a sourse, *Zhurn. Vych. Matemat. Matemat. Fiziki*, 1982, V.22, 1393–1400 (in Russian).

[5] Oron A. and Rosenau P., Some symmetries of the nonlinear heat and wave equations, *Phys. Lett. A.*, 1986, V.118, 172–176.

[6] Edwards M.P., Classifical symmetry reductions of nonlinear diffusion-convection equations, *Phys. Lett. A.*, 1994, V.190, 149–154.

[7] Cherniha R. and Serov M., Symmetries ansätze and exact solutions of nonlinear second-order evolution equations with convection terms, *Euro. J. of Appl. Math.*, 1998, V.9, 527–542.

[8] Gandarias M.L., Classifical point symmetries of a porous medium equation, *J. Phys. A: Math. Gen.*, 1996, V.29, 607–633.

[9] Zhdanov R.Z. and Lahno V.I., Group classification of heat conductivity equations with a nonlinear sourse, *J. Phys. A: Math. Gen.*, 1999, V.32, 7405–7418.

[10] Mubarakzyanov G.M., On solvable Lie algebras, *Izv. Vys. Ucheb. Zaved.*, 1963, V.32, N 1, 114–123 (in Russian).

[11] Mubarakzyanov G.M., The classification of the real structure of five-dimensional Lie algebras, *Izv. Vys. Ucheb. Zaved.*, 1963, V.34, N 3, 99–105 (in Russian).

[12] Turkowski P., Solvable Lie algebras of dimensional six, *J. Math. Phys.*, 1990, V.31, 1344–1350.

[13] Turkowski P., Low-dimensional real Lie algebras, *J. Math. Phys.*, 1988, V.29, 2139–2144.

[14] Lahno V. and Onyshchenko A., The Ovsiannikov's theorem on group classification of a linear hyperbolic type partial differential equation revisited, in Proceedings of Third International Conference "Symmetry in Nonlinear Mathematical Physics" (12–18 July, 1999, Kyiv), Editors A.G. Nikitin and V.M. Boyko, Kyiv, Institute of Mathematics, 2000, V.30, Part 1, 141–145.

[15] Ovsiannikov L.V., Group characteristics of S.A. Chaplygin equations, *J. Prikl. Mech. and Techn. Fiz.*, 1960, N 3, 126–145 (in Russian).

[16] Basarab–Horwath P., Lahno V. and Zhdanov R., The structure of Lie algebras and the classification problem for partial differential equation, *Acta Appl. Math.*, 2001, V.69, N 1, 43–94.

[17] Basarab–Horwath P., Lahno V. and Zhdanov R., Classifying evolution equations, *Nonlinear Analysis: Theory, Methods & Applications*, 2001, V.47, N 8, 5135–5144.