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P. M. NAGHDI'S NOTES ON CONTINUUM MECHANICS

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A List of General References

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Table of Contents

ME185

Part I: Kinematics

1. Bodies, configurations, motions, mass, and mass density.

A body is a set of particles. Sometimes in the literature a body *B* is referred to as a manifold of particles. Particles of a body *B* will be designated as X (see Fig. 1.1) and may be identified by any convenient system of labels, such as for example a set of colors. In mechanics the body is assumed to be smooth and, by assumption, can be put into correspondence with a domain of Euclidean space. Thus, by assumption, a particle X can be put into a one-to-one correspondence with the triples of real numbers X_1,X_2,X_3 in a region of Euclidean 3-space E^3. The mapping from the body manifold onto the domain of E^3 is assumed to be one-to-one, invertible, and differentiable as many times as desired; and for most purposes two or three times suffice.

Each part (or subset) *S* of the body *B* at each instant of time is assumed to be endowed with a positive measure $M(S)$, i.e., a real number > 0, called the *mass of the part S* and the whole body is assumed to be endowed with a nonnegative measure $M(B)$ called the *mass* of the body. We return to a further consideration of mass later in this section.

Bodies are seen only in their configurations, i.e., the regions of E^3 they occupy at each instant of time t ($-\infty < t < +\infty$). These configurations should not be confused with the bodies themselves.

Consider a configuration of the body *B* at time t in which configuration *B* occupies a region *R* in E^3 bounded by a closed surface ∂R (Fig. 1.2). Let **x** be the position vector of the *place* occupied by a typical particle X at time t. Since the body can be mapped smoothly onto a domain of E^3, we write

$$\mathbf{x} = \bar{\chi}(X,t) \ . \tag{1.1}$$

In (1.1), X refers to the particle, t is the time, **x** [the value of the function $\bar{\chi}$] is the *place* occupied by the particle X at time t and the mapping function $\bar{\chi}$ is assumed to be differentiable as many times as desired both with respect to X and t. Also, for each t, (1.1) is assumed to be invertible so that

$$X = \bar{\chi}^{-1}(\mathbf{x},t) \ , \tag{1.2}$$

where the symbol χ^{-1} designates the inverse mapping. The mapping (1.1) is called a *motion* of the body *B*. The description of motion in (1.1) is similar to that in particle mechanics and is traditionally referred to as the *material* description.

During a motion of the body *B*, a particle X (by occupying successive points in E^3) moves through space and describes a path C ; the equation of this path is parametrically represented by (1.1). The rate of change of place with time as X traverses C is called the velocity and is tangent to the curve C. Similarly, the second rate of change of place with time as X traverses C is the acceleration. Thus, in the material description of the motion, the velocity **v** and acceleration **a** are defined by

$$\mathbf{v} = \dot{\mathbf{x}} = \frac{\partial\bar{\chi}(X,t)}{\partial t} \quad , \quad \mathbf{a} = \ddot{\mathbf{x}} = \frac{\partial^2\bar{\chi}(X,t)}{\partial t^2} \ . \tag{1.3}$$

In the above formulae, the particle velocity **v** is the partial derivative of the function $\bar{\chi}$ with respect to t holding X fixed; and, similarly, the particle acceleration **a** is the second partial derivative of $\bar{\chi}$ with respect to t holding X fixed. Also, the superposed dot, which designates the material time differentiation, can be utilized in conjunction with any function f and signifies differentiation with respect to t holding X fixed.

Reference configuration. Often it is convenient to select one particular configuration and refer everything concerning the body and its motion to this configuration. The reference configuration need not necessarily be a configuration that is actually occupied by the body in any of its motions. In particular, the reference configuration need not be the *initial* configuration of the body.

Let κ_R be a reference configuration of *B* in which **X** is the position vector of the place occupied by the particle X (Fig. 1.3). Then, the mapping from X to the place **X** in the configuration κ_R may be written as

$$\mathbf{X} = \kappa_R (X) \tag{1.4}$$

which specifies the position vector **X** occupied by the particle X in the reference configuration κ_R. The mapping (1.4) is assumed to be differentiable as many times as desired and invertible. The inverse mapping is

$$X = \kappa_R^{-1} (\mathbf{X}) \ . \tag{1.5}$$

Then, the mapping (1.1) at time t from the place **x** to the particle X may be expressed in terms of (1.5), i.e.,

$$\mathbf{x} = \bar{\chi} [\kappa_R^{-1} (\mathbf{X}),t] = \bar{\chi}_{\kappa_R} (\mathbf{X},t) \ . \tag{1.6}$$

The second of (1.6), which involves the function $\bar{\chi}_{\kappa_R}$ (with a subscript κ_R, emphasizes that the motion described in this manner represents a sequence of mappings of the reference configuration κ_R. In the future, however, for simplicity’s sake we omit the subscript κ_R and write $(1.6)_2$ and its inverse as

$$\mathbf{x} = \chi(\mathbf{X},t) \quad , \quad \mathbf{X} = \chi^{-1}(\mathbf{x},t) \tag{1.7}$$

with the understanding that **X** in the argument (1.6) is the position vector of X in the reference configuration $\kappa_{R.}$ Returning to $(1.6)_2$, we observe that for each κ_R a different function $\bar{\chi}_{\kappa_R}$ results; and the choice of reference configuration, similarly to the choice of coordinates, is arbitrary and is introduced for convenience.

A necessary and sufficient condition for the invertibility of the mappings of $(1.7)_{1,2}$ is that the determinant J of the transformation from **X** to **x** be nonzero, i.e.,

$$J = \det(\frac{\partial \chi}{\partial \mathbf{X}}) = \det(\frac{\partial \chi_i}{\partial X_A}) \neq 0 \ . \tag{1.8}$$

Description of the motion. There are several methods of describing the motion of a body. Here, we describe three but note that, because of our smoothness assumptions, they are all equivalent.

It was noted earlier that the description utilized in (1.1) is the material description. In such a description one deals with abstract particles X which together with time t are the independent variables.

A description such as (1.7) in which the position $\mathbf{X}$ of X at some time, e.g., t = 0, is used as a label for the particle X is called the *referential* description. In the referential description, which is also known as *Lagrangian*, the independent variables are $\mathbf{X}$ and t. For some purposes, it is convenient to display and to utilize (1.7) in terms of its rectangular Cartesian coordinates. Thus, let $\mathbf{E}_K$ be constant orthonormal basis vectors associated with the reference configuration and similarly denote by $\mathbf{e}_k$ constant orthonormal basis vectors associated with the present configuration at time t. Then, the positions $\mathbf{X}$ and $\mathbf{x}$ referred to $\mathbf{E}_K$ and $\mathbf{e}_k$, respectively, are

$$\mathbf{X} = X_K \mathbf{E}_K \quad , \quad \mathbf{x} = x_k \mathbf{e}_k \tag{1.9}$$

where X_K are the rectangular Cartesian coordinates of the position $\mathbf{X}$ and similarly x_k are the rectangular Cartesian coordinates of the position $\mathbf{x}$. Referred to the basis $\mathbf{e}_k$, the component form of $(1.7)_1$ can be displayed as

$$x_k = \chi_k(X_K,t) \ , \tag{1.10}$$

where without loss in generality (since $\mathbf{E}_K$ are constant orthonormal basis) we have also replaced the argument $\mathbf{X}$ by its components X_K. In the future we often display the various formulae both in their "coordinate-free" forms, as well as their component forms.

The particle velocity and acceleration in a material description of motion are given, respectively, by $(1.3)_{1,2}$. The corresponding expressions for velocity and acceleration in the referential description (1.7) will have the same forms as $(1.3)_{1,2}$ but with the argument X replaced by $\mathbf{X}$:

$$\mathbf{v} = \dot{\mathbf{x}} = \frac{\partial \chi}{\partial t}(\mathbf{X},t) \quad , \quad \mathbf{a} = \ddot{\mathbf{x}} = \frac{\partial^2 \chi}{\partial t^2}(\mathbf{X},t) \ . \tag{1.11}$$

So far we have introduced two descriptions of motion. The third is called the *spatial* description in which attention is centered in the present configuration, i.e., the region of space occupied by the body at the present time t. In the spatial description, which is also known as *Eulerian,* the independent variables are the place **x** and the time t. To elaborate further, consider an entity f defined by

$$f = \hat{f}(X,t) \ . \tag{1.12}$$

Since (1.1) is invertible in the form (1.2), the function $\hat{f}(X,t)$ can be expressed as a different function of **x**,t, i.e.,

$$\hat{f}(X,t) = \hat{f}(\bar{\chi}^{-1}(\mathbf{x},t),t) = \tilde{f}(\mathbf{x},t) \ . \tag{1.13}$$

Moreover, the function $\tilde{f}$ is unique. It is clear that with the use of the transformation of the form (1.13), the velocity and acceleration in $(1.3)_{1,2}$ can be expressed in terms of different functions of **x**,t:

$$\mathbf{v} = \hat{\mathbf{v}}(X,t) = \tilde{\mathbf{v}}(\mathbf{x},t) = \tilde{v}_k \, \mathbf{e}_k \quad , \quad \mathbf{a} = \hat{\mathbf{a}}(X,t) = \tilde{\mathbf{a}}(\mathbf{x},t) = \tilde{a}_k \, \mathbf{e}_k \ . \tag{1.14}$$

In the above spatial description, the spatial form $\tilde{f}(\mathbf{x},t)$ on the right-hand side of (1.13) was obtained from a representation (1.12) in which $\hat{f}$ is a function of particle X and t. Analogous results hold if we start with $f = \hat{f}(\mathbf{X},t)$ and then use $(1.7)_2$ to obtain

$$f = \hat{f}(\mathbf{X},t) = \tilde{f}(\mathbf{x},t) \ . \tag{1.15}$$

Material derivative. Consider again a function such as $\hat{f}(X,t)$ in (1.12). The partial derivative of this function with respect to t holding the variable X fixed, denoted by $\dot{f}$, is called the *material derivative* of f. But f, with the use of $(1.13)_2$, can also be expressed as a different function of $\mathbf{x}$,t. Thus, using the chain rule for differentiation, we have

$$\dot{f} = \frac{\partial \hat{f}}{\partial t}\Big|_X = \frac{\partial \tilde{f}}{\partial \mathbf{x}} \cdot \frac{\partial \bar{\chi}}{\partial t} + \frac{\partial \tilde{f}}{\partial t} \ , \tag{1.16}$$

where on the left-hand side of the above we have temporarily emphasized that the variable X is held fixed when calculating the partial derivative of $\hat{f}$ with respect to t. With the use of the first of $(1.3)_1$, we may rewrite $(1.16)_2$ as

$$\dot{f} = \frac{\partial \hat{f}(\mathbf{X},t)}{\partial t}\Big|_X = \frac{\partial \tilde{f}}{\partial t} + \frac{\partial \tilde{f}}{\partial \mathbf{x}} \cdot \tilde{\mathbf{v}} = \frac{\partial \tilde{f}}{\partial t} + \tilde{v}_k \frac{\partial \tilde{f}}{\partial x_k} \tag{1.17}$$

which is the material derivative of $\tilde{f}(\mathbf{x},t)$. The results of the type (1.15) and (1.17) are applicable to any scalar-valued, vector-valued or tensor-valued field. In particular, consider the referential form of the velocity vector $\mathbf{v}$ which by $(1.11)_1$ and transformation of the type (1.15) is

$$\mathbf{v} = \hat{\mathbf{v}}(\mathbf{X},t) = \tilde{\mathbf{v}}(\mathbf{x},t) \ . \tag{1.18}$$

Then, the acceleration or the material derivative of $\mathbf{v}$ where these are regarded as functions of $\mathbf{x}$,t are

$$\mathbf{a} = \dot{\mathbf{v}} = \frac{\partial \tilde{\mathbf{v}}}{\partial t} + \frac{\partial \tilde{\mathbf{v}}}{\partial \mathbf{x}} \tilde{\mathbf{v}} = \tilde{\mathbf{a}}(\mathbf{x},t) \ . \tag{1.19}$$

Material curve, material surface and material volume. Let a body B with material points (or particles) X in a fixed reference configuration κ_R occupy a region R_R in E^3 bounded by a closed surface ∂R_R. Any subset S_R of B in E^3 will be designated by P_R ($\subseteq R_R$) bounded by a closed boundary surface ∂P_R. Because of the nature of the smoothness assumption imposed on the mapping $(1.7)_1$ from the reference configuration κ_R to the current configuration κ of B at time t, all curves, surfaces and volumes in κ_R are carried by the motion χ into curves, surfaces and volumes in the configuration κ. Thus, the region R_R with the closed boundary surface ∂R_R is mapped into a region R with corresponding closed boundary surface ∂R in κ. Moreover, the parts P_R ($\subseteq R_R$) bounded by ∂P_R is mapped into a part P ($\subseteq R$) bounded by a closed surface ∂P. The part P is occupied by the same material points (or particles) as those which occupied P_R and such a region or volume is called a *material volume.*

A surface in the reference configuration κ_R is defined by equations of the form

$$F(\mathbf{X}) = 0 \quad \text{or} \quad \mathbf{X} = \mathbf{X}(U_1,U_2) \ , \tag{1.20}$$

where U_1,U_2 are parametric variables on the surface. With the use of $(1.7)_2$, the surface (1.20) can be mapped into a corresponding surface

$$f(\mathbf{x},t) = F(\chi^{-1}(\mathbf{x},t)) = 0 \quad \text{or} \quad \mathbf{x}(U_1,U_2,t) = \chi[\mathbf{X}(U_1,U_2),t] \tag{1.21}$$

in the configuration κ at time t. The surface (1.21) is called a *material surface*, since its material points (or particles) are the same as those of the surface (1.20).

A curve in the reference configuration may be regarded as the intersection of two surfaces of the form

$$F(\mathbf{X}) = 0 \quad , \quad G(\mathbf{X}) = 0 \ , \tag{1.22}$$

and is defined by

$$\mathbf{X} = \mathbf{X}(U) \ , \tag{1.23}$$

where U is a parameter and can be identified with the arc length. The intersection of the surfaces (1.22) can be mapped into a curve in the configuration κ which is the intersection of the surfaces

$$f(\mathbf{x},t) = F(\chi^{-1}(\mathbf{x},t)) = 0 \quad , \quad g(\mathbf{x},t) = G(\chi^{-1}(\mathbf{x},t)) = 0 \tag{1.24}$$

or

$$\mathbf{x}(\mathrm{U},t) = \chi[\mathbf{X}(\mathrm{U}),t] \ . \tag{1.25}$$

The curve (1.25) is called a *material curve*, since its material points (particles) are the same as those of the curve (1.23).

Lagrange's criterion for a material surface. A surface $f(\mathbf{x},t) = 0$ is said to be material if and only if the material derivative of f vanishes, i.e.,

$$\dot{f} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \mathbf{x}} \cdot \mathbf{v} = 0 \ . \tag{1.26}$$

First we write $f(\mathbf{x},t) = F(\mathbf{X},t)$ with the use of $(1.7)_1$ and suppose that $f(\mathbf{x},t)$ is material. Then, $F(\mathbf{X},t)$ must be independent of t and hence $\dot{F} = 0$ and this implies that $\dot{f} = 0$.

Next, suppose that $\dot{f} = 0$. Then, it follows that $\dot{F} = 0$ and this implies that F must be independent of t and a function of $\mathbf{X}$ only so that $F(\mathbf{X}) = 0$ and after using $(1.7)_2$ we have the result that $F(\chi^{-1}(\mathbf{x},t)) = f(\mathbf{x},t) = 0$ is material.

In a similar manner, we may establish that the intersection of surfaces $f(\mathbf{x},t) = 0$ and $g(\mathbf{x},t) = 0$ is a material curve if and only if

$$\dot{f} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \mathbf{x}} \cdot \mathbf{v} = 0 \quad , \quad \dot{g} = \frac{\partial g}{\partial t} + \frac{\partial g}{\partial \mathbf{x}} \cdot \mathbf{v} = 0 \ . \tag{1.27}$$

Mass density. Before closing this section, consider again a configuration κ of the body B at time t in which B occupies a region of E^3 R bounded by a closed surface ∂R and recall that a part (or a subset) of B will be denoted by S. The part S in a neighborhood of the place $\mathbf{x}$ in κ occupies a region of space P ($\subseteq R$) bounded by a closed surface ∂P. We introduce explicitly the mass $M(S)$ of the part and the mass $M(B)$ of the whole body at time t. These measures are non-negative and may be functions of t. Assuming that the measure $M(S)$ is absolutely continuous, then the limit

$$\rho \;=\; \lim_{v\to 0} \frac{M(S_t)}{v} \tag{1.28}$$

exists, where v = v(*P*) is the volume of the region of space *P*. The scalar field $\rho = \rho(\mathbf{x},t)$ is called the *mass density* of the body in the configuration at time t. The mass density is nonnegative and is a function of both **x** and t. The mass of the part *S* of the body and the mass of the whole body, both at time t, can be expressed in terms of the mass density ρ:

$$M(S_t) \;=\; \int_P \rho\, dv \quad , \quad M(B_t) \;=\; \int_R \rho\, dv \;, \tag{1.29}$$

where the subscript t attached to *S* and *B* emphasizes that the left-hand sides of $(1.29)_{1,2}$ refer to the masses of *S* and *B* in the configuration κ at time t and dv is an element of volume in this configuration, the regions of integration being over *P* and *R* in E^3. Similarly, we define the mass density ρ_R in a reference configuration and write

$$M(S_R) \;=\; \int_{P_R} \rho_R\, dV_R \quad , \quad M(B_R) \;=\; \int_{R_R} \rho_R\, dV_R \;, \tag{1.30}$$

where dV_R denotes an element of volume in the reference configuration and P_R, R_R ($P_R \subseteq R_R$) are the regions of space occupied by the part and the whole body in the reference configuration. If the reference configuration is identified with the initial configuration at time t_o, then instead of (1.30) we write

$$M(S_o) \;=\; \int_{P_o} \rho_o\, dV \quad , \quad M(B_o) \;=\; \int_{R_o} \rho_o\, dV \;, \tag{1.31}$$

where ρ_o and dV are, respectively, the mass density and the volume element in the initial configuration while P_o and R_o ($P_o \subseteq R_o$) are the regions of space occupied, respectively, by the part and the whole body in the initial configuation.

It should be emphasized that, at this stage in our development, the mass of a part of the body and that of the whole body depend on the particular configuration occupied by the body.

2. Deformation gradient. Measures of strain. Rotation and stretch.

Henceforth we *identify* the particle (material point) X by its position vector **X** in a fixed reference configuration, which (for convenience) we take to be coincident with the initial configuration κ_o. The deformation gradient (also called displacement gradient) **F** relative to the reference position is defined by

$$\mathbf{F} = \mathbf{F}(\mathbf{X},t) = \frac{\partial\chi(\mathbf{X},t)}{\partial\mathbf{X}} \quad , \quad J = \det\mathbf{F} \neq 0 \ . \tag{2.1}$$

The components of **F** are designated by F_{iA} or sometimes written as $x_{i,A}$ with a comma designating partial differentiation. They are given by

$$F_{iA} = \frac{\partial\chi_i(X_B,t)}{\partial X_A} \tag{2.2}$$

and may be regarded as a 3×3 square matrix.

At a fixed time t, line elements d**x** are related to line elements d**X** in κ_o by

$$d\mathbf{x} = \mathbf{F}\,d\mathbf{X} \quad \text{or} \quad dx_i = x_{i,A}\,dX_A \ . \tag{2.3}$$

The deformation gradient **F** in (2.3) describes the *local* deformation of a particle (material point) whose position vector is **X** in κ_o. In other words, (2.3) is a linear transformation of a small neighborhood of **X** from the reference configuration κ_o into the current configuration κ at time t.

Let the magnitudes of d**X** and d**x** be denoted by dS and ds, respectively; and let **M** and **m** be the unit vectors in the directions of d**X** and d**x**, respectively. Then,

$$\begin{aligned} d\mathbf{X} &= \mathbf{M}dS \quad , \quad \mathbf{M}\cdot\mathbf{M} = 1 \ , \\ d\mathbf{x} &= \mathbf{m}\,ds \quad , \quad \mathbf{m}\cdot\mathbf{m} = 1 \ , \end{aligned} \tag{2.4}$$

or in component forms

$$dX_A = M_A \, dS \quad , \quad M_A M_A = 1 \ ,$$
$$dx_i = m_i \, ds \quad , \quad m_i m_i = 1 \ . \tag{2.5}$$

Measures of strain.

In general, the material line element d**X** undergoes both stretch and rotation. The ratio ds/dS , denoted by λ, is called the stretch of the line element, i.e.,

$$\lambda = \frac{ds}{dS} \ . \tag{2.6}$$

It then follows from (2.3)-(2.5) that

$$\mathbf{m} \, ds = d\mathbf{x} = \mathbf{F} \, d\mathbf{X} = \mathbf{F} \, \mathbf{M} \, dS \ ,$$
$$\text{or} \quad m_i \, ds = dx_i = x_{i,A} \, dX_A = x_{i,A} \, M_A \, dS \ . \tag{2.7}$$

Next, using (2.6), from (2.7) we have

$$\lambda \mathbf{m} = \mathbf{F} \, \mathbf{M} \quad \text{or} \quad \lambda m_i = x_{i,A} \, M_A \ . \tag{2.8}$$

By taking the scalar product of each side of (2.8) with itself we obtain

$$\lambda^2 = \mathbf{F} \, \mathbf{M} \cdot \mathbf{F} \, \mathbf{M} = \mathbf{M} \cdot \mathbf{C} \, \mathbf{M} \ , \tag{2.9}$$

where in writing $(2.9)_3$ we have also defined the tensor **C** by

$$\mathbf{C} = \mathbf{F}^T \, \mathbf{F} \ . \tag{2.10}$$

The component forms of the equations (2.9)-(2.10) are:

$$\lambda^2 = C_{AB} \, M_A \, M_B \ , \tag{2.9a}$$

and

$$C_{AB} = x_{i,A} \, x_{i,B} \ . \tag{2.10a}$$

The symmetric tensor **C** (with components C_{AB}) is called the *right Cauchy-Green tensor*.

Clearly, for a material line element which coincides with d**X** in the reference configuration κ_o, the value of the stretch λ in the direction **M** can be calculated from (2.9) once **C** is known. Alternatively, λ can be calculated in terms of the deformed line element d**x** in the current configuration κ. To see this, since **F** is invertible, we rewrite (2.3) as

$$\mathbf{F}^{-1}\, d\mathbf{x} = d\mathbf{X} \quad \text{or} \quad X_{A,i}\, dx_i = dX_A \ , \tag{2.11}$$

where the inverse function $\mathbf{F}^{-1}$ using $(1.7)_2$ is defined by

$$\mathbf{F}^{-1} = \frac{\partial \chi^{-1}}{\partial \mathbf{x}} \quad \text{or} \quad X_{A,i} = \frac{\partial \chi^{-1}_A}{\partial x_i} \ . \tag{2.12}$$

Then, from (2.4), (2.5), (2.11) and (2.12) we have

$$\begin{aligned} &\mathbf{M}\, dS = d\mathbf{X} = \mathbf{F}^{-1}\, d\mathbf{x} = \mathbf{F}^{-1}\, \mathbf{m}\, ds \ , \\ \text{or} \quad &M_A\, dS = dX_A = X_{A,i}\, dx_i = X_{A,i}\, m_i\, ds \ , \end{aligned} \tag{2.13}$$

which with the use of (2.6) can be written in the form

$$\lambda^{-1}\, \mathbf{M} = \mathbf{F}^{-1}\, \mathbf{m} \quad \text{or} \quad \lambda^{-1}\, M_A = X_{A,i}\, m_i \ . \tag{2.14}$$

The result (2.14) can also be obtained immediately by inverting (2.8). By taking the scalar product of each side of (2.14) with itself we obtain

$$\lambda^{-2} = \mathbf{F}^{-1}\mathbf{m} \cdot \mathbf{F}^{-1}\mathbf{m} = \mathbf{m} \cdot (\mathbf{F}^{-1})^T\, \mathbf{F}^{-1}\, \mathbf{m} = \mathbf{m} \cdot \mathbf{B}^{-1}\, \mathbf{m} \ , \tag{2.15}$$

where in writing $(2.15)_3$ we have also defined the tensor **B** by

$$\mathbf{B} = \mathbf{F}\,\mathbf{F}^T \ . \tag{2.16}$$

The component forms of equations (2.15)-(2.16) are:

$$\lambda^{-2} = c_{ij}\, m_i\, m_i \ , \tag{2.15a}$$

$$c_{ij} = b^{-1}_{ij} = X_{A,i}\, X_{A,j} \quad , \quad b_{ij} = x_{i,A}\, x_{j,A} \ , \tag{2.16a}$$

where the notation c_{ij} designates the components of the inverse of the tensor **B**. The symmetric tensor **B** whose components are given in (2.16a) is called the *left Cauchy-Green tensor.* The tensors **C** and **B** represent, respectively, the referential (or Lagrangian) and the spatial (or Eulerian) descriptions of strain. When the motion is rigid, both tensors become identity tensors, i.e., $\mathbf{C} = \mathbf{B} = \mathbf{I}$, where **I** is the unit tensor.

For some purposes, it is convenient to employ relative measures of strain such that these measures vanish when the motion is rigid. Recalling (2.9) and (2.15) we may write

$$\begin{gathered} \lambda^2 - 1 = \mathbf{M} \cdot (\mathbf{C} - \mathbf{I})\mathbf{M} = (C_{AB} - \delta_{AB}) M_A\, M_B \\ 1 - \lambda^{-2} = \mathbf{m} \cdot (\mathbf{I} - \mathbf{B}^{-1})\, \mathbf{m} = (\delta_{ij} - b^{-1}_{ij})\, m_i\, m_j \end{gathered} \tag{2.17}$$

or equivalently,

$$\begin{gathered} \lambda^2 - 1 = 2\mathbf{M} \cdot \mathbf{E}\, \mathbf{M} = 2E_{AB}\, M_A\, M_B \ , \\ 1 - \lambda^{-2} = 2\mathbf{m} \cdot \bar{\mathbf{E}}\, \mathbf{m} = 2e_{ij}\, m_i\, m_j \ , \end{gathered} \tag{2.18}$$

where

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}) \quad \text{or} \quad E_{AB} = \frac{1}{2}(C_{AB} - \delta_{AB}) \ , \tag{2.19}$$

$$\bar{\mathbf{E}} = \frac{1}{2}(\,\mathbf{I} - \mathbf{B}^{-1}) \quad \text{or} \quad e_{ij} = \frac{1}{2}(\delta_{ij} - c_{ij}) \ . \tag{2.20}$$

The relative measure **E** defined by (2.19) is known as the relative Lagrangian strain, and the corresponding relative spatial measure of strain $\bar{\mathbf{E}}$ is defined by (2.20).

Rotation and stretch.

The deformation gradient **F** in (2.3) describes the local deformation of a material line element at **X** from the reference configuration κ_o to the current configuration κ; it involves, in general, both rotation and stretch. Now, since by (1.8) the deformation gradient **F** is nonsingular, using the polar decomposition theorem it may be expressed in the polar forms

$$\mathbf{F} = \mathbf{R}\,\mathbf{U} = \mathbf{V}\,\mathbf{R} \quad \text{or} \quad x_{i,A} = R_{iB}\,U_{BA} = V_{ij}\,R_{jA} \ , \tag{2.21}$$

where **U** and **V** are positive definite symmetric tensors called, respectively, the right and left stretch tensors and **R** is the proper orthogonal tensor satisfying

$$\mathbf{R}\,\mathbf{R}^T = \mathbf{R}^T\,\mathbf{R} = \mathbf{I} \qquad \det \mathbf{R} = 1 \ , \tag{2.22}$$

or in component form

$$R_{iA}\,R_{iB} = \delta_{AB} \quad \text{or} \quad R_{iA}\,R_{jA} = \delta_{ij} \ . \tag{2.22a}$$

The effect of (2.21) is to replace the linear transformation (2.3) by two linear transformations: Either by

$$d\mathbf{X}' = \mathbf{U}\,d\mathbf{X} \ \text{ and } \ d\mathbf{x} = \mathbf{R}\,d\mathbf{X}' \quad \text{or} \quad dX'_B = U_{BA}\,dX_A \ \text{ and } \ dx_i = R_{iB}\,dX'_B \ , \tag{2.23}$$

or by

$$d\mathbf{x}' = \mathbf{R}\,d\mathbf{X} \ \text{ and } \ d\mathbf{x} = \mathbf{V}\,d\mathbf{x}' \quad \text{or} \quad dx'_j = R_{jA}\,dX_A \ \text{ and } \ dx_i = V_{ij}\,dx'_j \ . \tag{2.24}$$

Associated with each of the stretch tensors **U**, **V** are three positive principal values (eigenvalues) and three orthogonal principal directions (eigenvectors). These will be discussed later.

Here, it should be emphasized that, in general, **F**, **R**, **U**, **V** are all functions of **X** and t and they can vary from one material point to another during motion. In the remainder of this section, we provide physical interpretations for the decompositions (2.23) and (2.24).

(i) **Stretch followed by pure rotation.** In this case, let the deformation of d**X** into d**X′** involve only pure stretch and the deformation from d**X′** into d**x** be one of pure rotation. By $(2.21)_1$ and (2.23), as well as (2.9)-(2.10), we have

$$d\mathbf{x} \cdot d\mathbf{x} = \mathbf{R}\, d\mathbf{X}' \cdot \mathbf{R}\, d\mathbf{X}' = d\mathbf{X}' \cdot d\mathbf{X}' \ , \tag{2.25a}$$

or

$$dx_i\, dx_i = R_{iA}\, R_{iB}\, dX_A'\, dX_B' = \delta_{AB}\, dX_A'\, dX_B' = dX_A'\, dX_A' \ , \tag{2.25b}$$

and

$$\mathbf{C} = \mathbf{F}^T \mathbf{F} = (\mathbf{R}\,\mathbf{U})^T (\mathbf{R}\,\mathbf{U}) = \mathbf{U}^2 \ , \tag{2.26}$$

or

$$C_{AB} = x_{i,A}\, x_{i,B} = R_{iK}\, U_{KA}\, R_{iL}\, U_{LB} = \delta_{KL}\, U_{KA}\, U_{LB} = U_{KA}\, U_{KB} \ , \tag{2.27a}$$

$$\lambda^2 = U_{KA}\, U_{KB}\, M_A\, M_B = C_{AB}\, M_A\, M_B \ . \tag{2.27b}$$

According to (2.25), the length of the line element d**X′** is the same as the length of d**x** so that all the stretching is represented by **U** in (2.26). However, line elements are generally also rotated by the action of **U**. The part of the deformation described by **R** is a pure rotation.

(ii) **Pure rotation followed by stretch.** In this case, let the deformation of d**X** into d**x′** be one of pure rotation and the deformation from d**x′** to d**x** involve only stretch and rotation. By (2.21) and (2.24), as well as (2.15)-(2.16), we have

$$d\mathbf{x}' \cdot d\mathbf{x}' = \mathbf{R}\, d\mathbf{X} \cdot \mathbf{R}\, d\mathbf{X} = \mathbf{R}^T \mathbf{R}\, d\mathbf{X} \cdot d\mathbf{X} = d\mathbf{X} \cdot d\mathbf{X} \ , \tag{2.28}$$

or

$$dx_i' \, dx_i' \; = \; R_{iA} \, R_{iB} \, dX_A \, dX_B \; = \; \delta_{AB} \, dX_A \, dX_B \; = \; dX_A \, dX_A \tag{2.29}$$

and

$$\mathbf{B} \; = \; \mathbf{F} \, \mathbf{F}^T \; = \; (\mathbf{V} \, \mathbf{R}) \, (\mathbf{V} \, \mathbf{R})^T \; = \; \mathbf{V}^2 \;\; , \tag{2.30a}$$

$$\frac{1}{\lambda^2} \; = \; \mathbf{m} \cdot \mathbf{B}^{-1} \, \mathbf{m} \;\; , \tag{2.30b}$$

or

$$b_{ij} \; = \; x_{i,A} \, x_{j,A} \; = \; V_{ir} \, R_{rA} \, V_{js} \, R_{sA} \; = \; V_{ir} \, V_{jr} \tag{2.31a}$$

$$\lambda^{-2} \; = \; (\mathbf{B}^{-1})_{ij} \, m_i \, m_j \;\; . \tag{2.31b}$$

According to (2.28), the length of the line element d$\mathbf{x}'$ is the same as the length of d$\mathbf{X}$ so that the deformation described by $\mathbf{R}$ is one of pure rotation.

Keeping in mind the decomposition (2.21), it is evident from (2.26) and (2.30a) that $\mathbf{U} \; = \; (\mathbf{F}^T \, \mathbf{F})^{1/2}$ and $\mathbf{V} \; = \; (\mathbf{F} \, \mathbf{F}^T)^{1/2}$ involve the square root operation which often is difficult to execute and this is the main reason for the choice of $\mathbf{U}^2 \; = \; \mathbf{C}$ and $\mathbf{V}^2 \; = \; \mathbf{B}$ as the strain tensors. Moreover, since $\mathbf{U}^2$ is symmetric positive definite, it possesses a unique symmetric positive definite square root; a parallel remark applies to the square root of $\mathbf{V}^2$. In this regard the expressions for the square root of the stretch λ and its inverse, in the forms (2.9) and (2.15), are especially noteworthy.

Secs. 3 & 4. Further developments of kinematical results.

5. Velocity gradient. Rate of deformation. Vorticity.

Consider the spatial (Eulerian) form of the particle velocity in the form given by (1.14), $\mathbf{v} = \tilde{\mathbf{v}}(\mathbf{x},t)$ and define the spatial velocity gradient tensor by

$$\mathbf{L} = \text{grad}\ \tilde{\mathbf{v}}(\mathbf{x},t) \tag{5.1}$$

or

$$\frac{\partial \mathbf{v}}{\partial \mathbf{x}} = v_{i,j}\ \mathbf{e}_i \otimes \mathbf{e}_j \quad , \quad v_{i,j} = \frac{\partial v_i}{\partial x_j} \ , \tag{5.2}$$

where $v_{i,j}$ are the components of $\mathbf{L}$ referred to the orthonormal basis $\mathbf{e}_i$ and in recording (5.2) we have used the same symbol for the function $\mathbf{v}$ and its value. The tensor $\mathbf{L}$ can be uniquely decomposed into its symmetric part $\mathbf{D}$ and its skew-symmetric part $\mathbf{W}$ (see Appendix L, Sect. 8) as follows:

$$\mathbf{L} = \mathbf{D} + \mathbf{W} \ , \tag{5.3}$$

where

$$\mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T) = d_{ij}\ \mathbf{e}_i \otimes \mathbf{e}_j \quad , \quad \mathbf{W} = \frac{1}{2}(\mathbf{L} - \mathbf{L}^T) = w_{ij}\ \mathbf{e}_i \otimes \mathbf{e}_j \ . \tag{5.4}$$

The corresponding component forms of (5.3) and (5.4) are:

$$v_{i,j} = d_{ij} + w_{ij} \ , \tag{5.3a}$$

where

$$d_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}) = d_{ji} \quad , \quad w_{ij} = \frac{1}{2}(v_{i,j} - v_{j,i}) = -\, w_{ji} \ . \tag{5.4a}$$

The tensor $\mathbf{D}$ with components d_{ij} is called the *rate of deformation tensor* and $\mathbf{W}$ with components w_{ij} is called the *vorticity* or *the spin tensor.*

There is a one-to-one correspondence between a skew tensor and an axial vector (see Appendix L, Sect. 12). The axial vector $\mathbf{w}$ which corresponds to the skew vorticity tensor $\mathbf{W}$ in (5.3) or (5.3a) is defined by

$$\mathbf{W}\,\mathbf{z} = \mathbf{w} \times \mathbf{z} \quad \text{or} \quad \mathbf{w} = \frac{1}{2}\,\text{curl}\ \mathbf{v} \tag{5.5}$$

for every vector **z**. Referred to the basis $\mathbf{e}_i$, **w** reads as

$$\mathbf{w} = w_i \, \mathbf{e}_i \tag{5.6}$$

and the corresponding component forms of $(5.5)_{1,2}$ are:

$$w_{ij} = -\,\varepsilon_{ijk} \, w_k = \varepsilon_{jik} \, w_k \quad , \quad w_i = \frac{1}{2} \varepsilon_{ijk} \, w_{kj} = \frac{1}{2} \varepsilon_{ijk} \, v_{k,j} \ , \tag{5.5a}$$

where ε_{ijk} stand for the components of the permutation symbol. Motion in which $\mathbf{w} = \mathbf{0}$ is called *irrotational* motion. It is usual in some books to identify curl **v** (instead of $\frac{1}{2}$ curl **v** in $(5.5)_2$) as vorticity and denote it by **w** = curl **v**.

The foregoing results have been obtained starting from (5.1), where $\mathbf{v} = v(\mathbf{x},t)$. Alternatively, we may begin by recalling the referential (Lagrangian) form of the particle velocity in the form given by $(1.14)_1$, where $\mathbf{v} = \mathbf{v}(\mathbf{X},t)$. Then, the material derivative of the deformation gradient (2.1) is

$$\dot{\mathbf{F}} = \mathbf{L} \, \mathbf{F} \ , \tag{5.7}$$

or in component form

$$\overline{\dot{x_{i,A}}} = \dot{x}_{i,A} = v_{i,A} = v_{i,k} \, x_{k,A} \ . \tag{5.7a}$$

Additional useful results may be obtained from (5.7) or (5.7a). Thus, the material derivative of the Jacobian of transformation (1.8) is

$$\begin{aligned} \dot{J} &= \overline{\det \mathbf{F}}^{\,\cdot} = J \operatorname{tr} (\mathbf{L} \, \mathbf{F} \, \mathbf{F}^{-1}) = J \operatorname{tr} \mathbf{L} \\ &= J \operatorname{tr} \mathbf{D} = J \, v_{k,k} = J \, d_{kk} \end{aligned} \tag{5.8}$$

or

$$\dot{J} = J \, X_{A,i} \, v_{i,k} \, x_{k,A} = J \, \delta_{ik} \, v_{i,k} = J \, v_{k,k} \ . \tag{5.8a}$$

A motion which is volume-preserving, i.e., a motion corresponding to which the volume occupied by any material region remains unchanged, is called *isochoric*. For an isochoric motion,

$$J = 1 \quad , \quad \dot{J} = 0 \Rightarrow \operatorname{tr}\mathbf{D} = d_{kk} = \operatorname{div}\mathbf{v} = v_{k,k} = 0 \ . \tag{5.9}$$

Next, consider the material derivative of the tensor **C** defined by (2.10):

$$\dot{\mathbf{C}} = \overline{\mathbf{F}^T\mathbf{F}}^{\cdot} = \dot{\mathbf{F}}^T \mathbf{F} + \mathbf{F}^T \dot{\mathbf{F}} = \mathbf{F}^T(\mathbf{L}^T + \mathbf{L})\mathbf{F} = 2\mathbf{F}^T \mathbf{D} \mathbf{F} \ , \tag{5.10}$$

or in component form

$$\dot{C}_{AB} = \overline{x_{i,A}\, x_{i,B}}^{\cdot} = \overline{x_{i,A}}^{\cdot}\, x_{i,B} + x_{i,A}\, \overline{x_{i,B}}^{\cdot}$$

$$= v_{i,k}\, x_{k,A}\, x_{i,B} + v_{i,k}\, x_{k,B}\, x_{i,A}$$

$$= (v_{i,k} + v_{k,i})x_{k,A}\, x_{i,B} = 2d_{ik}\, x_{k,A}\, x_{i,B} \tag{5.10a}$$

In order to discuss a physical interpretation of the tensors **D** and **W** (or equivalently the axial vector **w**), we take the material derivative of (2.8) with respect to t and make use of (5.7) to obtain

$$\overline{\lambda\, \mathbf{m}}^{\cdot} = \dot{\lambda}\, \mathbf{m} + \lambda\, \dot{\mathbf{m}} = \overline{\mathbf{F}\,\mathbf{M}}^{\cdot} = \mathbf{L}\,\mathbf{F}\,\mathbf{M} = \lambda\, \mathbf{L}\, \mathbf{m} \ , \tag{5.11}$$

or in component form

$$\overline{\lambda\, m_i}^{\cdot} = \dot{\lambda}\, m_i + \lambda\, \dot{m}_i = \overline{x_{i,A}\, M_A}^{\cdot} = v_{i,k}\, x_{k,A}\, M_A = \lambda\, v_{i,k}\, m_k \ . \tag{5.11a}$$

The scalar product of (5.11) with the unit vector **m**, after also using the results $\mathbf{m} \cdot \mathbf{m} = 1$, $\mathbf{m} \cdot \dot{\mathbf{m}} = 0$, yields

$$\frac{\dot{\lambda}}{\lambda} = \overline{l\mathrm{n}\, \lambda}^{\cdot} = \mathbf{L}\, \mathbf{m} \cdot \mathbf{m} = (\mathbf{D} + \mathbf{W}) \cdot \mathbf{m} \otimes \mathbf{m} = \mathbf{D} \cdot \mathbf{m} \otimes \mathbf{m} \ , \tag{5.12}$$

or

$$\frac{\dot{\lambda}}{\lambda} = \overline{l\mathrm{n}\, \lambda}^{\cdot} = v_{i,k}\, m_i\, m_k = (d_{ik} + w_{ik})m_i\, m_k = d_{ik}\, m_i\, m_k \ . \tag{5.12a}$$

It follows from (5.12) that the rate of deformation **D** determines the material time derivative of the logarithmic stretch for a material line element having the direction **m** in the current cofiguration κ. Further, using (5.12), the result $(5.11)_2$ may be rewritten as

$$\dot{\mathbf{m}} = \mathbf{L}\,\mathbf{m} - \frac{\dot{\lambda}}{\lambda}\,\mathbf{m} = \mathbf{W}\,\mathbf{m} + [\mathbf{D} - (\frac{\dot{\lambda}}{\lambda})\,\mathbf{I}]\mathbf{m} \ , \qquad (5.13)$$

or from (5.11a)

$$\dot{m}_i = v_{i,j}\, m_j - (\frac{\dot{\lambda}}{\lambda}) m_i = w_{ij}\, m_j + [d_{ij} - (\frac{\dot{\lambda}}{\lambda})\delta_{ij}\,]\, m_j \ . \qquad (5.13a)$$

Now, let **m** be a principal direction of the eigenvectors associated with the rate of deformation tensor **D**. Then,

$$\mathbf{D}\,\mathbf{m} = \gamma\,\mathbf{m} \quad \text{or} \quad d_{ij}\, m_j = \gamma\, m_i \qquad (5.14)$$

where the associated scalar eigenvalue γ is

$$\gamma = \mathbf{D}\,\mathbf{m}\cdot\mathbf{m} = \mathbf{D}\cdot\mathbf{m}\otimes\mathbf{m} = d_{ij}\, m_i\, m_j = \frac{\dot{\lambda}}{\lambda} \ \text{(by (5.12a))} \qquad (5.15)$$

where we have also used (5.12). It follows from (5.14) and (5.15) that

$$\mathbf{D}\,\mathbf{m} = \frac{\dot{\lambda}}{\lambda}\,\mathbf{m} \quad \text{or} \quad d_{ij}\, m_j = \frac{\dot{\lambda}}{\lambda}\, m_i \ . \qquad (5.16)$$

Moreover, from combination of (5.13) and (5.16) we deduce that

$$\dot{\mathbf{m}} = \mathbf{W}\,\mathbf{m} = \mathbf{w}\times\mathbf{m} \quad \text{or} \quad \dot{m}_i = w_{ij}\, m_j = \varepsilon_{ikj}\, w_k\, m_j \ . \qquad (5.17)$$

Thus, the material time derivative of a unit vector **m** (which represents the direction of a material line in κ) in a principal direction of **D** is determined by (5.17). In other words, the axial vector **w** (or the vorticity) is the angular velocity of the line element which is parallel to a principal direction.

(For further results, see the Supplement to Part I, Section 5.)

6. Superposed rigid body motions.

A rigid motion is one that preserves relative distance. In this section, we examine the effect of such a motion on the various kinematic measures introduced in preceding sections. More specifically, we consider here motions which differ from a prescribed motion, such as $(1.7)_1$, only by superposed rigid body motions of the whole body, i.e., motions which in addition to a prescribed motion involve purely rigid motions of the body. For later reference, we recall here the motion defined in Eq. $(1.7)_1$, namely

$$\mathbf{x} = \chi\,(\mathbf{X}, t)\ \ .$$

Consider a material point (or particle) of the body, which in the present configuration at time t occupies the place $\mathbf{x}$ as specified by $(1.7)_1$. Suppose that under a superposed rigid body motion, the particle which is at $\mathbf{x}$ at time t moves to a place $\mathbf{x}^+$ at time $t^+ = t + a$, a being a constant. In what follows for all quantities associated with the superposed motion we use the same symbols as those associated with the motion $(1.7)_1$ to which we also attach a plus "+" sign. Thus, we introduce a vector function χ^+ and write

$$\mathbf{x}^+ = \hat{\chi}^+\,(\mathbf{X}, t^+) = \chi^+\,(\mathbf{X}, t)\ \ . \tag{6.1}$$

It is clear that the difference between the two functions $\hat{\chi}^+$ and χ^+ is due to the presence of the constant a in the argument of the former.

Similarly, consider another material point of the body which in the present configuration at time t occupies the place $\mathbf{y}$ specified by

$$\mathbf{y} = \chi\,(\mathbf{Y}, t)\ \ . \tag{6.2}$$

Suppose that under the same superposed rigid body motion, the particle which is at $\mathbf{y}$ at time t moves to a place $\mathbf{y}^+$ at time t^+. Then, corresponding to (6.1), we write

$$\mathbf{y}^+ = \hat{\chi}^+\,(\mathbf{Y}, t^+) = \chi^+\,(\mathbf{Y}, t) \tag{6.3}$$

Recalling the inverse relationship $\mathbf{X} = \chi^{-1}(\mathbf{x},\ t)$ and the analogous result for $\mathbf{Y} = \chi^{-1}(\mathbf{y},\ t)$, the function χ^+ on the right-hand sides (6.1) and (6.3) may be expressed as different functions of $\mathbf{x}$, t and **y**, t, respectively, i.e.,

$$\mathbf{x}^+ = \chi^+(\mathbf{X}, t) = \bar{\chi}^+(\mathbf{x}, t) \;\; , \;\; \mathbf{y}^+ = \chi^+(\mathbf{Y}, t) = \bar{\chi}^+(\mathbf{y}, t) \;\; . \tag{6.4}$$

During the superposed rigid body motions of the whole body, the magnitude of the relative displacement $\bar{\chi}^+(\mathbf{x}, t)$ - $\bar{\chi}^+(\mathbf{y}, t)$ must remain unaltered for all pairs of material points **X**, **Y** in the body and for all t in some finite time interval $[t_1, t_2]$. Hence,

$$[\bar{\chi}^+(\mathbf{x}, t) - \bar{\chi}^+(\mathbf{y}, t)] \cdot [\bar{\chi}^+(\mathbf{x}, t) - \bar{\chi}^+(\mathbf{y}, t)] = (\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) \tag{6.5}$$

for all **x**, **y** in the region *R* occupied by the body at time t.

Since **x**, **y** are independent, we may differentiate (6.5) successively with respect to **x** and **y** to obtain the following differential equation for $\bar{\chi}^+$:

$$\left[\frac{\partial \bar{\chi}^+(\mathbf{x}, t)}{\partial \mathbf{x}} \right]^T \left[\frac{\partial \bar{\chi}^+(\mathbf{y}, t)}{\partial \mathbf{y}} \right] = \mathbf{I} \;\; , \tag{6.6}$$

where **x**, **y** are any points in *R* and t is any time in the interval $[t_1, t_2]$. Since the tensor $\partial \bar{\chi}^+(\mathbf{y}, t)/\partial \mathbf{y}$ is invertible, (6.6) can be written in the alternative form

$$\left[\frac{\partial \bar{\chi}^+(\mathbf{x}, t)}{\partial \mathbf{x}} \right]^T = \left[\frac{\partial \bar{\chi}^+(\mathbf{y}, t)}{\partial \mathbf{y}} \right]^{-1} = \mathbf{Q}^T(t) \tag{6.7}$$

for all $\mathbf{x}, \mathbf{y} \subseteq R$ and t in $[t_{1,} \, t_2]$. Thus, each side of the equation (6.7) is a tensor function of t, say $Q^T(t)$, and we may set

$$\frac{\partial \bar{\chi}^+(\mathbf{x}, t)}{\partial \mathbf{x}} = \mathbf{Q}(t) \text{ for } \mathbf{x} \in R \;\; . \tag{6.8}$$

Therefore we also have $\dfrac{\partial \bar{\chi}^+(\mathbf{y}, t)}{\partial \mathbf{y}} = \mathbf{Q}(t)$. >From (6.6) or (6.7), we conclude that

$$\mathbf{Q}^T\mathbf{Q} = \mathbf{I} \;\; , \;\; \det \mathbf{Q} = \pm 1 \;\; , \tag{6.9}$$

and hence **Q** is an orthogonal tensor. The motion under consideration must include the particular case $\bar{\chi}^+(\mathbf{x}, t) = \mathbf{x}$ and for this particular case $\mathbf{Q} = \mathbf{I}$ and det $\mathbf{Q} = +1$. Since the motions are continuous, we must always have

$$\det \mathbf{Q} = 1 \tag{6.10}$$

and therefore $\mathbf{Q}(t)$ is a proper orthogonal tensor.

The differential equation (6.8) may be integrated to yield

$$\bar{\chi}^{+}(\mathbf{x}, t) = \mathbf{a}(t) + \mathbf{Q}(t)\mathbf{x} \ , \tag{6.11}$$

where $\mathbf{a}(t)$ is a vector function of time. The last result is a general solution of (6.5) for $\bar{\chi}^{+}(\mathbf{x}, t)$. For later convenience, we put

$$\mathbf{a}(t) = \mathbf{c}^{+}(t^{+}) - \mathbf{Q}(t)\mathbf{c}(t) \ , \tag{6.12}$$

where $\mathbf{c}^{+}$, $\mathbf{c}$ are vector functions of t^{+} and t, respectively. With the use of (6.12), the (6.11) can be expressed in the form

$$\mathbf{x}^{+} = \mathbf{c}^{+} + \mathbf{Q}(\mathbf{x} - \mathbf{c}) \ , \tag{6.13}$$

where

$$\mathbf{Q}^{T}\,\mathbf{Q} = \mathbf{Q}\,\mathbf{Q}^{T} = \mathbf{I} \ , \ \det \mathbf{Q} = 1 \ . \tag{6.14}$$

The transformation (6.13) is a rigid transformation since it is distance preserving, i.e.,

$$\begin{aligned} |\mathbf{x}^{+} - \mathbf{y}^{+}|^{2} = (\mathbf{x}^{+} - \mathbf{y}^{+}) \cdot (\mathbf{x}^{+} - \mathbf{y}^{+}) &= \mathbf{Q}(\mathbf{x} - \mathbf{y}) \cdot \mathbf{Q}(\mathbf{x} - \mathbf{y}) \\ &= (\mathbf{x} - \mathbf{y}) \cdot [\mathbf{Q}^{T}\mathbf{Q}(\mathbf{x} - \mathbf{y})] \\ &= (\mathbf{x} - \mathbf{y}) \cdot \mathbf{I}(\mathbf{x} - \mathbf{y}) \\ &= (\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) = |\mathbf{x} - \mathbf{y}|^{2} \ . \end{aligned} \tag{6.15}$$

The transformation (6.13) also preserves the angle between any two nonzero vectors $\mathbf{x} - \mathbf{y}$ and $\mathbf{x} - \mathbf{z}$ since

$$\begin{aligned} \cos\theta^{+} &= \frac{\mathbf{x}^{+} - \mathbf{y}^{+}}{|\mathbf{x}^{+}\mathbf{z}^{+}|} \\ &= \frac{\mathbf{Q}(\mathbf{x} - \mathbf{y})}{|\mathbf{x} - \mathbf{y}|} \cdot \mathbf{Q}(\mathbf{x} \\ &= \frac{(\mathbf{x} - \mathbf{y}) \cdot [\mathbf{Q}^{T}\mathbf{Q}(\mathbf{x} - \mathbf{z})]}{|\mathbf{x} - \mathbf{y}| \ |\mathbf{x} - \mathbf{z}|} \\ &= \frac{(\mathbf{x} - \mathbf{y})}{|\mathbf{x} - \mathbf{y}|} \cdot \frac{(\mathbf{x} - \mathbf{z})}{|\mathbf{x} - \mathbf{z}|} = \cos\theta \ , \end{aligned} \tag{6.16}$$

where $\mathbf{z}$ is any place in the configuration occupied by the body at time t which moves to $\mathbf{z}^+$ at time t^+ as a consequence of the rigid body motion. Since (6.13) is both length and angle preserving, it follows that the element of length, element of area and element of volume all remain unaltered under superposed rigid body motions.

The component forms of the various results between (6.5) and (6.10) proceed as follows. First, the component form of (6.5) is

$$[\bar{\chi}_i^+(x_j, t) - \bar{\chi}_i^+(y_j, t)][\bar{\chi}_i^+(x_j, t) - \bar{\chi}_i^+(y_j, t)] \tag{6.5a}$$

To record the results (6.6) in component form, we differentiate (6.5a) successively with respect to x_m and y_n and obtain

$$\frac{\partial\bar{\chi}_i^+(x_j\ t)}{\partial x_{mj}}, t)] = \delta_{im}(x_i - y_i)$$

and

$$\frac{\partial\bar{\chi}_i^+(x_j, t)}{\partial x_m}\frac{\partial\bar{\chi}_i(y_j, t)}{\partial y_n} \tag{6.6a}$$

Next, multiply (6.6a) by $\dfrac{\partial y_n}{\partial\bar{\chi}_k^+(y_j, t)}$ and use the chain rule for differentiation, i.e.,

$$\frac{\partial\bar{\chi}_i^+(y_j, t)}{\partial y_n}\frac{\partial y_n}{\partial\bar{\chi}_k^+(y_j, t)}$$

to deduce

$$\frac{\partial\bar{\chi}_k^+(x_j, t)}{\partial x_m} = \frac{\partial y_m}{\partial\bar{\chi}_k^+(y_j, t)}\ . \tag{6.7a}$$

Since the left-hand side of (6.7a) is independent of y_j and the right-hand side is independent of x_j, each side must be a function of time only and we have

$$\frac{\partial\bar{\chi}_k^+(x_j, t)}{\partial x_m} = Q_{km}(t)\ . \tag{6.8a}$$

Integration of the differential equation (6.8a) yields

$$x_k^+ = \bar{\chi}_k^+(x_j, t) = a_k(t) \tag{6.11a}$$

Since x_j is any point, we must also conclude from (6.11a) that

$$y_k^+ = \bar{\chi}_k(y_j, t) = a_k(t) + Q_{km}(t)y_m$$

and hence

$$\frac{\partial\bar{\chi}_k^+(y_j, t)}{\partial y_n} = Q_{kn}(t) \ \ .$$

Substitution of the last result and (6.8a) into (6.6a) results in the component for of

$$Q_{km}(t)Q_{kn}(t) = \delta_{mn} \ \ . \tag{6.9a}$$

Also, for the convenience of the reader, we also record here the component forms of most of the equations given above. Thus, for example, the component form of (6.1) is

$$x_i^+ = \hat{\chi}_i^+(X_A, t^+) = \chi_i^+(X_A,t) \ \ . \tag{6.1a}$$

Similarly, we identify the component form of (6.2), (6.3), etc., as

$$y_i = \chi_i(Y_A, t) \ \ , \tag{6.2a}$$

$$y_i^+ = \hat{\chi}_i^+(Y_A, t^+) = \chi_i^+(Y_A, t) \ \ , \tag{6.3a}$$

$$x_i^+ = \chi_i^+(X_A, t) = \bar{\chi}_i^+(x_j, t) \ \ , \ \ y_i^+ = \chi_i^+(Y_A, t) = \bar{\chi}_i^+(y_j, t) \ \ , \tag{6.4a}$$

$$[\bar{\chi}_i^+(x_j, t) - \bar{\chi}_i^+(y_j, t)][\bar{\chi}_i^+(x_j, t) - \bar{\chi}_i^+(y_j, t)] = (x_i - y_i)(x_i - y_i) \ \ , \tag{6.5a}$$

$$\frac{\partial\bar{\chi}_i^+(x_j, t)}{\partial x_m}\frac{\partial\bar{\chi}_i^+(y_j, t)}{\partial y_n} = \delta_{mn} \ \ , \tag{6.6a}$$

$$\frac{\partial\bar{\chi}_i^+(x_j, t)}{\partial x_k} = Q_{ik}(t) \ \ , \tag{6.8a}$$

$$Q_{im}(t)Q_{in}(t) = \delta_{mn} \ \ , \tag{6.9a}$$

$$\bar{\chi}_i^+(x_j, t) = a_i(t) + Q_{ik}(t)x_k \quad , \tag{6.11a}$$

$$a_i(t) = c_i^+(t^+) - Q_{ik}(t)c_k(t) \ , \tag{6.12a}$$

$$x_i^+ = c_i^+ + Q_{ij}(x_j - c_j) \quad , \tag{6.13a}$$

$$Q_{im}\, Q_{in} = \delta_{mn} = Q_{mi}\, Q_{ni} \quad , \tag{6.14a}$$

$$\begin{aligned}(x_i^+ - y_i^+)(x_i^+ - y_i^+)k) \\ = \delta_{jk}(x_j - y_j)(x_k - y_k) = (x_j - y_j)(x_j - y_j) \quad ,\end{aligned} \tag{6.15a}$$

$$\begin{aligned}|\mathbf{x}^+ - \mathbf{y}^+|\ |\mathbf{x}^+ - \mathbf{z}^+|\ \cos\theta^+ i^+) \\ = Q_{im}(x_m - y_m)Q_{in}(x_n - z_n) \\ = \delta_{mn}(x_m - y_m)(x_n - z_n) \\ = (x_m - y_m)(x_m - z_m) \\ = |\mathbf{x} - \mathbf{y}|\ |\mathbf{x} - \mathbf{z}|\ \cos\theta \ .\end{aligned} \tag{6.17a}$$

Recall

$$\mathbf{x}^+ = \hat{\chi}^+(\mathbf{X}, t^+) = \chi^+(\mathbf{X}, t) = \bar{\chi}^+(\mathbf{x}, t) \quad ,$$

or

$$\mathbf{x}^+ = \bar{\chi}^+(\mathbf{x}, t^+) = \mathbf{a}(t) + \mathbf{Q}(t)\mathbf{x} \quad . \tag{6.11}$$

Then,

$$\begin{aligned}\mathbf{v}^+ = \dot{\mathbf{x}}^+ = \frac{d\mathbf{x}^+}{dt^+} &= \frac{\partial\hat{\chi}^+(\mathbf{X}, t^+)}{\partial t^+} \\ &= \frac{\partial\chi^+(\mathbf{X}, t)}{\partial t}\frac{dt}{dt^+} \\ &= \frac{\partial\chi^+(\mathbf{X}, t)}{\partial t}\end{aligned} \tag{6.18}$$

Therefore

$$\mathbf{v}^+ = \dot{\mathbf{x}}^+ = \frac{d}{dt^+}(\mathbf{a}(t) + \mathbf{Q}(t)\mathbf{x})$$

$$= \frac{d}{dt}(\mathbf{a}(t) + \mathbf{Q}(t)\mathbf{x})\frac{dt}{dt^+}$$

$$= \dot{\mathbf{a}}(t) + \dot{\mathbf{Q}}(t)\mathbf{x} + \mathbf{Q}(t)\dot{\mathbf{x}}$$

$$= \dot{\mathbf{a}}(t) + \dot{\mathbf{Q}}(t)\mathbf{x} + \mathbf{Q}(t)\mathbf{v} \quad . \tag{6.19}$$

Let

$$\Omega\,(t) = \dot{\mathbf{Q}}(t)\mathbf{Q}^T(t) \quad . \tag{6.20}$$

Then

$$\Omega\,\mathbf{Q} = \dot{\mathbf{Q}}\;\mathbf{Q}^T\;\mathbf{Q} = \dot{\mathbf{Q}} \quad . \tag{6.21}$$

But, from (6.9) $\mathbf{Q}^T\;\mathbf{Q} = \mathbf{I}$, so that

$$\dot{\mathbf{Q}}^T(t)\mathbf{Q}(t) + \mathbf{Q}^T(t)\dot{\mathbf{Q}}(t) = \mathbf{O} \quad ,$$

$$[\Omega\,(t)\mathbf{Q}(t)]^T\mathbf{Q}(t) + \mathbf{Q}^T(t)\Omega(t)\mathbf{Q}(t) = \mathbf{O} \quad ,$$

$$\mathbf{Q}^T(t)\Omega^T(t)\mathbf{Q}(t) + \mathbf{Q}^T(t)\Omega(t)\mathbf{Q}(t) = \mathbf{O} \quad ,$$

$$\Omega^T(t) + \Omega(t) = \mathbf{O}$$

or

$$\Omega^T(t) = -\Omega(t) \quad . \tag{6.22}$$

Since $\boldsymbol{\Omega}$ (t) is skew-symmetric, it posseses an axial vector

$$\omega_i = -\frac{1}{2}\,\varepsilon_{ijk}\,\Omega_{jk} \quad , \tag{6.23}$$

or

$$\Omega_{jk} = -\varepsilon_{jki}\,\omega_i \quad .$$

Alternatively, since Ω is skew-symmetric, there exists a vector-valued function ω such that for any vector $\mathbf{V}$,

$$\Omega\ \mathbf{V} = \omega \times \mathbf{V} \ . \tag{6.24}$$

Returing to (6.19), we may write

$$\mathbf{v}^{+} = \dot{\mathbf{a}} + \Omega\ \mathbf{Q}\ \mathbf{x} + \mathbf{Q}\ \mathbf{v} \ . \tag{6.25}$$

So far we have been discussing <u>superposed</u> rigid body motions. In order to relate our results to the more familiar equations of rigid body dynamics, let us now consider a rigid motion defined by

$$\mathbf{x} = \mathbf{X} \ , \tag{6.26}$$

$$\mathbf{x}^+ = \mathbf{a}(t) + \mathbf{Q}(t)\mathbf{x} = \mathbf{a}(t) + \mathbf{Q}(t)\mathbf{X} \ . \tag{6.27}$$

The equation $(6.27)_1$ may be solved for $\mathbf{x}$ in the form

$$\mathbf{x} = \mathbf{Q}^T(t)(\mathbf{x}^+ - \mathbf{a}(t)) \ . \tag{6.28}$$

Then

$$\mathbf{v}^+ = \dot{\mathbf{a}}(t) + \dot{\mathbf{Q}}(t)\mathbf{x}$$

$$= \dot{\mathbf{a}}(t) + \Omega\ \mathbf{Q}(t)[\mathbf{Q}^T(t)(\mathbf{x}^+$$

$$= \dot{\mathbf{a}}(t) + \Omega\ (t)[\mathbf{x}^+ - \mathbf{a}(t)]$$

$$= \dot{\mathbf{a}}(t) + \boldsymbol{\omega}(t) \times [\mathbf{x}^+ - \mathbf{a}(t)] \ . \tag{6.29}$$

Thus $\boldsymbol{\omega}(t)$ is recognized to be the angular velocity of the body.

The component form of the foregoing equations are listed below:

$$x_i^+ = a_i(t) + Q_{ij}(t)x_j \ , \tag{6.11a}$$

$$v_i^+ = \dot{x}_i^+ = \dot{a}_i(t) + \dot{Q}_{ij}(t)x_j + Q_{ij}(t)\dot{x}_j \ , \tag{6.19a}$$

$$\Omega_{ik}(t) = \dot{Q}_{im}(t)Q_{km}(t) \ , \tag{6.20a}$$

$$\Omega_{ik}Q_{kl} = \dot{Q}_{im}Q_{km}Q_{kl} = \dot{Q}_{il} \ , \tag{6.21a}$$

Recalling

$$\dot{Q}_{il}\ Q_{jl} + Q_{il}\ \dot{Q}_{jl} = 0 \ ,$$

we obtain

$$\Omega_{ij}(t) = -\Omega_{ji}(t) \ , \tag{6.22a}$$

$$\Omega_{im}V_m = -\varepsilon_{imp}\ \omega_p\ V_m = \varepsilon_{ipm}\ \omega_p\ V_m \ , \tag{6.24a}$$

$$v_i^+ = \dot{a}_i + \Omega_{im}\ Q_{mn}\ x_n + Q_{im}\ v_m \ , \tag{6.25a}$$

$$x_i^+ = a_i(t) + Q_{im}\ x_m \ , \tag{6.27a}$$

$$Q_{ij}\ x_i^+ = Q_{ij}\ a_i + Q_{ij}\ Q_{im}\ x_m$$

$$Q_{ij}(x_i^+ - a_i) = \delta_{jm}\ x_m = x_j \ , \tag{6.28a}$$

$$v_i^+ = \dot{a}_i + \Omega_{ij}(x_j^+ - a_j)$$

$$= \dot{a}_i + \varepsilon_{ijk}\ \omega_j(x_k^+ - a_k) \ . \tag{6.29a}$$

We return now to the more general considerations of superposed rigid body motions. >From (6.25) and (6.11), it follows that

$$\mathbf{v}^+ = \dot{\mathbf{a}} + \Omega\ \mathbf{Q}\ \mathbf{x} + \mathbf{Q}\ \mathbf{v}$$

$$= \dot{\mathbf{a}} + \Omega\ \mathbf{Q}[\mathbf{Q}^T(\mathbf{x}^+ - \mathbf{a})] + \mathbf{Q}\ \mathbf{v}$$

$$= \dot{\mathbf{a}} + \Omega\ (\mathbf{x}^+ - \mathbf{a}) + \mathbf{Q}\ \mathbf{v} \ , \tag{6.30}$$

or in component form

$$v_i^+ = \dot{a}_i + \Omega_{ij}[x_j^+ - a_j] + Q_{ij}v_j \ . \tag{6.30a}$$

Then,

$$\frac{\partial v_i^+}{\partial x_m^+} = \Omega_{ij} \frac{\partial x_j^+}{\partial x_m^+} + Q_{ij} \frac{\partial v_j}{\partial x_n} \frac{\partial x_n}{\partial x_m^+}$$

$$= \Omega_{ij}\, \delta_{jm} + Q_{ij} \frac{\partial v_j}{\partial x_n} \frac{\partial}{\partial x_m^+} [Q_{ln}(x_l^+ - a_l(t))]$$

$$= \Omega_{im} + Q_{ij} \frac{\partial v_j}{\partial x_n} Q_{ln} \frac{\partial x_l^+}{\partial x_m^+}$$

$$= \Omega_{im} + Q_{ij} \frac{\partial v_j}{\partial x_n} Q_{ln}\, \delta_{lm}$$

$$= \Omega_{im} + Q_{ij} \frac{\partial v_j}{\partial x_n} Q_{mn}$$

$$= \Omega_{im} + Q_{ij}\, Q_{mn} \frac{\partial v_j}{\partial x_n} \quad , \qquad (6.31a)$$

or in direct notation

$$\frac{\partial \mathbf{v}^+}{\partial \mathbf{x}^+} = \Omega + \mathbf{Q} \frac{\partial \mathbf{v}}{\partial \mathbf{x}} \mathbf{Q}^T \quad . \qquad (6.31)$$

It follows from (6.31) and the results of section 5 that

$$\mathbf{D}^+ = \mathbf{Q}\ \mathbf{D}\ \mathbf{Q}^T \quad ,$$
$$\mathbf{W}^+ = \mathbf{Q}\ \mathbf{W}\ \mathbf{Q}^T + \Omega \quad , \qquad (6.32)$$

or, in indicial notation,

$$d_{ij}^+ = Q_{im}\, Q_{jn}\, d_{mn} \quad ,$$
$$w_{ij}^+ = Q_{im}\, Q_{jn}\, w_{mn} + \Omega_{ij} \quad . \qquad (6.32a)$$

In particular, suppose that the body in its superposed motion is at time t passing through the configuration κ with angular velocity Ω, so that

$$\mathbf{Q} = \mathbf{I} \quad , \quad \dot{\mathbf{Q}} = \Omega\ \mathbf{Q} = \Omega \quad . \qquad (6.33)$$

Then (6.32)

$$\mathbf{D}^{+} = \mathbf{D} \quad , \quad \mathbf{W}^{+} = \mathbf{W} + \Omega \quad . \tag{6.34}$$

7. Infinitesimal deformation and infinitesimal strain measures.

We recall from section 1 that a motion of the body is defined by

$$\mathbf{x} = \chi(\mathbf{X}, t) \;\text{ or }\; x_i = \chi_i(X_A, t) \;. \tag{7.1}$$

The deformation function χ in (7.1) may be written in terms of the relative displacement $\mathbf{u}$ (see section 4), i.e.,

$$\chi = \mathbf{X} + \mathbf{u}(\mathbf{X}, t) \;\text{ or }\; \chi_i = \delta_{iA}\, X_A + u_i(X_A, t) \;. \tag{7.2}$$

We further recall the relative strain measures

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}) \;\text{ or }\; E_{AB} = \frac{1}{2}(C_{AB} - \delta_{AB}) \;, \tag{7.3}$$

and

$$\mathbf{e} = \frac{1}{2}(\mathbf{I} - \mathbf{c}) \;\text{ or }\; e_{ij} = \frac{1}{2}(\delta_{ij} - c_{ij}) \;, \tag{7.4}$$

where

$$\mathbf{C} = \mathbf{F}^T \mathbf{F} \;\text{ or }\; C_{AB} = F_{iA}\, F_{iB} \tag{7.5}$$

and

$$\mathbf{c} = (\mathbf{F}^{-1})^T\mathbf{F}^{-1} = (\mathbf{F}\,\mathbf{F}^T)^{-1} = \mathbf{B}^{-1} \;\text{ or }\; c_{ij} = F^{-1}_{Ai}\, F^{-1}_{Aj} \;. \tag{7.6}$$

Also, the relative displacement gradient is defined by

$$\mathbf{H} = \mathbf{F} - \mathbf{I} \;\text{ or }\; u_{i,A} = F_{iA} - \delta_{iA} \;. \tag{7.7}$$

In order to obtain infinitesimal kinematical results from those of the finite theory discussed in sections 1-5, we introduce a measure of <u>smallness</u> by a nonnegative function

$$\varepsilon = \varepsilon(t) = \max_{K}\{ \sup_{\mathbf{X} \in R_o} |\mathbf{u}_{,K}| \} \quad (K = 1, 2, 3) \;, \tag{7.8}$$

where sup stands for the supremum (or the least upper bound) of a nonempty bounded set of real numbers. If $f(\mathbf{u}_{,K})$ is any scalar-, vector-, or tensor-valued function of $\mathbf{u}_{,K} = \{\mathbf{u}_{,1}, \mathbf{u}_{,2}, \mathbf{u}_{,3}\}$ defined in a neighborhood of $\mathbf{u}_{,K} = 0$ (K = 1, 2, 3) and satisfying the condition that there exists a nonnegative real constant C such that $|f(\mathbf{u}_{,K})| < C\varepsilon^n$ as $\varepsilon \to 0$, then we write $f = 0(\varepsilon^n)$ as $\varepsilon \to 0$. Thus, in particular, the components of $\mathbf{u}_{,K}$ referred to either $\mathbf{e}_i$ or $\mathbf{E}_A$ are of $0(\varepsilon)$ as $\varepsilon \to 0$, i.e.,

$$u_{i,A} = 0(\varepsilon) \text{ and } u_{B,A} = 0(\varepsilon) \text{ as } \varepsilon \to 0 \ . \tag{7.9}$$

The components E_{AB} of the relative strain measure $\mathbf{E}$ in (7.3) can also be expressed in terms of relative displacement gradients $(7.9)_2$ in the form

$$E_{AB} = \frac{1}{2}(u_{A,B} + u_{B,A} + u_{M,A}u_{M,B}) \ . \tag{7.10}$$

Clearly, if terms of $0(\varepsilon^2)$ as $\varepsilon \to 0$ can be neglected, we can write (7.10) approximately as

$$E_{AB} = \frac{1}{2}(u_{A,B} + u_{B,A}) = 0(\varepsilon) \text{ as } \varepsilon \to 0 \ . \tag{7.11}$$

Next, consider the expression

$$F_{iA}(\delta_{Aj} - \frac{\partial u_A}{\partial X_B}\delta_{Bj}) \tag{7.12}$$

and substitute for $F_{iA} = \delta_{iA} + \dfrac{\partial u_i}{\partial X_A}$ from $(7.7)_2$. Thus,

$$\begin{aligned} \text{Expression (7.12)} &= (\delta_{iA} + \frac{\partial u_i}{\partial X_A})(\delta_{Aj} - \frac{\partial u_A}{\partial X_B}\delta_{Bj}) \\ &= \delta_{ij} + \frac{\partial u_i}{\partial X_A}\delta_{Aj} - \frac{\partial u_i}{\partial X_B}\delta_{Bj} + 0(\varepsilon^2) \\ &= \delta_{ij} + 0(\varepsilon^2) \\ &= \delta_{ij} \ , \end{aligned} \tag{7.13}$$

where in writing $(7.13)_4$, terms of $0(\varepsilon^2)$ as $\varepsilon \to 0$ have been neglected. Hence, to the order ε^2 as $\varepsilon \to 0$, we can identify the coefficient of F_{iA} in (7.12) as the inverse of F_{iA}, i.e.,

$$F_{Ai}^{-1} = \frac{\partial X_A}{\partial x_{iBi}} \ . \tag{7.14}$$

With the use of (7.14) and the chain rule of differentiation, it can be readily verified that

$$\frac{\partial u_A}{\partial x_i} = \frac{\partial u_A}{\partial X_B}\frac{\partial X_B}{\partial x_i}\frac{\partial u_B}{\partial X_C}\delta_{Ci})$$

$$= \frac{\partial u_A}{\partial X_B}\delta_{Bi} + 0(\varepsilon^2) \text{ as } \varepsilon \to 0 \qquad (7.15)$$

and similarly

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial u_i}{\partial X_A}\frac{\partial X_A}{\partial x_j}\frac{\partial u_A}{\partial X_B}\delta_{Bj})$$

$$= \frac{\partial u_i}{\partial X_A}\delta_{Aj} + 0(\varepsilon^2) \text{ as } \varepsilon \to 0 \ . \qquad (7.16)$$

It follows from (7.15) and (7.16) that to $0(\varepsilon^2)$, it is immaterial whether the partial derivatives of the displacement field **u** is taken with respect to x_i or X_A so that $\frac{\partial u_1}{\partial x_1} = \frac{\partial u_1}{\partial X_1}$ to $0(\varepsilon^2)$ as $\varepsilon \to 0$. Hence in dealing with infinitesimal kinematics, it is not necessary to distinguish between Eulerian and Lagrangian form of kinematical quantities.

>From $(7.6)_2$ and (7.14), we have

$$c_{ij} = F^{-1}_{Ai}\, F^{-1}_{Aj} = (\delta_{Ai} - \frac{\partial u_A}{\partial X_B}\delta_{Bi})(\delta_{Aj} - \frac{\partial u_A}{\partial X_C}\delta_{Cj})$$

$$= \delta_{ij} - (\frac{\partial u_A}{\partial X_B}\delta_{Bi}\delta_{Aj} + \frac{\partial u_A}{\partial X_C}\delta_{Cj}\,\delta_{Ai}) + 0(\varepsilon^2) \text{ as } \varepsilon \to 0 \ . \qquad (7.17)$$

Hence, if terms of $0(\varepsilon^2)$ as $\varepsilon \to 0$ can be neglected, then

$$e_{ij} = \frac{1}{2}(\delta_{ij} - c_{ij})$$

$$= \frac{1}{2}(\frac{\partial u_A}{\partial X_B} + \frac{\partial u_B}{\partial X_A})\delta_{iA}\delta_{jB}\varepsilon \to 0 \ , \qquad (7.18)$$

so that any differences between the two strain measures disappear upon linearization.

In view of the remark made following (7.15), to the order of ε^2 we may display the components of the relative displacement gradients either as $u_{A,B}$ or $u_{i,j}$. For example, if we make the latter choice, then we may write

$$u_{i,j} = e_{ij} + \omega_{ij} = 0(\varepsilon) \ ,$$

$$e_{ij} = 0(\varepsilon) \ , \ \omega_{ij} = 0(\varepsilon) \ \text{as} \ \varepsilon \to 0 \ , \tag{7.19}$$

where e_{ij} and ω_{ij} are defined by

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \ , \ \omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i}) \ . \tag{7.20}$$

We observe here that $u_{i,i} = e_{ii} = 0(\varepsilon)$ is called the cubical dilatation and that $\omega_{ii} = 0$.

We consider now the linearized version of the Cauchy-Green measure C_{AB}, the stretch U_{AB} and the rotation R_{iA}. The Cauchy-Green measure C_{AB} defined by $(7.5)_2$ is related to E_{AB} by $C_{AB} = \delta_{AB} + 2E_{AB}$. But, when the approximation (7.11) is adopted, then C_{AB} can be written as

$$C_{AB} = U_{AC}\,U_{CB} = \delta_{AB} + 2E_{AB}$$

$$= \delta_{AB} + 0(\varepsilon) \ \text{as} \ \varepsilon \to 0 \tag{7.21}$$

and from an examination of

$$(\delta_{AC} + E_{AC})(\delta_{CB} + E_{CB}) = \delta_{AB} + 2E_{AB} + 0(\varepsilon^2) \ \text{as} \ \varepsilon \to 0 \tag{7.22}$$

to the order ε^2 we can identify U_{AB} as

$$U_{AB} = \delta_{AB} + E_{AB} = \delta_{AB} + 0(\varepsilon) \ \text{as} \ \varepsilon \to 0 \ . \tag{7.23}$$

Also, by considering an expression of the form $U_{AB}(\delta_{BC}$neglected,then.EQI(7.24)$U^{-1}_{AB} = \delta_{AB} - E_{AB} = \delta_{AB} + 0(\varepsilon) \ \text{as} \ \varepsilon \to 0$.

We now turn to the rotation tensor R_{iA}. Recalling the polar decomposition theorem, the components of the rotation tensor **R** can be written as

$$R_{iA} = F_{iB}\,U^{-1}_{BA} \ . \tag{7.25}$$

It then follows from (7.7), (7.24) and (7.11) that

$$R_{iA} = (\delta_{iB} + u_{i,B})[\delta_{BA} - \frac{1}{2}(u_{A,B} + u_{B,A})]$$

$$= \delta_{iA} + \frac{1}{2}[u_{i,A} - u_{A,B}\delta_{iB}]$$

$$= \delta_{iA} + \frac{1}{2}(\frac{\partial u_i}{\partial X_A} - \frac{\partial u_A}{\partial x_i})$$

$$= \delta_{iA} + \frac{1}{2}(\frac{\partial u_i}{\partial x_A} - \frac{\partial u_A}{\partial x_i}) \ , \qquad (7.26)$$

where terms of $0(\varepsilon^2)$ as $\varepsilon \to 0$ have been neglected and where use has been made of the result (7.15). To the order of approximation considered, we may also write (7.26) as

$$\mathbf{R} = \mathbf{I} + \boldsymbol{\Omega} \ \text{ or } \ R_{ij} = \delta_{ij} + \omega_{ij} \ , \qquad (7.27)$$

where

$$\boldsymbol{\Omega} = -\boldsymbol{\Omega}^T = \frac{1}{2}(\mathbf{H} - \mathbf{H}^T)_{j,i}) \ , \qquad (7.28)$$

which is consistent with $(7.20)_2$.

In the remainder of this section, we employ the linearization procedure discussed above and obtain the linearized version of such expressions as

$$J = \det \mathbf{F} = \det(x_{i,A}) \qquad (7.29)$$

and its inverse. For this purpose, we first recall that J can be represented as

$$J = \frac{1}{6} \varepsilon_{ijk} \varepsilon_{LMN} F_{iL} F_{jM} F_{kN} \ . \qquad (7.30)$$

Substituting the components of the displacement gradient in terms of $u_{i,A}$ given by $(7.7)_2$ in (7.30) we get

$$J = \frac{1}{6}\varepsilon_{ijk}\,\varepsilon_{LMN}(\delta_{iL} + u_{i,L})(\delta_{jM} + u_{j,M})(\delta_{k,N} + u_{kN})$$

$$= \frac{1}{6}\varepsilon_{ijk}\,\varepsilon_{LMN}[\delta_{iL}\,\delta_{jM}\,\delta_{kN} + 3\delta_{iL}\,\delta_{jM}\,u_{k,N} + 0(\varepsilon^2)]$$

$$= 1 + \delta_{kN}\,u_{k,N} = 1 + u_{k,k} = 1 + e_{kk} \quad \text{as} \quad \varepsilon \to 0 \quad , \tag{7.31}$$

where in obtaining $(7.31)_3$ use has been made of the identities $\frac{1}{6} = \varepsilon_{ijk}\,\varepsilon_{ijk}$ and $\varepsilon_{ijk}\,\varepsilon_{ijm} = 2\delta_{km}$ and where terms of $0(\varepsilon^2)$ as $\varepsilon \to 0$ have been neglected. The inverse of (7.31) is given by

$$\begin{aligned} J^{-1} &= [1 + e_{ii} + \cdots]^{-1} \\ &= 1 - e_{ii} + (e_{ii})^2 + \cdots \\ &= 1 - e_{ii} \quad \text{as} \quad \varepsilon \to 0 \end{aligned} \tag{7.32}$$

and again in writing (7.32) terms of $0(\varepsilon^2)$ as $\varepsilon \to 0$ have been neglected.

(For a discussion of the interpretation of the infinitesimal strain measures, see Supplement to Part I, Section 7.)

8. The transport theorem.

Let S be an arbitrary part (or subset) of the body B and suppose that S occupies a region P_o, with a closed boundary surface ∂P_o, in a fixed reference configuration. Similarly let P, with closed boundary surface ∂P, be the region occupied by S in the configuration at time t.

Let ϕ be any scalar-valued or tensor-valued field with the following representations:

$$\phi = \tilde{\phi}(\mathbf{x}, t) = \tilde{\phi}(\chi(\mathbf{X}, t), t) \tag{8.1}$$

and consider the volume integral

$$I = \int_P \tilde{\phi}(\mathbf{x}, t)dv = \int_{P_o} \hat{\phi}(\mathbf{X}, t)J\ dV \ \ , \tag{8.2}$$

where we have used dv = J dV, J = det F > 0. Often we shall encounter an expression of the type (8.2) and we need to calculate its time derivative dI/dt. Thus, we write

$$\begin{aligned}
\frac{dI}{dt} &= \frac{d}{dt}\int_{P_o} \hat{\phi}(\mathbf{X}, t)J\ dV \\
&= \int_{P_o} \frac{d}{dt}(\hat{\phi}(\mathbf{X}, t)J)dV \\
&= \int_{P_o} [\frac{\partial\hat{\phi}}{\partial t}(\mathbf{X}, t)J + \hat{\phi}(\mathbf{X}, t)\dot{J}]dV \\
&= \int_{P_o} [\frac{\partial\hat{\phi}}{\partial t}(\mathbf{X}, t)J + Jv_{i,i}\hat{\phi}(\mathbf{X}, t)]dV \\
&= \int_{P_o} [\frac{\partial\hat{\phi}(\mathbf{X}, t)}{\partial t} + \hat{\phi}(\mathbf{X}, t)v_{i,i}]JdV \\
&= \int_{P_o} [\dot{\phi} + \hat{\phi}(\mathbf{X}, t)\text{div}\ \mathbf{v}]JdV \\
&= \int_{P} [\dot{\phi} + \tilde{\phi}(\mathbf{x}, t)\text{div}
\end{aligned} \tag{8.3}$$

where in the fourth of (8.3) we have also used $\dot{J} = J\, v_{i,i}$ and where

$$\dot{\phi} = \frac{\partial\hat{\phi}(\mathbf{X}, t)}{\partial t} = \frac{\partial\tilde{\phi}(\mathbf{x}, t)}{\partial t} + \frac{\partial\tilde{\phi}(\mathbf{x}, t)}{\partial x_i} v_i \ . \tag{8.4}$$

It then follows that the time derivative of $(8.2)_1$ is given by

$$\frac{d}{dt} \int_P \tilde{\phi}(\mathbf{x}, t) dv = \int_P [\dot{\phi} + \tilde{\phi}(\mathbf{x}, t) \mathrm{div}\, \mathbf{v}] dv \ . \tag{8.5}$$

>From the result (8.5) follows the various expressions given below:

$$\frac{d}{dt} \int_P \tilde{\phi}(\mathbf{x}, t) dv = \int_P [\frac{\partial\tilde{\phi}(\mathbf{x}, t)}{\partial t} + \frac{\partial\tilde{\phi}(\mathbf{x}, t)}{\partial x_i} v_i$$

$$= \int_P [\frac{\partial\tilde{\phi}(\mathbf{x}, t)}{\partial t} + \frac{\partial(\tilde{\phi}(\mathbf{x}, t) v_i)}{\partial x_i}] dv$$

$$= \int_P \frac{\partial\tilde{\phi}(\mathbf{x}, t)}{\partial t} dv + \int_{\partial P} \tilde{\phi}(\mathbf{x}, t) v_i n_i da$$

$$= \int_P \frac{\partial\tilde{\phi}(\mathbf{x}, t)}{\partial t} dv + \int_{\partial P} \tilde{\phi}(\mathbf{x}, t) \mathbf{v} \cdot \mathbf{n}\, da \ , \tag{8.6}$$

where the divergence theorem has been used. Next, apply $(8.6)_4$ to a fixed spatial region $\bar{P}$ in the configuration at time t and write

$$\frac{d}{dt} \int_{\bar{P}} \tilde{\phi}(\mathbf{x}, t) dv = \int_{\bar{P}} \frac{\partial\tilde{\phi}(\mathbf{x}, t)}{\partial t} dv + \int_{\partial\bar{P}} \tilde{\phi}(\mathbf{x}, t) \mathbf{v} \cdot \mathbf{n}\, da$$

$$= \frac{\partial}{\partial t} \int_{\bar{P}} \tilde{\phi}(\mathbf{x}, t) dv + \int_{\partial\bar{P}} \tilde{\phi}(\mathbf{x},t) \mathbf{v} \cdot \mathbf{n}\, da \ . \tag{8.7}$$

The form $(8.7)_2$ is another statement of the transport theorem. It is used when it is convenient to focus attention on a fixed region of space $\bar{P}$ at time t and consider the motion of the body over this region. This form $(8.7)_2$ can also be deduced directly from $(8.6)_4$ without the introduction of the fixed spatial region $\bar{P}$ by the following line of argument: Since in the calculation of the first integral in $(8.6)_4$ the variable **x** (and hence the spatial region *P*) is fixed, the operator ∂/∂t commutes with the volume integral in $(8.6)_4$ over a region *P* and the form $(8.7)_2$ with $\bar{P}$ replaced by *P* follows from $(8.6)_4$. The commuting of ∂/∂t with the volume integral in $(8.6)_4$ may be contrasted with a similar calculation involving the volume integral in $(8.3)_2$. In the latter integral, since **X** (and hence the material region P_o, is fixed, the operator d/dt commutes with the volume integral in $(8.3)_2$ over the region P_o.

Part II: Conservation Laws and Some Related Results

1. Conservation of mass.

We recall from section 1 of Part I that the mass of any part P of the body, i.e., M is a continuous function of its volume and that there exists a scalar mass density ρ such that

$$M(S_t) = \int_P \rho \, dv \ , \ P \subseteq R \ , \tag{1}$$

where $\rho = \rho(\mathbf{x}, t)$ depends on the particular configuration occupying the region of space R and dv is the element of volume in the present configuration. A statement of the conservation of mass for any material part of the body in the present configuration is as follows:

$$\frac{d}{dt} \int_P \rho \, dv = 0 \ . \tag{2}$$

By the transport theorem, (2) can be written as $\int_P [\dot{\rho} + \rho v_{k,k}] dv = 0$. The last result must hold for <u>all</u> arbitrary parts P; hence, by the argument outlined previously, assuming that ρ is continuously differentiable we have

$$\dot{\rho} + \rho v_{k,k} = 0 \ , \tag{3}$$

where a superposed dot denotes material derivative, i.e., $\dot{\rho} = \frac{\partial \rho}{\partial t} + v_k \frac{\partial \rho}{\partial x_k}$. The result (3) can also be expressed as

$$\frac{\partial \rho}{\partial t} + (\rho v_k)_{,k} = 0 \text{ or } \frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{v}) = 0 \ . \tag{4}$$

Equation (3) or (4) represents the local equation for conservation of mass. It is also referred to as <u>spatial</u> form of the "continuity equation."

Another form of the principle of conservation of mass may be stated as:

$$\int_P \rho \, dv = \int_{P_o} \rho_o \, dV \ , \tag{5}$$

where P_o is the material part in the initial reference configuration, dV is the element of volume in the reference configuration and ρ_o denotes the mass density in the reference configuration. Using $dv = J\,dV$, $J = \det \mathbf{F} = \det x_{i,A}$, from (5) we get

$$\rho = J^{-1}\,\rho_o \;\text{ or }\; \rho = (\det \mathbf{F})^{-1}\,\rho_o \;,\; \rho_o = J\rho \;, \tag{6}$$

which is the material form of continuity equation.

Appendix to Sec. 1 of Part II

Theorem: If $\phi(\mathbf{x}, t)$ is continuous in R and

$$\int_P \phi \, dv = 0 \qquad (1)$$

for every part $P \subseteq R$, then the necessary and sufficient condition for the validity of (1) is that

$$\phi = 0 \text{ in } R \ . \qquad (2)$$

Proof: Let us first recall the definition of continuity.

Definition: A function $\phi(\mathbf{x}, t)$ is said to be continuous in a region R if for every $\mathbf{x} \in R$ and every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|\phi(\mathbf{x}, t) - \phi({}_o\mathbf{x}, t)| < \varepsilon$ whenever $|\mathbf{x} - {}_o\mathbf{x}| < \delta$.

Sufficiency: If $\phi = 0$ in R, then (1) is trivially satisfied.

Necessity: (Proof by contradiction). Suppose that there is a point ${}_o\mathbf{x} \in P$ ($\subseteq R$) at which $\phi_o = \phi({}_o\mathbf{x}, t) > 0$. Then, by the continuity of ϕ, there exists a $\delta > 0$ such that

$$|\phi(\mathbf{x}, t) - \phi_o| < \frac{\phi_o}{2} \text{ for all } |\mathbf{x} - {}_o\mathbf{x}| < \delta \ . \qquad (3)$$

Now let P_δ be the region $|\mathbf{x} - {}_o\mathbf{x}| < \delta$ and V_δ be the volume of this region so that

$$V_\delta = \int_{P_\delta} dv > 0 \ . \qquad (4)$$

>From (3) and the fact that $\phi_o > 0$ we have

$$\phi > \frac{\phi_o}{2} \text{ in } P_\delta \ . \qquad (5)$$

Now let P be a part which contains P_δ, then

$$\int_{P_\delta} \phi \, dv > \int_{P_\delta} \tfrac{1}{2} \phi_o \, dv = \tfrac{1}{2} \phi_o V_\delta > 0 \ . \qquad (6)$$

Since the result (6) contradicts (1), it follows that there can exist no point ${}_o\mathbf{x}$ such that $\phi({}_o\mathbf{x}, t) >$

0. Similarly, we realize that if $\phi_o = \phi({}_o\mathbf{x}, t) < 0$, then by the continuity of ϕ there exists a $\delta > 0$ such that

$$|\phi(\mathbf{x}, t) - \phi_o| < -\frac{\phi_o}{2} \quad \text{for all } |\mathbf{x} - {}_o\mathbf{x}| < \delta \ . \tag{7}$$

Thus we can conclude that

$$\phi(\mathbf{x}, t) < \frac{\phi_o}{2} \quad \text{for all } \mathbf{x} \in P_\delta \, . \tag{8}$$

Hence

$$\int_{P_\delta} \phi \, dv < \int_{P_\delta} \tfrac{1}{2} \phi_o \, dv = \tfrac{1}{2} \phi_o V_\delta < 0 \ . \tag{9}$$

Since (9) contradicts (1), again it follows that no point ${}_o\mathbf{x}$ exists for which $\phi({}_o\mathbf{x}, t) < 0$. Combining this result with that above we conclude that (2) is also a necessary condition for the validity of (1).

2. Forces and couples. Euler's laws.

We admit two types of forces, namely the body force per unit mass **b** and the surface force (or contact force) per unit area) as follows:

$$\begin{aligned} \mathbf{b} &= \mathbf{b}(\mathbf{X},t) = \text{body force per unit mass}, \\ \mathbf{t} &= \mathbf{t}(\mathbf{X},t\,;\,\mathbf{n}) = \text{contact force per unit area.} \end{aligned} \tag{1}$$

It is important to note that the surface force **t** depends on the orientation of the surface area (with outward unit normal **n**) upon which it acts. Similar to the definitions (1) above, we could also admit body couple per unit mass **c** and surface couple per unit area **m**, but these are ruled out in the construction of classical continuum mechanics.

We also note here the following:

$$\begin{aligned} &\text{momentum per unit mass} = \mathbf{v} = \dot{\mathbf{x}} \\ &\text{moment of momentum per unit mass} = \mathbf{x} \times \mathbf{v} = \mathbf{x} \times \dot{\mathbf{x}} \end{aligned} \tag{2}$$

We consider now the basic balance laws (also called conservation laws) for momentum and moment of momentum (also called angular momentum) in the context of purely mechanical theory. These balance laws, which are known as Euler's laws, may be stated (in words) as follows:

$$\begin{aligned} &\left\{\begin{matrix}\text{Rate of change of momentum} \\ \text{for any part of the body}\end{matrix}\right\} \left\{\begin{matrix}\text{All external forces acting} \\ \text{on the part}\end{matrix}\right\}, \\ &\left\{\begin{matrix}\text{Rate of change of moment of momentum} \\ \text{for any part of the body}\end{matrix}\right\} \left\{\begin{matrix}\text{Moment of all external forces} \\ \text{acting on the part}\end{matrix}\right\}. \end{aligned} \tag{3}$$

The above laws can be translated into mathematical forms both with respect to the current configuration (Eulerian form) or with respect to the reference configuration (Lagrangian form). In this section we limit the discussion to the Eulerian (or spatial) form of the balance laws (3). Thus, remembering the definitions $(2)_{1,2}$, the momentum for any part P of the body occupying a region P with boundary surface ∂P and the moment of momentum for any part P are:

$$\int_P \mathbf{v}\,\mathrm{dm} \ , \ \int_P \mathbf{x}\times\mathbf{v}\,\mathrm{dm} \ , \tag{4}$$

where $\mathrm{dm} = \rho\mathrm{dv}$ is the element of mass. Also, the total force acting on P and the moment of total force acting on P are:

$$\int_P \mathbf{b}\,\mathrm{dm} + \int_{\partial P} \mathbf{t}\,\mathrm{da} \ , \ \int_P (\mathbf{x}\times\mathbf{b})\mathrm{dm} + \int_{\partial P} (\mathbf{x}\times\mathbf{t})\mathrm{da} \ . \tag{5}$$

Keeping the expressions (4) and (5) in mind, the spatial (or Eulerian) form of the balance laws (3) are:

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_P \mathbf{v}\,\mathrm{dm} = \int_P \mathbf{b}\,\mathrm{dm} + \int_{\partial P} \mathbf{t}\,\mathrm{da} \ , \tag{6}$$

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_P \mathbf{x}\times\mathbf{v}\,\mathrm{dm} = \int_P \mathbf{x}\times\mathbf{b}\,\mathrm{dm} + \int_{\partial P} \mathbf{x}\times\mathbf{t}\,\mathrm{da} \ . \tag{7}$$

In the next few sections of Part II we exploit the implications of (6) - (7) and also derive their local forms.

3. Further consideration of the stress vector. Existence of the stress tensor and its relationship to the stress vector.

Consider an arbitrary part of the material region of the body B which occupies a part P in the present configuration at time t. Let P be divided into two regions P_1, P_2 separated by a surface σ (see Fig. 3.1). Further, let ∂P_1, ∂P_2 refer to the boundaries of P_1, P_2, respectively; and let $\partial P'$, $\partial P''$ be the portions of the boundaries of P_1, P_2 such that

$$\begin{aligned} \partial P' &= \partial P_1 \cap \partial P \;, \\ \partial P'' &= \partial P_2 \cap \partial P \;. \end{aligned} \tag{1}$$

Thus, a summary of the above description is as follows:

$$\begin{aligned} P = P_1 \cup P_2 \;\;&,\;\; \partial P = \partial P' \cup \partial P'' \;, \\ \partial P_1 = \partial P' \cup \sigma \;\;&,\;\; \partial P_2 = \partial P'' \cup \sigma \;. \end{aligned} \tag{2}$$

Now recall the linear momentum principle, i.e.,

$$\frac{d}{dt} \int_P \rho\, \dot{\mathbf{x}}\, dv = \int_P \rho\, \mathbf{b}\, dv + \int_{\partial P} \mathbf{t}_{(\mathbf{n})} da \tag{3}$$

or with the use of dm = ρ dv in the form:

$$\frac{d}{dt} \int_P \dot{\mathbf{x}}\, dm = \int_P \mathbf{b}\, dm + \int_{\partial P} \mathbf{t}_{(\mathbf{n})}\, da \tag{4}$$

which holds for an arbitrary material region $P \subseteq R$. Application of (4) separately to the parts P_1, P_2 and again to $P_1 \cup P_2 = P$ yields

$$\frac{d}{dt} \int_{P_1} \dot{\mathbf{x}}\, dm - \int_{P_1} \mathbf{b}\, dm - \int_{\partial P_1} \mathbf{t}_{(\mathbf{n})} da = \mathbf{0} \;, \tag{5}$$

$$\frac{d}{dt} \int_{P_2} \dot{\mathbf{x}}\, dm - \int_{P_2} \mathbf{b}\, dm - \int_{\partial P_2} \mathbf{t}_{(\mathbf{n})} da = \mathbf{0} \tag{6}$$

and

$$\frac{d}{dt} \int_{P_1 \cup P_2} \dot{\mathbf{x}}\, dm - \int_{P_1 \cup P_2} \mathbf{b}\, dm - \int_{\partial P' \cup \partial P''} \mathbf{t}_{(\mathbf{n})} da = \mathbf{0} \;. \tag{7}$$

The stress vector $\mathbf{t}_{(\mathbf{n})}$ in (5) acting over the boundary ∂P_1 results from contact forces exerted by the material on one side of the boundary (exterior to P_1) on the material of the other side. Similar remarks hold for the stress vector in (6) and in (7). We emphasize that the stress vector in (5) over $\partial P' \cup \sigma$ represents the contact force exerted on P_1 across the surface, etc. The appropriate normals associated with $\mathbf{t}_{(\mathbf{n})}$ over the surface σ are equal and opposite in sign. To elaborate, let $\mathbf{n}$ be the outward unit normal at a point on σ when σ is a portion of ∂P_1. Then, the outward unit normal at the same point on σ when σ is a portion of ∂P_2 is $-\mathbf{n}$.

>From combination of (5) and (6) and after subtraction from (7), we obtain the following equation:

$$\int_\sigma [\mathbf{t}_{(\mathbf{n})} + \mathbf{t}_{(-\mathbf{n})}]\mathrm{da} = \mathbf{0} \tag{8}$$

over the arbitrary surface σ. Assuming that the stress vector is a continuous function of position and $\mathbf{n}$, it follows that

$$\mathbf{t}_{(\mathbf{n})} = -\mathbf{t}_{(-\mathbf{n})} \quad \text{or} \quad \mathbf{t}(\mathbf{x}\,,\,\mathrm{t}\,;\,\mathbf{n}) = -\,\mathbf{t}(\mathbf{x}\,,\,\mathrm{t}\,;\,-\mathbf{n})\,. \tag{9}$$

According to the result (9), the stress vectors acting on opposite sides of the same surface at a given point are equal in magnitude and opposite in direction. The result (9) is known as Cauchy's lemma.

Again we suppose that a body *B* is mapped into the present configuration κ, at time t, which occupies the region *R*. Consider some interior particle X_o of *B* having position vector ${}_o\mathbf{x}$ in *R*. Construct at ${}_o\mathbf{x}$ a tetrahedron *T*, lying entirely within *R*, and in such a way that the side i is perpendicular to the $\mathbf{e}_i$-direction (see Fig. 3.2) and inclined plane - with outward unit normal ${}_o\mathbf{n}$ - falls in the octant where x_1 , x_2, x_3 are all positive. We refer to the side i of *T* by S_i and to the inclined plane by *S*, respectively. Let h denote the height of the tetrahedron, i.e., the perpendicular distance from ${}_o\mathbf{x}$ to the inclined face *S*, and let S denote the area of this face. Then, the areas of the three orthogonal faces S_i are

$$\mathrm{S_i} = \mathrm{S}\;{}_o\mathbf{n}\cdot\mathbf{e}_i = \mathrm{S}\;{}_o\mathrm{n_i} \tag{10}$$

and the volume of the tetrahedron is

$$V_{tet} = \frac{1}{3}\, h\, S\ . \tag{11}$$

Recalling that $dm = \rho dv$, by virtue of the transport theorem and the conservation of mass (see Sec. 1 of part II), the principle of linear momentum (4) reduces to

$$\int_P \ddot{\mathbf{x}}\, dm = \int_P \mathbf{b}\, dm + \int_{\partial P} \mathbf{t}_{(\mathbf{n})} da\ . \tag{12}$$

Next, apply (12) to the material region *T* and obtain

$$\int_T \ddot{\mathbf{x}}\, dm = \int_T \mathbf{b}\, dm + \int_{\partial T} \mathbf{t}_{(\mathbf{n})} da\ . \tag{13}$$

Observing that the surface integral in (13) represents contributions from all four boundary planes of *T*, we have

$$\int_{\partial T} \mathbf{t}_{(\mathbf{n})} da = \sum_{i=1}^{3} \int_{S_i} \mathbf{t}_{(-\mathbf{e}_i)} da + \int_S \mathbf{t}_{({}_o\mathbf{n})} da\ . \tag{14}$$

Now according to Cauchy's lemma in (9)

$$\mathbf{t}_{(-\mathbf{e}_i)} = -\,\mathbf{t}_{(\mathbf{e}_i)} \tag{15}$$

and then with the use of (14) and (15), the linear momentum equation (13) can be rewritten in the form

$$\int_T (\ddot{\mathbf{x}} - \mathbf{b}) dm = \int_S \mathbf{t}_{({}_o\mathbf{n})} da - \int_{\sum_{i=1}^{3} S_i} \mathbf{t}_{(\mathbf{e}_i)} da\ . \tag{16}$$

Now recall that $dm = \rho dv$ and that ρ is already assumed to be bounded (see Sec. 1 of Part I). Further, we assume that the fields $\ddot{\mathbf{x}}$ and $\mathbf{b}$ are bounded. Then, since (by a theorem of analysis)

$$\left| \int f\, dv \right| \leq \int |f|\, dv$$

where $|f|$ denotes the absolute value of f, we obtain the following estimate for the integral on the left-hand side of (16):

$$| \int_T \rho(\ddot{\mathbf{x}} - \mathbf{b})dv \; | \leq \int_T \; | \rho(\ddot{\mathbf{x}} - \mathbf{b}) \; | dv$$

$$= \int_T K \, dv$$

$$= K^* \int_T dv = K^* \, \frac{Sh}{3}$$

where use has been made of the mean value theorem for integrals, we have set $K(\mathbf{x}, t) = | \rho(\ddot{\mathbf{x}} - \mathbf{b}) |$ and K^* stands for some specific interior value of K in *T*. Hence, we may conclude that there exists a fixed, positive real number K^* such that

$$| \int_T \rho(\ddot{\mathbf{x}} - \mathbf{b})dv | \leq K^* \, \frac{Sh}{3} \quad . \tag{17}$$

Next, assume that the stress vector field is continuous in both **x** and **n**. Then, by the mean value theorem for integrals, the two surface integrals in (16) yield:

$$\int_{\sum_{i=1}^{3} S_i} \mathbf{t}_{(\mathbf{e}_i)}da = \mathbf{t}_i^* S_i = \mathbf{t}_i^* \, S \; {}_o n_i \tag{18}$$

and

$$\int_S \mathbf{t}_{({}_o n)}da = \mathbf{t}^*_{({}_o \mathbf{n})} \, S \; , \tag{19}$$

where in writing $(18)_2$ use has been made of (10) and where $\mathbf{t}_i^*$ and $\mathbf{t}^*$ stand for some specific interior values (at the point ${}_o\mathbf{x}$) of the stress vectors on the respective faces S_i and on the plane *S* of *T*. Each stress vector $\mathbf{t}_i^*$ acts on that side of coordinate plane i which is associated with the outward unit normal $\mathbf{e}_i$. It follows from (16), (17), (18) and (19) that (since K^* is fixed)

$$\frac{1}{3} \, K^*Sh \geq | \int_T \rho(\ddot{\mathbf{x}}_{(o\mathbf{n})}da - \int_{\sum_{i=1}^{3} S_i} \mathbf{t}_{(\mathbf{e}_i)}da \; |$$

$$= | \mathbf{t}^*_{({}_o\mathbf{n})} \, S - \mathbf{t}_i^* \, S \; {}_o n_i |$$

$$= S \, | \mathbf{t}^*_{({}_o\mathbf{n})} - \mathbf{t}_i^* \; {}_o n_i \; | \; .$$

Therefore,

$$|\,\mathbf{t}^*_{({}_o\mathbf{n})} - \mathbf{t}^*_i\ {}_on_i\,| \leq \frac{1}{3}\,K^*h\ . \tag{20}$$

Consider now a sequence of tetrahedra T_1, T_2, ..., of diminishing heights $h_1 > h_2 >$ ..., each member of which is similar to T with three mutually orthogonal faces having outward unit normals $-\mathbf{e}_i$ and an inclined plane with outward unit normal ${}_o\mathbf{n}$. Next, apply (20) to each member of the sequence and, in the limit as $h \to 0$, obtain

$$|\,\mathbf{t}^*_{({}_o\mathbf{n})} - \mathbf{t}^*_i\ {}_on_i\,| \leq 0\ , \tag{21}$$

where the stress vectors in (21) are evaluated at the point ${}_o\mathbf{x}$ which is the common vertex of the family of tetrahedra. It follows from (21) that

$$\mathbf{t}^*_{({}_o\mathbf{n})} = \mathbf{t}^*_i\ {}_on_i\ . \tag{22}$$

Since (22) must hold at any point ${}_o\mathbf{x}$ and corresponding to any direction ${}_o\mathbf{n}$, without ambiguity, we suppress henceforth the designation star, replace ${}_o\mathbf{x}$ by $\mathbf{x}$, ${}_o\mathbf{n}$ by $\mathbf{n}$ and write

$$\mathbf{t}_{(\mathbf{n})} = \mathbf{t}_i n_i\ . \tag{23}$$

We now define t_{ki} by

$$t_{ki} = \mathbf{t}_i \cdot \mathbf{e}_k \quad \text{or} \quad \mathbf{t}_i = t_{ki}\,\mathbf{e}_k\ . \tag{24}$$

Let t_k denote the components of the stress vector $\mathbf{t}_{(\mathbf{n})}$, i.e., $t_k = \mathbf{t}_{(\mathbf{n})} \cdot \mathbf{e}_k$. Then, from (23) and (24) we have

$$t_k = \mathbf{t} \cdot \mathbf{e}_k = \mathbf{t}_i n_i \cdot \mathbf{e}_k = t_{ki} n_i\ , \tag{25}$$

where in $(25)_1$ we have written for simplicity $\mathbf{t}$ in place of $\mathbf{t}_{(\mathbf{n})}$. Thus, if $\mathbf{t} = \mathbf{t}(\mathbf{x}\ , t\ ; \mathbf{n})$ is the stress vector at the point $\mathbf{x}$ acting on a surface whose outward unit normal is $\mathbf{n}$, then it is clear that $t_{ki} = t_{ki}(\mathbf{x}\ , t)$ are defined at $\mathbf{x}$ and are independent of $\mathbf{n}$. The relation (25) establishes the <u>existence</u> of the set of quantities t_{ki}. It remains to show that t_{ki} are Cartesian components of a second order tensor. To this end, consider the transformation of two sets of Cartesian coordinates and recall

$$x_k' = a_{jk}x_j \ , \ \mathbf{e}_k' = a_{jk}\mathbf{e}_j \ , \ n_k' = a_{jk}n_j \ , \text{ etc. ,} \tag{26}$$

where

$$a_{ik} = \mathbf{e}_i \cdot \mathbf{e}_k' \ . \tag{27}$$

From

$$\mathbf{t} = t_k\mathbf{e}_k = t_k'\mathbf{e}_k' = t_{ki}n_i\mathbf{e}_k = t_{ki}'n_i'\mathbf{e}_k' \tag{28}$$

and with the use of $n_i' = a_{ri}n_r$, $\mathbf{e}_k' = a_{sk}\,\mathbf{e}_s$ where a_{ri} are components of an orthogonal tensor, we obtain $(t_{sr} - a_{ri}a_{sk}t_{ki}')n_r\mathbf{e}_s$. But $\mathbf{e}_s$ are linearly independent. Therefore,

$$(t_{sr} - a_{ri}a_{sk}t_{ki}')n_r = 0 \ , \tag{29}$$

which holds for all **n** and the quantity in parentheses is independent of **n**. Hence,

$$t_{sr} = a_{sk}a_{ri}t_{ki}' \ , \tag{30}$$

which establishes the tensor character of t_{ki}. The second order tensor whose Cartesian components are defined by (24) is called the Cauchy stress tensor.

With reference to a rectangular Cartesian system of spatial coordinates, consider a parallelepiped shown in Fig. 3.3 and recall the formula

$$\mathbf{t}_i = t_{ki}\mathbf{e}_k \tag{31}$$

In (31), $\mathbf{t}_i$ is the stress vector acting on the face whose outward unit normal is $\mathbf{e}_i$. For example, on the front face of the parallelepiped in Fig. 3.3, the outward unit normal coincides with $\mathbf{e}_1$; and thus, from (31) with i = 1, the stress vector on this face is given by

$$\mathbf{t}_1 = t_{k1}\mathbf{e}_k = t_{11}\mathbf{e}_1 + t_{21}\mathbf{e}_2 + t_{31}\mathbf{e}_3 \ .$$

Keeping the above in mind, an examination of the subscripts of the stress tensor t_{ki} in (31) easily reveals that (i) the second index "i" refers to the stress vector $\mathbf{t}_i$ on the face whose outward unit normal is $\mathbf{e}_i$, while (ii) the first index "k" refers to the component of $\mathbf{t}_i$ in the coordinate directions. Utilizing this convention, which stems from the definition of (24), the tractions (i.e., the forces per unit area) are sketched in Fig. 3.3.

4. Derivation of spatial (or Eulerian) form of the equations of motion.

We derive in this section the spatial (or Eulerian) form of the equations of motion from the integral balance laws (6) and (7) of section 2. We begin with the balance of linear momentum (6) in section 2 of Part II, which for the present purpose can be written as (recall that dm = ρdv):

$$\frac{d}{dt}\int_{P}\rho\mathbf{v}dv = \int_{P}\rho\mathbf{b}dv + \int_{\partial P}\mathbf{t}_{(\mathbf{n})}da \quad . \tag{1}$$

By the transport theorem discussed in Part I, the left-hand side of (1) is given by

$$\begin{aligned} \text{LHS of (1)} &= \int_{P}[\dot{\overline{(\rho\mathbf{v})}} + \rho\mathbf{v}\operatorname{div}\mathbf{v}]\, dv \\ &= \int_{P}[\dot{\rho}\mathbf{v} + \rho\dot{\mathbf{v}} + \rho\mathbf{v}\operatorname{div}\mathbf{v}]\, dv \\ &= \int_{P}\{\rho\dot{\mathbf{v}} + \mathbf{v}[\dot{\rho} + \rho\operatorname{div}\mathbf{v}]\, dv \\ &= \int_{P}\rho\mathbf{a}dv \quad , \end{aligned} \tag{2}$$

where in arriving at the results, $(2)_4$, we have also used the local conservation of mass given by (3) of section 1. Next, we consider the surface integral in (1), substitute the relation (23) of section 3, namely $\mathbf{t} = \mathbf{t}_i n_i$, and use the divergence theorem obtain

$$\int_{\partial P}\mathbf{t}da = \int_{\partial P}\mathbf{t}_i n_i da = \int_{P}\mathbf{t}_{i,i}\, dv \quad , \tag{3}$$

where a comma following a subscript indicates partial differentiation.

After substitution of the results (2) and (3) into (1), the resulting equation can be put in the form

$$\int_{P}[\mathbf{t}_{i,i} + \rho\mathbf{b} - \rho\mathbf{a}]\, dv = 0 \quad , \tag{4}$$

which must hold for all arbitrary material volumes. Then, since the integrand is a continuous function, by the usual procedure we arrive at the local form

$$\mathbf{t}_{i,i} + \rho\mathbf{b} = \rho\mathbf{a} \quad . \tag{5}$$

Next, we turn to the balance of moment of momentum (7) in section 2 of Part II, which can be rewritten as

$$\frac{d}{dt}\int_P (\mathbf{x} \times \rho\mathbf{v})\, dv = \int_P (\mathbf{x} \times \rho\mathbf{b})\, dv + \int_{\partial P} \mathbf{x} \times \mathbf{t}_{(\mathbf{n})}\, da \tag{6}$$

Again by the transport theorem, the left-hand side of (6) can be shown to yield

$$\text{LHS of (6)} = \int_P \mathbf{x} \times \rho\mathbf{a}\, dv \quad . \tag{7}$$

Similarly, the surface integral in (6) after substitution from (23) of section 3 and the use of the divergence theorem gives

$$\begin{aligned} \int_{\partial P} \mathbf{x} \times \mathbf{t}_{(\mathbf{n})}\, da &= \int_{\partial P} (\mathbf{x} \times \mathbf{t}_i n_i)\, da = \int_P (\mathbf{x} \times \mathbf{t}_i)_{,i}\, dv \\ &= \int_P [(\mathbf{x}_{,i} \times \mathbf{t}_i) + \mathbf{x} \times \mathbf{t}_{i,i}]\, dv \end{aligned} \tag{8}$$

Introduction of (7) and (8) into (6) results in

$$\int_P [(\mathbf{x}_{,i} \times \mathbf{t}_i) + \mathbf{x} \times (\mathbf{t}_{i,i} + \rho\mathbf{b} - \rho\mathbf{a})]\, dv = 0 \quad ,$$

or

$$\int_P \mathbf{x}_{,i} \times \mathbf{t}_i\, dv = 0 \quad , \tag{9}$$

where in obtaining the last result we have also used (5). Since (9) must hold for all arbitrary material volumes *P* and since the integrand is continuous, by the usual argument we conclude that

$$\mathbf{x},_i \times \mathbf{t}_i = 0 \ ,$$

or equivalently

$$\mathbf{e}_i \times \mathbf{t}_i = 0 \ . \tag{10}$$

The two results (5) and (10) represent the consequences of the balance of linear momentum and moment of momentum. Although the foregoing derivation in vectorial form leading to equations of motion (5) and the restriction (10) is simple and attractive, it conceals some of the features of the stress tensor introduced earlier in section 3. Thus, we now recall the relation (24) for $\mathbf{t}_i = t_{ki}\mathbf{e}_k$ and substitute this into (5) and (10) to obtain the alternative forms of the equations of motion and the restriction on $\mathbf{t}_i$. With the use of the identities

$$\begin{aligned} \mathbf{t}_{i,i} &= (t_{ki}\mathbf{e}_k),_i = t_{ki,i}\mathbf{e}_k \ , \\ \mathbf{e}_i \times \mathbf{t}_i &= \mathbf{e}_i \times t_{ki}\mathbf{e}_k = \varepsilon_{ikj}t_{ki}\mathbf{e}_j \ , \end{aligned} \tag{11}$$

the component forms of (6) and (10) result in

$$t_{ki,i} + \rho b_k = \rho a_k \ , \tag{12}$$

$$t_{ki} = t_{ik} \ . \tag{13}$$

Thus, it is easily seen that the vector equation (5) is equivalent to the three scalar equations of motion (12). Similarly, the restriction (10) on the three vectors $\mathbf{t}_i$ implies the symmetry of the stress tensor t_{ki}. This last observation reveals the fact that the three scalar equations of motion involve only six components of the stress instead of nine.

5. Derivation of the equations of motion in referential (or Lagrangian) form.

In previous developments, the stress vector $\mathbf{t}$ (and therefore the stress tensor t_{ij}) acting on P are measured per unit area of surfaces in the present configuration at time t. In view of the transformation $\mathbf{x} = \chi(\mathbf{X}, t)$ all surfaces in R can be mapped into corresponding surfaces in R_o occupied by the body in the reference configuration. For some purposes, it is more convenient to measure the stress vector and the stress tensor acting on P per unit area of surfaces in P_o. Corresponding to an arbitrary part P with boundary ∂P in the present configuration, we have a part P_o with boundary ∂P_o in the reference configuration (see Fig. 1.1). We denote the outward unit normal to ∂P_o by $\mathbf{N}$, where

$$\mathbf{N} = N_A \mathbf{e}_A \ . \tag{1}$$

We denote the stress vector acting on ∂P, but measured per unit area of the surface ∂P_o in the reference configuration by $\mathbf{p}$.

In terms of quantities measured in the reference configuration, the momentum principles (i.e., linear and moment of momentum) are:

$$\int_{P_o} \rho_o \, \mathbf{a} \, dV = \int_{P_o} \rho_o \, \mathbf{b} \, dV + \int_{\partial P_o} \mathbf{p} \, dA \ , \tag{2}$$

and

$$\int_{P_o} (\mathbf{x} \times \rho_o \, \mathbf{a}) dV = \int_{P_o} (\mathbf{x} \times \rho_o \, \mathbf{b}) dV + \int_{\partial P_o} \mathbf{x} \times \mathbf{p} \, dA \tag{3}$$

where $\mathbf{p} = \mathbf{p}(\mathbf{X}, t\, ; \mathbf{N})$ depends on position, time and the unit normal $\mathbf{N}$ to ∂P_o. By a procedure similar to that used previously (Part II: Sec. 3), we can prove that

$$\mathbf{p} = N_A \mathbf{p}_A \ , \ \mathbf{p}_A = p_{iA} \mathbf{e}_i \ , \tag{4}$$

$$\mathbf{p} = p_{iA} \, N_A \, \mathbf{e}_i \ , \tag{5}$$

$$\mathbf{p} = \mathbf{P} \, \mathbf{N} \ . \tag{5a}$$

The vectors $\mathbf{p}_A$ $(A = 1,2,3)$ represent stress vectors acting across surfaces at a point P in the present configuration κ which were originally coordinate planes $X_A = \text{const.}$ through the corresponding point P_o in the reference configuration and measured per unit area of these planes (Fig. 5.1). Also, the components p_{iA} (i.e., of the stress tensor **P**) represent surface forces acting in the present configuration, but measured per unit area of an X_A-plane in the reference configuration and resolved parallel to the $\mathbf{e}_i$-directions. Introducing (5) into (2), by a procedure similar to that used previously (Part II, Sec. 4), we obtain

$$p_{iA,A} + \rho_o b_i = \rho_o a_i \quad \text{or} \quad \mathbf{p}_{A,A} + \rho_o \mathbf{b} = \rho_o \mathbf{a} \,, \tag{6}$$

Then, from (3), (5), and after using (6), we also obtain

$$p_{iA} x_{j,A} = p_{jA} x_{i,A} \,. \tag{7}$$

Introducing the notation s_{AB} through

$$p_{iB} = x_{i,A} s_{AB} \,, \tag{8}$$

it follows from (7) that

$$s_{AB} = s_{BA} \,. \tag{9}$$

The last two are called, respectively, the symmetric and the nonsymmetric Piola-Kirchhoff stress.

Also, it can be verified that

$$J\mathbf{T} = \mathbf{P}\,\mathbf{F}^T = \mathbf{F}\,\mathbf{S}\,\mathbf{F}^T \quad \text{or} \quad Jt_{ik} = x_{i,A} p_{kA} = x_{k,A} p_{iA} = x_{i,A} x_{k,B} s_{AB} \,, \tag{10}$$

which relates the Cauchy stress **T** to the stress tensors **P** and **S**. To verify the truth of (10), we may start with

$$d\mathbf{f} = \mathbf{t}\, da = \mathbf{p}\, dA \tag{11}$$

which represents an element of contact force acting on the current configuration of the body *B* in terms of both the spatial and referential description of the stress vector. Using the derived relation between the area elements da and dA, namely $\mathbf{n}\, da = J\, F^{-T}\, \mathbf{N}\, dA$, we have

$$\begin{aligned} d\mathbf{f} &= \mathbf{t}\, da = \mathbf{T}\, \mathbf{n}\, da \\ &= \mathbf{p}\, dA = \mathbf{P}\, \mathbf{N}\, dA \ . \end{aligned} \tag{12}$$

Hence $(J\,\mathbf{T}\,\mathbf{F}^{-T} - \mathbf{P})\mathbf{N}\, dA = \mathbf{0}$ for all $\mathbf{N}$ and we may conclude that $J\,\mathbf{T}\,\mathbf{F}^{-T} - \mathbf{P} = \mathbf{0}$ or

$$J\,\mathbf{T} = \mathbf{P}\,\mathbf{F}^{T} \ . \tag{13}$$

Recalling (8) or $\mathbf{P} = \mathbf{F}\,\mathbf{S}$, we are led to the results $(10)_1$.

The coordinate-free forms of the various results between (6)-(10) may be displayed as follows:

$$\mathrm{Div}\,\mathbf{P} + \rho_o\,\mathbf{b} = \rho_o\,\dot{\mathbf{v}} \ , \tag{6a}$$

$$\mathbf{P}\,\mathbf{F}^{T} = \mathbf{F}\,\mathbf{P}^{T} \ , \tag{7a}$$

$$\mathbf{P} = \mathbf{F}\,\mathbf{S} \ , \tag{8a}$$

$$\mathbf{S} = \mathbf{S}^{T} \ . \tag{9a}$$

6. Invariance under superposed rigid body motions.

It was established previously that a particle X of a body, which in the reference configuration is at **X** and at time t in κ occupies the place $\mathbf{x} = \chi(\mathbf{X}, t)$, under superposed rigid motions of the whole body at a different time $t^+ = t + a$ occupies in κ^+ the place $\mathbf{x}^+ = \bar{\chi}^+(\mathbf{X}, t^{+)}$ specified by

$$\mathbf{x}^+ = \mathbf{a} + \mathbf{Q}\,\mathbf{x} \quad \text{or} \quad x_i^+ = a_i + Q_{ij}x_j \ . \tag{1}$$

In (1), **a** is a vector-valued function of t, **Q** is a proper orthogonal tensor-valued function of t and a is a constant. The vector **a** can be interpreted as a rigid body translation and **Q** as a rotation tensor.

We have already studied how various kinematical quantities transform under superposed rigid body motions and shall presently extend these considerations to the dynamical quantities which appear in the equations of motion, namely

$$t_{ij,j} + \rho b_i = \rho \dot{v}_i \ , \ t_{ij} = t_{ji} \ . \tag{2}$$

However, we first need to establish some further kinematical results:

(a) **Further kinematical results:**

Recall that an element of area dA of a material surface having outward unit normal vector **N** in the reference configuration of a body is deformed at time t into the element of area da whose outward unit normal vector is **n**. Then,

$$da_k = J\frac{\partial X_K}{\partial x_k} dA_K \ , \tag{3}$$

where

$$da_k = da\, n_k \ , \ dA_K = dA\, N_K \ . \tag{4}$$

Under the motion $\bar{\chi}^+$ the element of area dA is deformed into da^+ and **N** is deformed into $\mathbf{n}^+$ so that

$$da_k^+ = J^+\frac{\partial X_K}{\partial x_k^+} dA_K \ , \tag{5}$$

and

$$da_k^+ = da^+ n_k^+ \quad . \tag{6}$$

It follows from $(1)_2$ that

$$\frac{\partial x_i^+}{\partial x_k} = Q_{ij}\frac{\partial x_j}{\partial x_k} = Q_{ij}\delta_{jk} = Q_{ik} \tag{7}$$

and consequently

$$F_{iA}^+ = \frac{\partial x_i^+}{\partial X_A} j \tag{8}$$

i.e.,

$$\mathbf{F}^+ = \mathbf{Q}\,\mathbf{F} \quad . \tag{9}$$

Therefore,

$$\begin{aligned} J^+ &= \det \mathbf{F}^+ = \det(\mathbf{Q}\,\mathbf{F}) \\ &= (\det \mathbf{Q})(\det \mathbf{F}) \\ &= +1 \det \mathbf{F} \\ &= J \quad . \end{aligned} \tag{10}$$

Also, by the law of conservation of mass,

$$\rho_o = \rho J = \rho^+ J^+ \quad , \tag{11}$$

which together with $(10)_5$ implies

$$\rho^+ = \rho \quad . \tag{12}$$

Returning again to $(1)_2$, we see that

$$\begin{aligned} Q_{ik}\, x_i^+ &= Q_{ik}\, a_i + Q_{ik}Q_{ij}x_j \\ &= Q_{ik}a_i + \delta_{kj}x_j \\ &= Q_{ik}a_i + x_k \quad , \end{aligned} \tag{13}$$

$$x_k = Q_{ik}(x_i^+ - a_i) \quad \text{or} \quad \mathbf{x} = \mathbf{Q}^T(\mathbf{x}^+ - \mathbf{a}) \quad . \tag{14}$$

Therefore

$$\frac{\partial x_k}{\partial x_i^+} = Q_{ik} \quad , \tag{15}$$

and consequently

$$\frac{\partial X_K}{\partial x_k^+} = \frac{\partial X_K}{\partial x_j}\frac{\partial x_j}{\partial x_k^+} = \frac{\partial X_K}{\partial x_j}Q_{kj} \tag{16}$$

which, in direct notation, reads

$$(\mathbf{F}^+)^{-1} = \mathbf{F}^{-1}\mathbf{Q}^T \tag{17}$$

and could also have been deduced by taking the inverse of both sides of (9).

Substituting the results $(10)_5$ and $(16)_2$ into (5) yields

$$\begin{aligned} da_k^+ &= J\frac{\partial X_K}{\partial x_j}Q_{kj}dA_K \\ &= Q_{kj}da_j \quad , \end{aligned} \tag{18}$$

where (3) has been used in completing the last step. The combination of $(18)_2$ with $(4)_1$ and (6) gives

$$da^+n_k^+ = Q_{kj}da\, n_j \quad . \tag{19}$$

Now square both sides of (19) and remember that **n** and $\mathbf{n}^+$ are unit vectors and **Q** is a proper orthogonal tensor so that

$$\begin{aligned} (da^+)^2 n_k^+ n_k^+ &= (da^+)^2 \\ &= Q_{kj}\, n_j\, Q_{kl}\, n_l\, (da)^2 \\ &= \delta_{jl}\, n_j\, n_l\, (da)^2 \\ &= n_j\, n_j\, (da)^2 = (da)^2 \quad . \end{aligned} \tag{20}$$

Since area is always a positive number, it follws from $(20)_5$ that

$$da^+ = da \tag{21}$$

and hence from (19) that

$$n_k^+ = Q_{kj}\, n_j \quad \text{or} \quad \mathbf{n}^+ = \mathbf{Q}\,\mathbf{n} \ . \tag{22}$$

It is worthwhile noticing that we have been able to deduce the behavior of $\mathbf{F}$, $\mathbf{F}^{-1}$, J, ρ, da and $\mathbf{n}$ under superposed rigid body motions without making any additional physical assumptions in (a).

(b) **The stress vector and the stress tensor.**

It is clear from the developments of section 2 that not all kinematical quantities transform according to formulae of the type

$$\mathbf{u}^+ = \mathbf{Q}(t)\mathbf{u} \ , \ \mathbf{U}^+ = \mathbf{Q}(t)\mathbf{U}\,\mathbf{Q}^T(t) \tag{23}$$

under superposed rigid body motions, where $\mathbf{u}$ and $\mathbf{U}$ in (23) stand for a vector and a second order tensor field, respectively. We investigate now the relationships between† $\mathbf{t}$ and $\mathbf{t}^+$ and between $\mathbf{T}$ and $\mathbf{T}^+$. For convenience, we recall the formulae for t_i and t_{ij}, i.e.,

$$\mathbf{t} = \mathbf{t}(\mathbf{X}, t; \mathbf{n}) \ , \ t_i = t_{ij} n_j \ , \tag{24}$$

$$\mathbf{t}^+ = \mathbf{t}^+(\mathbf{X}, t^+; \mathbf{n}^+) \ , \ t_i^+ \tag{25}$$

The mechanical fields which enter the linear and moment of momentum principles are the stress vector and the body force per unit mass. We first consider the former, which in the motion $\chi(\mathbf{X}, t)$ is a vector field defined by $(24)_1$. Consider now a second motion which differs from the first only by superposed rigid body motions specified in the form (1). The second motion imparts a change in the orientation of the body, so that the outward unit normal vector $\mathbf{n}$ to the same material surface (in the configuration κ) becomes the unit normal vector $\mathbf{n}^+$ (in the configuration κ^+). The stress vector in the second motion takes the form indicated by $(25)_1$ and

† Our development follows Naghdi (1972, pp. 484-486). See also Green and Naghdi (1979).

we have already seen that the outward unit normal vector $\mathbf{n}^+$ transforms according to (22). The geometrical properties of the transformation (1) were discussed in section 2. Keeping these in mind and recalling that $\mathbf{t}$ is linear in $\mathbf{n}$, it is reasonable to expect for the stress vector $\mathbf{t}^+(\mathbf{X}, t; \mathbf{n}^+)$ (i) to have the same magnitude as $\mathbf{t}(\mathbf{X}, t; \mathbf{n})$ and (ii) to have the same relative orientation to $\mathbf{n}^+$ as $\mathbf{t}$ has relative to $\mathbf{n}$. These remarks lead us to introduce the following assumption:

$$\mathbf{t}^+ = \mathbf{Q}\,\mathbf{t} \ , \ \ t_i^+ = Q_{ij} t_j \ . \tag{26}$$

We observe that (26) implies that

$$|\mathbf{t}^+|^2 = \mathbf{t}^+ \cdot \mathbf{t}^+ = t_i^+ \, t_i^+ = Q_{ij}\, Q_{ik}\, t_j\, t_k = \delta_{jk}\, t_j\, t_k = t_j\, t_j = |\mathbf{t}|^2 \tag{27}$$

and, recalling (22), we also have

$$\mathbf{t}^+ \cdot \mathbf{n}^+ = t_i^+ \, n_i^+ = Q_{ij}\, Q_{ik}\, t_j\, n_k = \delta_{jk}\, t_j\, n_k = t_j\, n_j = \mathbf{t} \cdot \mathbf{n} \ . \tag{28}$$

Now

$$\mathbf{t}^+ \cdot \mathbf{n}^+ = |\mathbf{t}^+| \ |\mathbf{n}^+| \cos\theta^+ = |\mathbf{t}^+| \cos\theta^+ \tag{29}$$

and

$$\mathbf{t} \cdot \mathbf{n} = |\mathbf{t}| \ |\mathbf{n}| \cos\theta = |\mathbf{t}| \cos\theta \ ,$$

which together with $(27)_7$ and $(28)_5$ lead to

$$\cos\theta^+ = \cos\theta \ . \tag{30}$$

The results (27) and (30) verify the motivating remarks made prior to (26) to the effect that (i) the magnitude of $\mathbf{t}$ is the same as the magnitude of $\mathbf{t}^+$ and (ii) the magnitude of the angle between $\mathbf{t}$ and the outward unit normal $\mathbf{n}$ is the same as the magnitude of the angle between $\mathbf{t}^+$ and $\mathbf{n}^+$.

We now turn attention to the stress tensor. >From the results $(24)_2$, $(25)_2$ and (22) we have

$$t_i^+ = t_{ij}^+ \, n_j^+ = t_{ij}^+ \, Q_{jk}\, n_k \ . \tag{31}$$

But, by assumption (26), we also have

$$t_i^+ = Q_{ij}\, t_j = Q_{ij}\, t_{jk}\, n_k \ . \tag{32}$$

Combination of (31) and (32) yields

$$(t^{+}_{ij}\, Q_{jk} - Q_{ij}\, t_{jk}) n_k = 0 \quad . \tag{33}$$

The last result must hold for all n_k and the coefficient of n_k is independent of n_k. Therefore

$$t^{+}_{ij}\, Q_{jk} - Q_{ij}\, t_{jk} = 0 \quad ,$$

$$t^{+}_{ij}\, Q_{jk}\, Q_{mk} - Q_{mk}\, Q_{ij}\, t_{jk} = 0 \, ,$$

$$t^{+}_{im} = Q_{ij}\, Q_{mk}\, t_{jk} \quad ,$$

and hence

$$t^{+}_{ij} = Q_{im}\, Q_{jn}\, t_{mn} \tag{34}$$

or, in direct notation,

$$\mathbf{T}^{+} = \mathbf{Q}\,\mathbf{T}\,\mathbf{Q}^{T} \quad . \tag{35}$$

A scalar, vector or tensor quantity which under superposed rigid body motions transforms as (12), (22) or (35), respectively, is called <u>objective</u>. Of course, not all physical quantities are objective. For example

$$\left.\begin{aligned} \mathbf{v}^{+} &= \dot{\mathbf{a}} + \dot{\mathbf{Q}}\,\mathbf{x} + \mathbf{Q}\,\dot{\mathbf{x}} \quad , \\ \mathbf{W}^{+} &= \mathbf{Q}\,\mathbf{W}\,\mathbf{Q}^{T} + \Omega \end{aligned}\right\} \tag{36}$$

are clearly not objective.

(c) **Body forces and accelerations.**

The equations of motion for the two motions χ and $\bar{\chi}^{+}$ of the body are, respectively, given by

$$\left.\begin{aligned} \frac{\partial t_{ij}}{\partial x_j} &= \rho(\dot{v}_i - b_i) \quad , \\ \frac{\partial t^{+}_{ij}}{\partial x^{+}_j} &= \rho^{+}(\dot{v}^{+}_i - b^{+}_i) \quad . \end{aligned}\right\} \tag{37}$$

But

$$
\begin{aligned}
\frac{\partial t^+_{ij}}{\partial x^+_j} &= \frac{\partial}{\partial x_k}(Q_{im}\, Q_{jn}\, t_{mn})\, \frac{\partial x_k}{\partial x^+_j} \\
&= Q_{im}\, Q_{jn}\, \frac{\partial t_{mn}}{\partial x_k}\, Q_{jk} \\
&= Q_{im}\, \delta_{kn}\, \frac{\partial t_{mn}}{\partial x_k} \\
&= Q_{im}\, \frac{\partial t_{mn}}{\partial x_n} \quad , \qquad (38)
\end{aligned}
$$

where (15) has been employed in $(38)_2$. It follows from $(38)_4$ and (37) that

$$\rho^+(\dot{v}^+_i - b^+_i) = Q_{im}\, \rho(\dot{v}_m - b_m) \quad , \qquad (39)$$

which with the aid of (12) becomes

$$\dot{v}^+_i - b^+_i = Q_{im}\,(\dot{v}_m - b_m) \qquad (40)$$

or

$$\dot{\mathbf{v}}^+ - \mathbf{b}^+ = \mathbf{Q}(\dot{\mathbf{v}} - \mathbf{b}) \quad .$$

References

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7. The principle of balance of energy.

We begin by assuming the existence of a scalar potential function per unit mass $\varepsilon = \varepsilon(\mathbf{x},t)$, called the specific internal energy. The internal energy for each part P in the present configuration is defined by the volume integral

$$\int_P \rho\, \varepsilon\, dv \,. \tag{1}$$

We introduce a scalar field $r = r(\mathbf{x},t)$ per unit mass per unit time, called the specific heat supply (or heat absorption), as well as the heat flux across a surface ∂P (with the outward unit normal $\mathbf{n}$) by the scalar $h = h(\mathbf{x},t;\mathbf{n})$ per unit area per unit time. The integrals

$$H = \int_P \rho\, r\, dv - \int_{\partial P} h\, da \,, \tag{2}$$

where ∂P is the boundary of P, defines the heat per unit time entering the part P in the present configuration. The first term on the right-hand side of (2) represents the heat transmitted into P by radiation and the second term the heat entering P by conduction. We also recall that the kinetic energy for each part P in the present configuration is defined as

$$\int_P \frac{1}{2}\, \rho\, \mathbf{v} \cdot \mathbf{v}\, dv \,. \tag{3}$$

We record now the rate of work by the body and contact forces, i.e., $R = R_b + R_c$, where

$$R_b = \int_P \rho\, \mathbf{b} \cdot \mathbf{v}\, dv \;, \qquad R_c = \int_{\partial P} \mathbf{t} \cdot \mathbf{v}\, da \,. \tag{4}$$

In (4), the scalar $\mathbf{b} \cdot \mathbf{v}$ which occurs in the volume integral represents the rate of work per unit mass by the body forces. Similarly, $\mathbf{t} \cdot \mathbf{v}$ is a scalar representing rate of work per unit area by the contact force $\mathbf{t}$. Each of the integrals in (4) is a rate of work term and thus has the dimension of "work per unit time".

With the foregoing background, the law of balance of energy may be stated as follows: The rate of increase of internal energy plus kinetic energy is equal to the rate of work by body force and contact force plus energies due to heat per unit time entering the body. Thus, we write

$$\frac{d}{dt}\int_P \rho\left[\varepsilon + \frac{1}{2}\,\mathbf{v}\cdot\mathbf{v}\right] dv = \int_P \rho r\, dv + \int_P \rho\mathbf{b}\cdot\mathbf{v}\, dv$$

$$+ \int_{\partial P} \mathbf{t}\cdot\mathbf{v}\, da - \int_{\partial P} h\, da \,. \tag{5}$$

We observe that the negative sign in front of the last integral in (5), as well as in (2), is in accord with the convention that heat $h = h(\mathbf{x},t;\mathbf{n})$ is assumed to flow into the surface in the direction opposite to that of the outward unit normal to ∂P.

By application of (5) to an elementary tetrahedron, it can be shown that $h_{(\mathbf{n})} = -h_{(-\mathbf{n})}$ or $h(\mathbf{x},t;\mathbf{n}) = -h(\mathbf{x},t;-\mathbf{n})$, together with

$$h = q_i n_i = \mathbf{q}\cdot\mathbf{n} \;\; , \;\; \mathbf{q} = q_i\mathbf{e}_i \,. \tag{6}$$

The last surface integral in (5) can then be transformed into a volume integral as follows (using the divergence theorem):

$$\int_{\partial P} h\, da = \int_{\partial P} q_i n_i\, da = \int_P q_{i,i}\, dv \,. \tag{7}$$

Similarly, using $\mathbf{t} = t_{ki} n_i \mathbf{e}_k$ and the divergence theorem

$$\int_{\partial P} \mathbf{t}\cdot\mathbf{v}\, da = \int_{\partial P} (\mathbf{t}_i\cdot\mathbf{v})_{,i}\, dv$$

$$= \int_P \left[t_{ki,i} v_k + t_{ki} v_{k,i}\right] dv \,. \tag{8}$$

By the transport theorem, the left-hand side of (5) can be expressed

$$\frac{d}{dt}\int_P \rho\left[\varepsilon + \frac{1}{2}\,\mathbf{v}\cdot\mathbf{v}\right] dv = \int_P \{\rho(\dot{\varepsilon} + \mathbf{v}\cdot\dot{\mathbf{v}}) + (\varepsilon + \frac{1}{2}\,\mathbf{v}\cdot\mathbf{v})(\dot{\rho} + \rho v_{m,m})\}\, dv$$

$$= \int_P \rho(\dot{\varepsilon} + \mathbf{v}\cdot\dot{\mathbf{v}}) \,, \tag{9}$$

where in obtaining the last result conservation of mass has been used.

Now, substitute (7), (8) and (9) into (5) and rearrange the terms to obtain

$$0 = \int_P \{\rho\, r - q_{i,i} - \rho\, \dot{\varepsilon} + t_{ki}v_{k,i} + v_k[t_{ki,i} + \rho(b_k - \dot{v}_k)]\}\, dv\ ,$$

and after using also the equations of motion we get

$$\int_P \{\rho\, r - q_{i,i} - \rho\, \dot{\varepsilon} + t_{ki}v_{k,i}\}\, dv = 0\ , \tag{10}$$

which holds for all arbitrary parts *P*. Hence, it follows that the local energy equation is

$$\rho\, r - q_{i,i} - \rho\, \dot{\varepsilon} + \mathrm{P} = 0\ , \tag{11}$$

where the stress power P is defined by

$$\mathrm{P} = t_{ki}v_{k,i} = t_{ki}d_{ki}\ , \tag{12}$$

and where we have made use of the fact that $v_{k,i} = d_{ki} + w_{ki}$ and t_{ki} is symmetric.

Part III: Examples of Constitutive Equations and Applications

Inviscid and viscous fluid and nonlinear and linear elasticity

Examples of Constitutive Equations

Generally materials are classified as solids or fluids. However, we also encounter materials which do not fall into either of these categories and which possess properties common to both solids and fluids. A strict definition of these classifications is not attempted here, but we mention here some of the main characteristic features of solids and contrast these with corresponding features in the case of fluids. One of the main characteristic features of a solid is that it has a reference state (or a reference configuration); and, in general, the deformed state (or configuration) is not too far from the reference state. Moreover, regardless of the amount of deformation, to some extent a solid remembers its reference state. A fluid, on the other hand, does not possess a reference state (or configuration) and does not necessarily remain near such a state in any motion.

1. Inviscid fluid

In hydrodynamics, an inviscid or an ideal fluid (including an ideal gas) is conceived of as a medium possessing the following properties: (i) it cannot sustain shearing motion and (ii) the stress vector (or the force intensity) acting on any surface in the fluid is always directed along the normal to the surface, i.e., $\mathbf{t}$ is parallel to $\mathbf{n}$. However, it is desirable to begin from a more general viewpoint. Thus, we assume that an inviscid fluid is characterized by a constitutive equation of the form

$$\mathbf{T} = \hat{\mathbf{T}}(\rho) \quad \text{or} \quad t_{ij} = \hat{t}_{ij}(\rho) \quad . \tag{1.1}$$

In (1.1), the response function $\hat{\mathbf{T}}$ is a single-valued function of its argument and ρ is the mass density of the fluid. Morever, it is instructive to consider an even more general assumption and suppose the constitutive response function to depend also on the velocity vector† $\mathbf{v}$. This is because at this stage it is not clear why the explicit dependence on $\mathbf{v}$ (or even $\mathbf{x}$) should be ruled

† We rule out inclusion of the velocity gradient which gives rise to a shearing motion. The latter is not compatible with the properties of an ideal fluid stipulated in the opening paragraph of section 1.

out. With this background, we begin by examining the more general constitutive assumption:

$$\mathbf{T} = \hat{\mathbf{T}}(\rho, \mathbf{v}) \quad \text{or} \quad t_{ij} = \hat{t}_{ij}(\rho, v_k) \quad . \tag{1.2}$$

It is convenient to recall here (from Parts I & II) that under a superposed rigid body motion the position $\mathbf{x}$ moves to $\mathbf{x}^+$ in accordance with

$$\begin{aligned} \mathbf{x}^+(\tau) &= \mathbf{a}(\tau) + \mathbf{Q}(\tau)\mathbf{x} \ , \qquad (\tau \le t) \\ \mathbf{v}^+(\tau) &= \mathbf{a}'(\tau) + \mathbf{Q}(\tau)\mathbf{v}(\tau) + \mathbf{Q}'(\tau)\mathbf{x} \ , \end{aligned} \tag{1.3}$$

where $\mathbf{Q}' = \Omega\,\mathbf{Q}$, Ω is the rigid body angular velocity, the temporary notation $\mathbf{Q}'(\tau)$ is defined by $Q'(\tau) = \frac{d}{d\tau}Q(\tau)$ with τ being real. Consider now a special case of (1.3) for which at time $\tau = t$, $\mathbf{a}'(t) = \dot{\mathbf{a}}(t)$ is a constant vector and Ω is a constant Ω_o, such that at time t

$$\dot{\mathbf{a}}(t) = \mathbf{c} \ , \ \dot{\mathbf{Q}}(t) = \Omega_o\,\mathbf{Q}(t) \ ,$$

where $\mathbf{c}$ is a constant vector and Ω_o is a constant rigid body angular velocity tensor. We note for later reference that when the above special motion represents a rigid body translational velocity, we obtain

$$\begin{aligned} &\mathbf{Q}(t) = \mathbf{I} \ , \ \dot{\mathbf{Q}}(t) = \mathbf{0} \ , \\ &\mathbf{v}^+ = \mathbf{c} + \mathbf{v} \ . \end{aligned} \tag{1.4}$$

Also, when the special motion involves only a constant rigid body angular velocity, we have

$$\begin{aligned} &\mathbf{c} = \mathbf{0} \ , \ \mathbf{Q}(t) = \mathbf{I} \ , \ \dot{\mathbf{Q}}(t) = \Omega_o \ , \\ &\mathbf{v}^+ = \mathbf{v} + \Omega_o\mathbf{x} \ , \ \mathbf{L}^+ = \mathbf{L} + \Omega_o \end{aligned} \tag{1.5}$$

We now examine $(1.2)_1$ under a constant rigid body translational velocity of the form (1.4). Since the constitutive equation $(1.2)_1$ must hold for all motions, including a s.r.b.m., and since on physical grounds the response function $\hat{\mathbf{T}}$ (but not its arguments) must remain unaltered under such superposed rigid body motions, we have

$$\mathbf{T}^+ = \hat{\mathbf{T}}(\rho^+, \mathbf{v}^+) = \hat{\mathbf{T}}(\rho^+, \mathbf{v} + \mathbf{c}) \quad . \tag{1.6}$$

Now under the special motion (1.4), the stress tensor transforms as

$$\mathbf{T}^+ = \mathbf{T} \quad . \tag{1.7}$$

Substituting $(1.2)_1$ and $(1.6)_2$ with (1.7) we get

$$\hat{\mathbf{T}}(\rho, \mathbf{v}) = \hat{\mathbf{T}}(\rho, \mathbf{v} + \mathbf{c}) \quad , \tag{1.8}$$

which must hold for all constant values $\mathbf{c}$. It follows that (1.8) can be satisfied only if the dependence of $\hat{\mathbf{T}}$ on $\mathbf{v}$ is suppressed and we are left with the original assumption (1.1). It should be evident that the same type of conclusion can be reached if we had initially included in (1.2) also dependence on the current position $\mathbf{x}$.

Once more we examine the constitutive equation (1.1) under a general s.r.b.m., and recall that under such motions

$$\mathbf{T}^+ = \mathbf{Q}\,\mathbf{T}\,\mathbf{Q}^T \quad .$$

Introducing the assumption (1.1) into the above invariance requirement, we obtain

$$\hat{\mathbf{T}}(\rho^+) = \mathbf{Q}\,\hat{\mathbf{T}}\,(\rho)\,\mathbf{Q}^T \quad , \tag{1.9}$$

which must hold for all proper orthogonal tensors $\mathbf{Q}$. Since the condition (1.9) is also unaltered if $\mathbf{Q}$ is replaced by $-\mathbf{Q}$, it follows that (1.9) must hold for all orthogonal $\mathbf{Q}$ and not just the proper orthogonal ones. Hence, $\hat{\mathbf{T}}(\rho)$ is an isotropic tensor and can only be a scalar multiple of the identity tensor $\mathbf{I}$ (see the Supplement), i.e.,

$$\mathbf{T} = \hat{\mathbf{T}}(\rho) = -p(\rho)\mathbf{I} \quad \text{or} \quad t_{ij} = \hat{t}_{ij} = -p(\rho)\delta_{ij} \quad . \tag{1.10}$$

In (1.10), p is a scalar function called the pressure and the minus sign is introduced by convention, in order to conform to the traditional form in which the stress tensor $\mathbf{T} = -p\mathbf{I}$ for an inviscid fluid is displayed in the fluid dynamics literature.

Substitution of the result (1.10) into the relation $\mathbf{t} = \mathbf{T}\,\mathbf{n}$ (see Part II) yields

$$\mathbf{t} = -p(\rho)\,\mathbf{n} \quad \text{or} \quad t_k = -p(\rho)n_k \quad , \tag{1.11}$$

which shows that the stress vector acting on a surface in an inviscid fluid is always directed along the normal to the surface. In most books on fluid dynamics, $(1.11)_1$ is taken as a defining property of an inviscid fluid, as was also indicated in the opending paragraph of this section. If (1.11) instead of (1.1) is taken as a starting point, then with the use of $\mathbf{t} = \mathbf{T}\,\mathbf{n}$ we have $\mathbf{T}\,\mathbf{n} = -p(\rho)\mathbf{n}$ or $(\mathbf{T} + p\mathbf{I})\mathbf{n} = 0$ which must hold for all outward unit normal $\mathbf{n}$, and we may conclude that $\mathbf{T} = -p\mathrm{I}$ in agreement with the earlier result (1.10) for the stress tensor in an inviscid fluid.

We are now in a position to obtain the differential equation of motion for the determination of the velocity field $\mathbf{v}$ in an inviscid fluid. Thus, after substituting (1.10) into the equations of motion (see Part II), we arrive at

$$-\mathrm{grad}\ p + \rho\mathbf{b} = \rho\dot{\mathbf{v}} \quad \text{or} \quad p_{,i} + \rho b_i = \rho\dot{v}_i \ . \tag{1.12}$$

For a specified body force $\mathbf{b}$, the vector equation $(1.12)_1$ [or $(1.12)_2$], together with the equation for mass conservation represent four scalar equations for the determination of the four unknown $(p, \mathbf{v})$. The simple structure of (1.12) at first sight may be somewhat deceptive: these equations are difficult to solve by analytical procedures (or even numerical methods). This is because of the nonlinearity due to the convective terms in $\dot{\mathbf{v}}$.

2. Viscous fluid.

Although our main objective here is the development of linear constitutive equations for a (Newtonian) viscous fluid, it is enlightening to start with a more general constitutive assumption. For a viscous fluid we assume that the stress tensor **T** depends on the present value of the mass density ρ, the velocity vector **v** and the velocity gradient tensor **L**. Thus, we write

$$\mathbf{T} = \bar{\mathbf{T}}(\rho, \mathbf{v}, \mathbf{L}) \;\text{ or }\; t_{ij} = \bar{t}_{ij}(\rho, v_k, v_{k,l}) \;, \tag{2.1}$$

where $\bar{\mathbf{T}}$ is a single-valued function of its arguments and satisfies any other continuity or differentiability conditions that may be required in subsequent analysis. Alternatively, for convenience and without loss in generality, we recall that **L = D + W** and rewrite (2.1) as

$$\mathbf{T} = \tilde{\mathbf{T}}(\rho, \mathbf{v}, \mathbf{D}, \mathbf{W}) \;\text{ or }\; t_{ij} = \tilde{t}_{ij}(\rho, v_k, d_{kl}, w_{kl}) \;. \tag{2.2}$$

The constitutive assumption $(2.2)_1$ [or $(2.2)_2$] must hold for all admissible motions, including such superposed rigid body motions as those specified by (1.4) and (1.5). First, by a procedure parallel to that discussed in the previous section (see the development between Eqs. (1.6)-(1.8) of section 1) we may suppress the dependence of the response function in $(2.2)_1$ on **v** and arrive at

$$\mathbf{T} = \hat{\mathbf{T}}(\rho, \mathbf{D}, \mathbf{W}) \;\text{ or }\; t_{ij} = \hat{t}_{ij}(\rho, d_{kl}, w_{kl}) \;. \tag{2.3}$$

Next, from the invariance requirement for **T**, namely $\mathbf{T}^+ = \mathbf{Q}\,\mathbf{T}\,\mathbf{Q}^T$, we obtain the restriction

$$\hat{\mathbf{T}}(\rho^+, \mathbf{D}^+, \mathbf{W}^+) = \mathbf{Q}\,\hat{\mathbf{T}}(\rho, \mathbf{D}, \mathbf{W})\,\mathbf{Q}^T \;, \tag{2.4}$$

which must hold for all proper orthogonal tensors **Q**. Consider now the special superposed rigid body motion specified by (1.5). Under this special motion (2.4) becomes

$$\hat{\mathbf{T}}(\rho, \mathbf{D}\,, \mathbf{W} + \Omega_o) = \hat{\mathbf{T}}(\rho, \mathbf{D}, \mathbf{W}) \;, \tag{2.5}$$

so that the response function $\hat{\mathbf{T}}$ cannot depend on **W** and the constitutive assumption (2.3) is reduced to

$$\mathbf{T} = \hat{\mathbf{T}}(\rho, \mathbf{D}) \;\text{ or }\; t_{ij} = \hat{t}_{ij}(\rho, d_{kl}) \;, \tag{2.6}$$

where $\hat{\mathbf{T}}$ in (2.6) is a different function from that in (2.3). By imposing on (2.6) the invariance requirement we arrive at

$$\hat{\mathbf{T}}(\rho^+, \mathbf{D}^+) = \mathbf{Q}\, \hat{\mathbf{T}}(\rho, \mathbf{D})\mathbf{Q}^T \,. \tag{2.7}$$

The restriction (2.7) must hold for all proper orthogonal tensors **Q**. Since the condition (2.7) is also unaltered if **Q** is replaced by – **Q**, it follows that (2.7) holds for all orthogonal **Q** and not just the proper orthogonal ones.

A viscous fluid characterized by the nonlinear constitutive equation $(2.6)_1$ is known as the Reiner-Rivlin fluid. We do not pursue further discussions of the general forms $(2.6)_1$, but in the remainder of this section consider in some detail the (Newtonian) linear viscous fluid as a special case in which the response function $\hat{\mathbf{T}}$ is a linear function of **D** with coefficients which depend on ρ.

In the discussion of the linear viscous fluid, it is convenient, in what follows, to carry out the details of the development of the constitutive equations in terms of their tensor components. Thus, for a linear viscous fluid, we specifiy $\hat{t}_{ij}$ on the right-hand side of $(2.6)_2$ to be linear in d_{kl} in the form

$$\hat{t}_{ij}(\rho, d_{kl}) = a_{ij} + b_{ijkl}\, d_{kl} \;\;, \tag{2.8}$$

the coefficients a_{ij} and b_{ijkl} are functions of ρ and satisfy the symmetry conditions

$$a_{ij} = a_{ji} \;\;,\;\; b_{ijkl} = b_{jikl} \;\;,\;\; b_{ijkl} = b_{ijlk} \;\;, \tag{2.9}$$

where the condition $(2.9)_3$ arises from the symmetry of d_{kl} and is assumed without loss in generality.

The response function (2.8) must satisfy the invariance condition (2.7) which in component form is

$$\hat{t}_{ij}(\rho, \overset{+}{d}_{kl}) = Q_{im}\, Q_{jn}\, \hat{t}_{mn}\, (\rho, d_{kl}) \;\;, \tag{2.10}$$

where in recording the above we have also used $\rho^+ = \rho$ in the argument of the left-hand side of (2.10). Introducing (2.8) into (2.10) we obtain the restriction

$$a_{ij} + b_{ijkl}\, Q_{kp}\, Q_{lq}\, d_{pq} = Q_{im}\, Q_{jn}\, (a_{mn} + b_{mnpq}\, d_{pq}) \tag{2.11}$$

which holds for all arbitrary d_{kl}. Considering separately the cases when $d_{kl} = 0$ and $d_{kl} \neq 0$, it follows from (2.11) that

$$a_{ij} = Q_{im}\, Q_{jn}\, a_{mn} \ , \tag{2.12}$$

and

$$b_{ijkl} = Q_{im}\, Q_{jn}\, Q_{kp}\, Q_{lq}\, b_{mnpq} \ ,$$

where in obtaining $(2.12)_2$ we have made use of the orthogonality property of Q_{ij}. Now according to the theorems on isotropic tensors (see the Supplement) the coefficients a_{ij} and b_{ijkl} are, respectively, isotropic tensors of order two and four and hence must have the forms

$$\begin{aligned} a_{ij} &= -p\delta_{ij} \\ b_{ijkl} &= \lambda\delta_{ij}\, \delta_{kl} + \mu\delta_{ik}\, \delta_{jl} + \gamma\delta_{il}\, \delta_{jk} \ , \end{aligned} \tag{2.13}$$

where p, λ, μ, γ are functions of ρ only and the use of $-p$ instead of p in $(2.13)_1$ is for later convenience and in order to conform to known expressions in the fluid dynamics literature. If we appeal to the symmetry condition $(2.9)_2$, then the right-hand side of $(2.13)_2$ must be symmetric in the pair of indices (ij) and $(2.13)_2$ reduces to

$$b_{ijkl} = \lambda\delta_{ij}\, \delta_{kl} + \mu(\delta_{ik}\, \delta_{jl} + \delta_{il}\, \delta_{jk}) \ . \tag{2.14}$$

It should be noted that the expression (2.14) implies a further symmetry restriction so that b_{ijkl} satisfies

$$b_{ijkl} = b_{klij} \ , \tag{2.15}$$

in addition to $(2.9)_{2,3}$. Substitution of $(2.13)_1$ and (2.14) into (2.8) finally yields

$$t_{ij} = -p\delta_{ij} + \lambda\delta_{ij}\, d_{kk} + 2\mu d_{ij} \ , \tag{2.16}$$

as the constitutive equation for a (Newtonian) linear viscous fluid. The scalar p in (2.16) is the pressure and λ, μ are called the viscosity coefficients.

The differential equations of motion for a (Newtonian) linear viscous fluid in terms of the velocity field can be obtained by substituting (2.16) into Cauchy's stress equations of motion and leads to

$$-p_{,i} + \lambda d_{kk,i} + 2\mu d_{ij,j} + \rho b_i = \rho \dot{v}_i \quad .$$

After recalling the kinematical relations

$$d_{kk} = v_{k,k} \quad , \quad d_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i})$$

and making use of $v_{j,ij} = v_{j,ji}$, the above differential equations of motion become

$$-p_{,i} + (\lambda + \mu)\, v_{k,ki} + \mu v_{i,kk} + \rho b_i = \rho \dot{v}_i \quad , \tag{2.17a}$$

or

$$-\nabla p + (\lambda + \mu)\nabla(\nabla \cdot \mathbf{v}) + \mu \nabla^2 \mathbf{v} + \rho \mathbf{b} = \rho \dot{\mathbf{v}} \quad . \tag{2.17b}$$

The system of differential equations (2.17) and the mass conservation represent four scalar equations for the determination of the four unknowns, (p, **v**). It should be noted in the absence of viscous affects ($\lambda = \mu = 0$), (2.17) reduces to those for an inviscid fluid (see Eqs. (1.12)).

3. Elastic solids: Nonlinear constitutive equations.

We consider here general constitutive equations for an elastic solid in the context of the purely mechanical theory.

A material is defined by a constitutive assumption. Moreover, an elastic material in the purely mechanical theory may be defined in terms of a constitutive equation for the stress. We return to this later but first we need to establish some preliminaries.

We recall the expression for the rate of work by the body and contact forces as follows (see Sec. 7 of Part II):

$$R = R_b + R_c \ , \tag{3.1}$$

$$R_b = \int_{\mathcal{P}} \rho \mathbf{b} \cdot \mathbf{v} \, dv \ , \ R_c = \int_{\partial \mathcal{P}} \mathbf{t} \cdot \mathbf{v} \, da \ . \tag{3.2}$$

Also, the local equations of motion (in terms of the symmetric stress tensor of Cauchy) are:

$$t_{ki,i} + \rho b_k = \rho \dot{v}_k \ , \ t_{ki} = t_{ik} \ . \tag{3.3}$$

Consider now the expression for R_c, use the divergence theorem to transform the surface integral into a volume integral and make use of the equations of motion $(3.3)_1$ to obtain:

$$\begin{aligned} R_c = \int_{\partial \mathcal{P}} \mathbf{t} \cdot \mathbf{v} \, da &= \int_{\partial \mathcal{P}} t_k v_k da = \int_{\partial \mathcal{P}} t_{ki} n_i v_k da = \int_{\mathcal{P}} (t_{ki} v_k)_{,i} dv \\ &= \int_{\mathcal{P}} t_{ki,i} v_k dv + \int_{\mathcal{P}} t_{ki} v_{k,i} dv \\ &= \int_{\mathcal{P}} \rho \dot{v}_k v_k dv - \int_{\mathcal{P}} \rho b_k v_k dv + \int_{\mathcal{P}} t_{ki} v_{k,i} dv \ . \end{aligned} \tag{3.4}$$

Then, after using $(3.2)_1$

$$R_c = \int_{\mathcal{P}} \rho \dot{v}_k v_k dv - R_b + \int_{\mathcal{P}} t_{ki} v_{k,i} dv \ . \tag{3.5}$$

From this result and (3.1) follows that

$$R = \frac{d}{dt} K(\mathcal{S}_t) + \int_{\mathcal{P}} t_{ki} v_{k,i} dv \ , \tag{3.6}$$

where $K(S_t) = \int_P \frac{1}{2}\rho \mathbf{v} \cdot \mathbf{v} \, dv$ is the kinetic energy for the material region occupied by *P* in the present configuration κ .

We define an elastic body as one whose response depends on deformation gradient or on an appropriate measure of strain. We assume the existence of a strain energy or a stored energy per unit mass ψ such that

$$t_{ki} v_{k,i} = \rho \dot{\psi} \ , \tag{3.7}$$

and we define the (total) strain energy for each part *P* in the present configuration by

$$U(S_t) = \int_P \rho \, \psi \, dv \ . \tag{3.8}$$

Then, by (3.6), (3.7), (3.8) the conservation of mass and the transport theorem we have

$$R(S_t) = \frac{d}{dt}[K(S_t) + U(S_t)] \ . \tag{3.9}$$

We have thus proved the following theorem: The rate of work by contact and body forces is equal to the sum of the rate of kinetic energy and the rate of the strain energy.

We now return to (3.7) and for an elastic body assume that the strain energy density $\psi = \psi'(\mathbf{F})$ [or $\psi = \psi'(x_{i,A})$].

Alternatively, we may assume that ψ depends on a measure of strain such as $\mathbf{E}$ and write

$$\psi = \bar{\psi}(\mathbf{E}) \ \text{ or } \ \psi = \bar{\psi}(E_{AB}) \ . \tag{3.10}$$

Now the material derivative of ψ is

$$\dot{\psi} = \frac{\partial \bar{\psi}}{\partial E_{AB}} \dot{E}_{AB} x_{i,B} \ , \tag{3.11}$$

where in obtaining (3.11) we have used (from Part I) the expressions

$$2E_{AB} = C_{AB} - \delta_{AB} \ , \ C_{AB} = x_{i,A} x_{i,B} \ ,$$

$$\dot{E}_{AB} = d_{ik} x_{k,A} x_{i,B} \ , \ \dot{C}_{AB} = 2\dot{E}_{AB} \ .$$

Introducing (3.11) into (3.7), we obtain

$$t_{ki}d_{ki} = \rho \frac{\partial \bar{\psi}}{\partial E_{AB}}$$

or

$$(t_{ki} - \rho \frac{\partial \bar{\psi}}{\partial E_{AB}} x_{i,A}x_{k,B})d_{ki} = 0 \ , \qquad (3.12)$$

which must hold for all d_{ki}. Hence, we conclude that

$$t_{ki} = \rho x_{i,A}x_{k,B} \frac{\partial \bar{\psi}}{\partial E_{AB}} \ \text{ or } \ \mathbf{T} = \rho \mathbf{F} \frac{\partial \bar{\psi}}{\partial \mathbf{E}} \mathbf{F}^T \ . \qquad (3.13)$$

If instead of the constitutive assumption (3.10), we begin with the alternative assumption

$$\psi = \hat{\psi}(\mathbf{C}) \ \text{ or } \ \psi = \hat{\psi}(C_{AB}) \ , \qquad (3.14)$$

then in a manner analogous to the development that led to (3.14) we deduce that

$$t_{ki} = 2\rho \ x_{i,A}x_{k,B} \frac{\partial \hat{\psi}}{\partial C_{AB}} \ \text{ or } \ \mathbf{T} = 2\rho \ \mathbf{F} \frac{\partial \hat{\psi}}{\partial \mathbf{C}} \mathbf{F}^T \ . \qquad (3.15)$$

4. Elastic solids: Linear constitutive equations.

We recall that in the context of infinitesimal kinematics discussed previously (Part I, section 7), a function f is said to be of $0(\varepsilon^n)$ as $\varepsilon \to 0$ if there exists a nonnegative real constant C such that $|f| < C\varepsilon^n$ as $\varepsilon \to 0$. Moreover, for infinitesimal deformation gradients, the distinction between Lagrangian and Eulerian descriptions disappears; and that, to within $0(\varepsilon^2)$, the relative displacement gradient can be written as

$$u_{i,j} = e_{ij} + \omega_{ij} \ , \tag{4.1}$$

where

$$e_{ij} = \frac{1}{2}\,(u_{i,j} + u_{j,i}) = 0(\varepsilon) \ , \ \ \omega_{ij} = \frac{1}{2}\,(u_{i,j} - u_{j,i}) = 0(\varepsilon) \ ,$$

$$\text{as } \varepsilon \to 0 \ . \tag{4.2}$$

We further assume that all derivatives (with respect to both coordinates and time) of kinematical quantities, *e.g.*, u_i, e_{ij}, ω_{ij}, are also of $0(\varepsilon)$ as $\varepsilon \to 0$.

We now introduce additional assumptions in order to obtain a <u>complete</u> linear theory. Thus, we assume that the fixed reference configuration (which was identified with the initial configuration) is "stress-free", *i.e.*, in this configuration the stress vector **t** and the body force **b** are zero. We also assume that the body force **b** (or its components b_i) and the stress vector **t** (or its components t_i) -- when expressed in suitable non-dimensional form -- as well as the components of the stress tensor t_{ki}, all are of $0(\varepsilon)$ as $\varepsilon \to 0$.

Recalling that for infinitesimal motion (see Eq. (7.32) of Part 1, section 7)

$$J^{-1} = 1 - e_{ii} = 1 - 0(\varepsilon) \text{ as } \varepsilon \to 0 \ , \tag{4.3}$$

from the referential form of the conservation of mass, namely $\rho J = \rho_o$, we conclude that

$$\rho = \rho_o - 0(\varepsilon) \text{ as } \varepsilon \to 0 \ . \tag{4.4}$$

Keeping in mind that **t**, **b**, $\ddot{\mathbf{u}}$ are all of $0(\varepsilon)$ as $\varepsilon \to 0$, with the help of (4.4) and after the neglect of terms of $0(\varepsilon^2)$ as $\varepsilon \to 0$, the equations of motion in the linear theory reduce to

$$t_{ki,i} + \rho_o b_k = \rho_o \dot{v}_k \ , \tag{4.5}$$

$$t_{ki} = t_{ik} \ , \tag{4.6}$$

where $v_i = \dot{u}_i$ is the particle velocity, a comma denotes partial differentiation and the $\dot{v}_i$ on the right-hand side of (4.5) stands for $\frac{\partial v_i}{\partial t}$. It should be observed that each term in (4.5) is of $0(\varepsilon)$ as $\varepsilon \to 0$.

The equations of motion of the linear theory can also be obtained from the results of the alternative derivation (see Part II, section 5). Thus, in terms of the non-symmetric Piola-Kirchhoff stress tensor p_{iA}, the equations of motion are

$$p_{iA,A} + \rho_o b_i = \rho_o \dot{v}_i \ , \tag{4.7}$$

$$p_{iA} x_{j,A} = p_{jA} x_{i,A} \ , \tag{4.8}$$

and we also recall the relations

$$p_{iB} = x_{i,A} s_{AB} \ , \quad s_{AB} = s_{BA} \ , \tag{4.9}$$

$$J t_{ki} = x_{k,A} p_{iA} = x_{k,A} x_{i,B} s_{AB} \ . \tag{4.10}$$

Since $J = 1 + 0(\varepsilon)$, $x_{i,A} = \delta_{iA} + u_{i,A} = \delta_{iA} + 0(\varepsilon)$, $X_{A,i} = \delta_{Ai} + 0(\varepsilon)$ as $\varepsilon \to 0$, we have

$$\begin{aligned} p_{iA} &= J t_{ki} X_{A,k} \\ &= t_{ki} \delta_{Ak} + 0(\varepsilon^2) \quad \text{as} \quad \varepsilon \to 0 \ . \end{aligned} \tag{4.11}$$

Similarly, since now $p_{iA} = 0(\varepsilon)$ as $\varepsilon \to 0$, $(4.9)_1$ after using (4.11) yields

$$\begin{aligned} s_{AB} &= p_{iB} X_{A,i} \\ &= p_{iB} \delta_{Ai} + 0(\varepsilon^2) \quad \text{as} \quad \varepsilon \to 0 \ . \end{aligned} \tag{4.12}$$

It follows from (4.11) and (4.12) that if terms of $0(\varepsilon^2)$ as $\varepsilon \to 0$ are neglected, then p_{iA}, s_{AB}, t_{ij}

are all equal and the distinction between the Cauchy stress and the Piola-Kirchhoff stresses disappears. Hence, the linearized version of the equations of motion (4.7) can also be written as

$$s_{AB,B} + \rho_o b_A = \rho_o \ddot{u}_A \ . \tag{4.13}$$

We now turn our attention to the linearized version of the constitutive equation discussed in the previous section. We recall that in the context of nonlinear elasticity, the constitutive equations for the stress tensor may be expressed as

$$t_{ki} = \rho x_{i,A} x_{k,B} \frac{\partial \bar{\psi}}{\partial E_{AB}} \ . \tag{4.14}$$

In the linear theory discussed above the components of the stress tensor $t_{ki} = 0(\varepsilon)$ as $\varepsilon \to 0$. Since the right-hand side of (4.14) must also be of $0(\varepsilon)$ as $\varepsilon \to 0$ in the linear theory, for an elastic body which is initially "stress-free", after linearization of (4.14), it will suffice to assume that $\bar{\psi}$ is quadratic in the infinitesimal strain e_{ij}. To justify this stipulation, we proceed to obtain the linearized version of (4.14). Recall that in the infinitesimal theory,

$$J = 1 + 0(\varepsilon)$$

$$x_{i,A} = \delta_{iA} + 0(\varepsilon)$$

$$e_{ij} = E_{AB}\delta_{Ai}\delta_{Bj} + 0(\varepsilon^2) \ .$$

Using these expressions, we may linearize equation (4.14) to obtain

$$t_{ki} = \rho x_{i,A} x_{k,B} \frac{d\bar{\psi}}{dE_{AB}} = \rho_o \left(1 - 0(\varepsilon)\right) \left(\delta_{iA} + 0(\varepsilon)\right) \left(\delta_{kB} + 0(\varepsilon)\right) \frac{d\bar{\psi}}{dE_{AB}}$$

$$= \rho_o \frac{d\bar{\psi}}{dE_{AB}} \delta_{iA}\delta_{kB} + 0(\varepsilon^2) = \rho_o \frac{\partial\bar{\psi}}{\partial e_{mn}} \frac{\partial e_{mn}}{\partial E_{AB}} \delta_{iA}\delta_{kB} + 0(\varepsilon^2)$$

$$= \rho_o \frac{\partial\bar{\psi}}{\partial e_{mn}} \frac{\partial(E_{PQ}\delta_{Pm}\delta_{Qn})}{\partial E_{AB}} \delta_{iA}\delta_{kB}$$

$$= \rho_o \frac{\partial\bar{\psi}}{\partial e_{mn}} \frac{\partial E_{PQ}}{\partial E_{AB}} \delta_{Pm}\delta_{Qn}\delta_{iA}\delta_{kB} \ . \tag{4.15}$$

Since E_{AB} is a symmetric second order tensor, we have

$$\frac{\partial E_{PQ}}{\partial E_{AB}} = \frac{1}{2}\left(\frac{\partial E_{PQ}}{\partial E_{AB}} + \frac{\partial E_{PQ}}{\partial E_{BA}}\right) = \frac{1}{2}\left(\delta_{PA}\delta_{QB} + \delta_{PB}\delta_{QA}\right)$$

Supplements to the Main Text

ME 185 Class Notes

Supplements to the Main text in Class Notes

Rather than writing expressions in terms of components of vectors and tensors, it is sometimes convenient to use a coordinate free notation. In coordinate free notation, a vector is referred to by the symbol **v**, whereas in indicial notation this vector is written in terms of its components v_i with respect to a basis $\mathbf{e}_i$. The values of the components v_i are dependent on the choice of basis. Similarly, a tensor may be referred to in coordinate free notation as **T** or in indicial notation as T_{ij} with respect to a given basis. Notice that tensor multiplication is not commutative; however, the components of a tensor in indicial notation are commutative. The following relationships between indicial and coordinate free notation are sometimes useful:

Coordinate Free Notation	Indicial Notation
v (a vector)	v_i
A (a second order tensor)	A_{ij}
$\mathbf{Av}$	$A_{ij}v_j$
$\mathbf{A}^T\mathbf{v}$	$A_{ji}v_j$
$\mathbf{AB}$	$A_{ij}B_{jk}$
$\mathbf{A}^T\mathbf{B}$	$A_{ji}B_{jk}$
$\mathbf{AB}^T$	$A_{ij}B_{kj}$

Examples from kinematics:

$\mathbf{C} = \mathbf{F}^T\mathbf{F}$	$C_{AB} = F_{iA}F_{iB}$
$\mathbf{B} = \mathbf{FF}^T$	$B_{ij} = F_{iA}F_{jA}$
$\mathbf{U}^2 = \mathbf{C}$	$U_{AD}U_{DB} = C_{AB}$

Eigenvalues, eigenvectors, and characteristic equations.

In general, a second order tensor **T** operating on a vector **v** results in another vector (say) **w** which is not necessarily in the same direction and does not necessarily have the same magnitude as **v**. This operation is expressed symbolically as

$$\mathbf{Tv} = \mathbf{w} \quad \text{or} \quad T_{ij}v_j = w_i \ . \tag{1}$$

However, if a nonzero vector **v** is such that when **T** operates on it the resulting vector **w** is parallel to **v**, then **v** is called an eigenvector of the tensor **T**.

In this case, we can write

$$\mathbf{Tv} = \beta\mathbf{v} \quad \text{or} \quad T_{ij}v_j = \beta v_i \ , \tag{2}$$

where the scalar β is called an eigenvalue corresponding to the eigenvector **v**. From (2), we can write

$$(\mathbf{T} - \beta\mathbf{I})\mathbf{v} = \mathbf{0} \quad \text{or} \quad (T_{ij} - \beta\delta_{ij})v_j = 0 \ , \tag{3}$$

where **I** is the identity tensor. Equation (3) represents three equations which must be satisfied by all three components of **v** in order that **v** be an eigenvector of the tensor **T**. According to a theorem of linear algebra, a homogeneous system of three equations for three unknowns (such as equation (3)), has a non-trivial solution if and only if the determinant of the matrix of coefficients vanishes, i.e. if

$$\det(\mathbf{T} - \beta\mathbf{I}) = 0 \quad \text{or} \quad \det(T_{ij} - \beta\delta_{ij}) = 0 \ . \tag{4}$$

Equation (4) represents a cubic equation in β, which may be written as

$$\phi(\beta) = -\beta^3 + I_1\beta^2 - I_2\beta + I_3 = 0 \ . \tag{5}$$

Equation (5) is called the characteristic equation of **T** and I_1 , I_2 , and I_3 are called the principal invariants of **T**. Expressions for the principal invariants can be obtained by solving the system (4), giving

$$I_1 = \mathrm{tr}\mathbf{T} \qquad\qquad I_1 = T_{ii}$$

$$I_2 = \frac{1}{2}(\mathrm{tr}^2\mathbf{T} - \mathrm{tr}\mathbf{T}^2) \qquad \text{or} \qquad I_2 = \frac{1}{2}(T_{ii}T_{jj} - T_{ij}T_{ji}) \tag{6}$$

$$I_3 = \det\mathbf{T} \qquad\qquad I_3 = \det(T_{ij}) = \frac{1}{6}\varepsilon_{ijk}\varepsilon_{lmn}T_{il}T_{jm}T_{kn}\ .$$

In general equation (5) has three roots which are not necessarily real and are not necessarily distinct. However, in the special case where **T** is symmetric, we can prove some additional theorems.

Theorem 1: If T is symmetric, then all the roots of the characteristic equation (5) are real.

Proof: We will prove this theorem by contradiction. Suppose first that the roots of (5) are not real. According to a theorem of algebra, the complex roots of a polynomial with real coefficients must occur in conjugate pairs. Therefore, if there exists a complex root $\beta^{(1)}$ of (5), then there must also exist a root $\beta^{(2)}$ of (5) which is a conjugate of $\beta^{(1)}$. Thus $\beta^{(1)}$ and $\beta^{(2)}$ must be of the form

$$\begin{aligned} \beta^{(1)} &= \mu + i\gamma \\ \beta^{(2)} &= \mu - i\gamma\,, \end{aligned} \tag{7}$$

where $i^2 = -1$. The eigenvectors which correspond to the eigenvalues $\beta^{(1)}$ and $\beta^{(2)}$ are of the form

$$\mathbf{v}^{(1)} = \boldsymbol{\alpha} + i\boldsymbol{\delta} \qquad\qquad v_j^{(1)} = \alpha_j + i\delta_j$$

$$\text{or} \tag{8}$$

$$\mathbf{v}^{(2)} = \boldsymbol{\alpha} - i\boldsymbol{\delta} \qquad\qquad v_j^{(2)} = \alpha_j - i\delta_j\ .$$

Since equation (2) must hold for all eigenvectors and their corresponding eigenvalues, we can write

$$\mathbf{T}\mathbf{v}^{(1)} = \beta^{(1)}\mathbf{v}^{(1)} \qquad \text{or} \qquad T_{ij}v_j^{(1)} = \beta^{(1)}v_i^{(1)}$$

$$\mathbf{T}\mathbf{v}^{(2)} = \beta^{(2)}\mathbf{v}^{(2)} \qquad\qquad T_{ij}v_j^{(2)} = \beta^{(2)}v_i^{(2)} \,. \tag{9}$$

Multiply $(9)_1$ by $v_i^{(2)}$ and $(9)_2$ by $v_i^{(1)}$ to obtain (in indicial notation)

$$\beta^{(1)}v_i^{(1)}v_i^{(2)} = T_{ij}v_j^{(1)}v_i^{(2)}$$
$$\beta^{(2)}v_i^{(2)}v_i^{(1)} = T_{ij}v_j^{(2)}v_i^{(1)} = T_{ji}v_j^{(2)}v_i^{(1)} = T_{ij}v_j^{(1)}v_i^{(2)} \;. \tag{10}$$

In obtaining $(10)_2$, we have made use of the facts that T_{ij} is symmetric and that i and j are dummy variables. Subtracting $(10)_2$ from $(10)_1$ gives

$$(\beta^{(1)} - \beta^{(2)})v_i^{(1)}v_i^{\,(2)} = 0 \;. \tag{11}$$

Substituting (7) and (8) into (11), we can write

$$0 = 2\gamma i(\alpha_j + i\delta_j)\,(\alpha_j - i\delta_j) = 2\gamma i(\alpha_j\alpha_j + \delta_j\,\delta_j) \;. \tag{12}$$

The term in parenthesis in (12) is always greater than zero since it is the sum of two squared real numbers. Therefore, we must have $\gamma = 0$. Equation (7) then implies that $\beta^{(1)}$ and $\beta^{(2)}$ are both real, which is a contradiction of our initial assumption. We can therefore conclude that if **T** is symmetric, all the roots of its characteristic equation are real.

Theorem 2: The eigenvectors which correspond to distinct eigenvalues of a symmetric tensor are orthogonal.

Proof: Suppose that $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$ are eigenvectors of a symmetric tensor **T** corresponding to distinct eigenvalues $\beta^{(1)}$ and $\beta^{(2)}$, i.e. $\beta^{(1)} \neq \beta^{(2)}$. As in the previous proof, we can show that equation (11) must be satisfied. Since $\beta^{(1)} \neq \beta^{(2)}$, we must have

$$\mathbf{v}^{(1)} \cdot \mathbf{v}^{(2)} = 0 \quad \text{or} \quad v_i^{(1)} v_i^{(2)} = 0 \,. \tag{13}$$

Equation (13) is exactly the definition of orthogonal vectors, thus the eigenvectors $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$ are orthogonal.

Theorem 3: Every second order symmetric tensor has three linearly independent principal directions.

Proof: For the proof of this theorem, see Advanced Engineering Mathematics, Wylie, p. 541.

Remark: In the preceding development, we can, without loss of generality, replace the eigenvector $\mathbf{v}$ with a unit vector $\mathbf{m}$, where

$$\mathbf{m} = \mathbf{v}/|\mathbf{v}| \,. \tag{14}$$

The vector $\mathbf{m}$ is called the normalized eigenvector.

Using theorems 1, 2 and 3 and the previous remark, we can state that any symmetric tensor $\mathbf{T}$ has three real eigenvalues $\beta^{(1)}$, $\beta^{(2)}$, and $\beta^{(3)}$ and three orthonormal eigenvectors $\mathbf{m}^{(1)}$, $\mathbf{m}^{(2)}$, and $\mathbf{m}^{(3)}$. It can be shown, using a theorem of linear algebra, that since $\mathbf{T}$ has three linearly independent eigenvectors, it can be transformed into a diagonalized form, denoted by Λ, using the transformation

$$T'_{ij} = \Lambda_{ij} = a_{mi} a_{nj} T_{mn} \,,$$

where

$$a_{ij} = \mathbf{e}_i \cdot \mathbf{m}^{(j)} \quad \text{and} \quad x_i = a_{ij} x'_j \,. \tag{15}$$

Here the primed quantities correspond to the transformed system. It can also be shown that the diagonal members of Λ are simply the eigenvalues, so that

$$(\Lambda_{ij}) = \begin{bmatrix} \beta^{(1)} & 0 & 0 \\ 0 & \beta^{(2)} & 0 \\ 0 & 0 & \beta^{(3)} \end{bmatrix} \quad . \tag{16}$$

Quadratic forms and positive definiteness.

Associated with any symmetric second order tensor **T** is a scalar valued function of a vector Q(**v**), where **v** is an arbitrary vector, defined by

$$Q(\mathbf{v}) = (\mathbf{Tv}) \cdot \mathbf{v} = T_{ij}\, v_j v_i = \mathbf{v} \cdot (\mathbf{Tv}) \; . \tag{17}$$

We call Q(**v**) a quadratic form. The tensor **T** is said to be positive definite if for all $\mathbf{v} \neq \mathbf{0}$, $Q(\mathbf{v}) > 0$.

Theorem 4: A symmetric second order tensor **T** is positive definite if and only if all the eigenvalues of **T** are positive.

Proof: Again using a theorem from linear algebra, one can conclude that since the eigenvectors $\mathbf{m}^{(1)}$, $\mathbf{m}^{(2)}$, and $\mathbf{m}^{(3)}$ are orthogonal in a 3-dimensional space, then any arbitrary vector **w** in that space may be represented as a linear combination of $\mathbf{m}^{(1)}$, $\mathbf{m}^{(2)}$, and $\mathbf{m}^{(3)}$. Thus, we can write

$$\mathbf{w} = a_1\mathbf{m}^{(1)} + a_2\mathbf{m}^{(2)} + a_3\mathbf{m}^{(3)} \; . \tag{18}$$

The eigenvalues of **T** corresponding to $\mathbf{m}^{(1)}$, $\mathbf{m}^{(2)}$, and $\mathbf{m}^{(3)}$ are $\beta^{(1)}$, $\beta^{(2)}$, and $\beta^{(3)}$, as before, so equation (2) gives

$$\mathbf{Tm}^{(i)} = \beta^{(i)}\mathbf{m}^{(i)} \; . \tag{19}$$

Using (18) and (19), we can write

$$\mathbf{Tw} = a_1\beta^{(1)}\mathbf{m}^{(1)} + a_2\beta^{(2)}\mathbf{m}^{(2)} + a_3\beta^{(3)}\mathbf{m}^{(3)} . \tag{20}$$

Substituting (20) into (17) gives

$$Q(\mathbf{w}) = T_{ij}w_jw_i = a_1^2\beta^{(1)} + a_2^2\beta^{(2)} + a_3^2\beta^{(3)} . \tag{21}$$

Necessity: If **T** is positive definite, then $Q(\mathbf{w}) > 0$ for all **w**. Since a_1 , a_2 , and a_3 are all real numbers, the square of each of these terms must be positive (or zero). If $\mathbf{w} = a_1\mathbf{m}^{(1)}$, then $Q(\mathbf{w}) > 0$ implies from (21) that $\beta^{(1)}a_1^2 > 0$, or that $\beta^{(1)} > 0$. Since **w** is arbitrary, a similar argument shows that $\beta^{(2)}$ and $\beta^{(3)}$ are both greater than zero. Thus, if **T** is positive definite, then all the eigenvalues of **T** are positive.

Sufficiency: If all the eigenvalues of **T** are greater than zero, then by (21) it follows that $Q(\mathbf{w}) > 0$ for all $\mathbf{w} \neq 0$. Therefore, **T** is positive definite if all the eigenvalues of **T** are positive.

Theorem 5: If **B** is a positive definite symmetric tensor, then there exists a unique symmetric positive definite square root **T** of **B** such that $\mathbf{T}^2 = \mathbf{B}$.

Remarks on the calculation of the square root of B:

Let $B_{ik} = a_{ip}a_{kq}\Lambda_{pq}$ (see (15)), where Λ is a diagonal matrix as in equation (16). Define a tensor $\Lambda^{\frac{1}{2}}_{pq}$ as

$$(\Lambda^{\frac{1}{2}})_{pq} = \begin{bmatrix} \sqrt{\beta^{(1)}} & 0 & 0 \\ 0 & \sqrt{\beta^{(2)}} & 0 \\ 0 & 0 & \sqrt{\beta^{(3)}} \end{bmatrix} ,$$

so that

$$\Lambda^{\frac{1}{2}}_{pq}\Lambda^{\frac{1}{2}}_{qs} = \Lambda_{ps} \ . \tag{22}$$

Defining $T_{ij} = a_{ip}a_{jq}\Lambda^{\frac{1}{2}}_{pq}$, we can show that

$$\begin{aligned} T_{ij}T_{jk} &= a_{ip}a_{jq}\,\Lambda^{\frac{1}{2}}_{pq}a_{jm}a_{kn}\Lambda^{\frac{1}{2}}_{mn} = a_{ip}a_{jq}a_{jm}a_{kn}\Lambda^{\frac{1}{2}}_{pq}\Lambda^{\frac{1}{2}}_{mn} \\ &= a_{ip}a_{kn}\delta_{qm}\,\Lambda^{\frac{1}{2}}_{pq}\Lambda^{\frac{1}{2}}_{mn} = a_{ip}a_{kn}\Lambda_{pn} = B_{ik} \end{aligned}$$

or

$$\mathbf{T}^2 = \mathbf{B} \ . \tag{23}$$

Theorem 6: Polar decomposition theorem (see class notes section I/3)

Any non-singular (invertible) tensor $\mathbf{F}$ may be uniquely expressed as

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR} \ , \tag{24}$$

where $\mathbf{R}$ is orthogonal and $\mathbf{U}$ and $\mathbf{V}$ are symmetric positive definite tensors.

Proof: Existence

Define $\mathbf{U} = (\mathbf{F}^T\mathbf{F})^{\frac{1}{2}}$ and $\mathbf{R} = \mathbf{F}\mathbf{U}^{-1}$. We first show that $\mathbf{U}$ is symmetric postive definite. To show that $\mathbf{U}^2$ is positive definite, consider the quadratic form

$$Q(\mathbf{v}) = \mathbf{v} \cdot \mathbf{U}^2\mathbf{v} = \mathbf{v} \cdot \mathbf{F}^T\mathbf{F}\mathbf{v} = \mathbf{F}\mathbf{v} \cdot \mathbf{F}\mathbf{v} \ .$$

Since $\mathbf{F}$ is invertible, then $\mathbf{F}\mathbf{v} \neq \mathbf{0}$ if $\mathbf{v} \neq \mathbf{0}$. Assuming that $\mathbf{F}$ and $\mathbf{v}$ are both real, it follows that $Q(\mathbf{v}) > 0$ if $\mathbf{v} = \mathbf{0}$. Thus, $\mathbf{U}^2$ is positive definite. Also, since $\mathbf{F}^T\mathbf{F}$ is symmetric, then $\mathbf{U}^2$ must be symmetric. From theorem 5, we know that $\mathbf{U}^2$ must have a unique positive definite square root $\mathbf{U}$, where $\mathbf{U}$ is also symmetric. To show that $\mathbf{R}$ is orthogonal, note that

$$\mathbf{R}^T\mathbf{R} = (\mathbf{U}^{-1})^T\mathbf{F}^T\mathbf{F}\mathbf{U}^{-1} = \mathbf{U}^{-1}\mathbf{F}^T\mathbf{F}\mathbf{U}^{-1} = \mathbf{U}^{-1}\mathbf{U}\mathbf{U}\mathbf{U}^{-1} = \mathbf{I} \ .$$

Using the definitions of **U** and **R**, we can write

$$\mathbf{R}\mathbf{U} = \mathbf{F}\mathbf{U}^{-1}\mathbf{U} = \mathbf{F} \ .$$

Thus, we have shown the existence of the decomposition (24).

Uniqueness

To show uniqueness of the decomposition (24), assume first that two such decompositions exist, so that $\mathbf{F} = \mathbf{RU} = \mathbf{R}^*\mathbf{U}^*$. The star is used here to denote an alternative decomposition. Notice that

$$\mathbf{F}^T\mathbf{F} = \mathbf{U}^T\mathbf{R}^T\mathbf{RU} = \mathbf{U}^2$$

and

$$\mathbf{F}^T\mathbf{F} = \mathbf{U}^{*T}\mathbf{R}^{*T}\mathbf{R}^*\mathbf{U}^* = \mathbf{U}^{*2} ,$$

thus $\mathbf{U}^2 = \mathbf{U}^{*2}$. By theorem 5, the square roots of $\mathbf{U}^2$ and $\mathbf{U}^{*2}$ are unique, so we must have $\mathbf{U} = \mathbf{U}^*$. Notice that

$$\mathbf{0} = \mathbf{F} - \mathbf{F} = \mathbf{RU} - \mathbf{R}^*\mathbf{U}^* = (\mathbf{R} - \mathbf{R}^*)\mathbf{U} .$$

Since $\mathbf{U}$ is non-singular, $\mathbf{U}^{-1}$ exists. Thus,

$$\mathbf{0} = (\mathbf{R} - \mathbf{R}^*)\mathbf{U}\mathbf{U}^{-1} = \mathbf{R} - \mathbf{R}^* ,$$

and so $\mathbf{R} = \mathbf{R}^*$. Thus, the decomposition $\mathbf{F} = \mathbf{RU}$ is unique. Similarly, one can prove the existence and uniqueness of the decomposition $\mathbf{F} = \mathbf{V}\overline{\mathbf{R}}$, where we temporarily allow $\overline{\mathbf{R}}$ to be different from $\mathbf{R}$. The relationship between $\mathbf{V}$, $\mathbf{U}$, $\mathbf{R}$, and $\overline{\mathbf{R}}$ may be determined by observing that

$$\mathbf{F} = \mathbf{RU} = (\mathbf{RUR}^T)\mathbf{R} = \mathbf{V}\overline{\mathbf{R}} ,$$

where $\mathbf{RUR}^T$ is symmetric and positive definite. Since the decomposition of $\mathbf{F}$ is unique, we conclude that

$$\mathbf{V} = \mathbf{RUR}^T , \quad \overline{\mathbf{R}} = \mathbf{R} , \quad \text{and} \quad \mathbf{U} = \mathbf{R}^T\mathbf{VR} . \tag{25}$$

Extremal properties of quadratic forms.

Consider the symmetric tensor **T** and the quadratic form

$$Q(\mathbf{v}) = \mathbf{v} \cdot \mathbf{T}\mathbf{v} = T_{ij}v_jv_i \, . \tag{26}$$

We wish to know the extreme values of Q(**v**) subject to the constraint that **v** be a unit vector (i.e. $\mathbf{v} \cdot \mathbf{v} = v_i\, v_i = 1$). Using the method of Lagrange multipliers, the conditions for the extremum of (26) are

$$\frac{\partial}{\partial v_k} \{T_{ij}v_iv_j - \beta(v_iv_i - 1)\} = 0 \, , \tag{27}$$

where β is the undetermined multiplier and $\mathbf{v} \cdot \mathbf{v} - 1 = 0$ is the constraint equation. Performing the differentiation, (27) becomes

$$T_{ij}\delta_{ik}v_j + T_{ij}v_i\delta_{jk} - 2\beta v_i\delta_{ik} = 0$$

or

$$T_{kj}v_j + T_{ik}v_i - 2\beta v_k = 0 \, . \tag{28}$$

Since **T** is symmetric, (28) becomes

$$T_{kj}v_j - \beta v_k = 0 \, . \tag{29}$$

Therefore, Q(**v**) attains extreme values when **v** is a eigenvector of **T**. Using (2), it is clear that if **m** is a unit eigenvector of **T** with corresponding eigenvalue β, then

$$Q(\mathbf{m}) = \mathbf{m} \cdot (\mathbf{T}\mathbf{m}) = \beta \, . \tag{30}$$

In summary, the quadratic form obtains its extremum values when it is operating on an eigenvector. If the eigenvectors are normal, these extremum values are simply the corresponding eigenvalues.

Relationship between eigenvalues of U, C, V, B, E, and D and the principal stretch.

(see class notes sections I/4 and I/5)

The principal directions are the directions in which the stretch attains an extreme value. The principal stretch λ is the extreme value of stretch in a given principal direction. As derived in section I/4, the eigenvalues of **U** and **V** are equal to the principal values of stretch and the eigenvalues of **C** and **B** are equal to λ^2. It was also shown in this section that the eigenvectors of **U** and **C** are coincident and that the eigenvectors of **V** and **B** coincide.

The relative strain tensor **E** is given by

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}) \,. \tag{31}$$

The principal values of strain, denoted by β, are equal to the eigenvalues of **E**. The principal directions of strain, denoted by **m**, are equal to the eigenvectors of **E**. From (2), we have

$$\mathbf{Em} = \beta\mathbf{m} \,. \tag{32}$$

Substituting (31) into (32) gives

$$\frac{1}{2}(\mathbf{C} - \mathbf{I})\mathbf{m} = \beta\mathbf{m} \,,$$

or

$$\mathbf{Cm} = (2\beta + 1)\mathbf{m} \,. \tag{33}$$

Equation (33) implies that **m** is an eigenvector of **C** as well as of **E** and that $2\beta + 1$ is an eigenvalue of **C**. As previously noted, the eigenvalue of **C** is λ^2, so

$$2\beta + 1 = \lambda^2 \quad \text{or} \quad \beta = \frac{1}{2}(\lambda^2 - 1) \,. \tag{34}$$

Thus, the eigenvectors of **U** , **C** , and **E** coincide and the principal value of strain is related to the principal value of stretch by equation (34).

As was derived in section I/5, the eigenvalues and eigenvectors of the rate of deformation tensor **D** are given by

$$\mathbf{D}\mathbf{m} = \frac{\dot{\lambda}}{\lambda}\mathbf{m} = \frac{d}{dt}(\ln \lambda)\,\mathbf{m} \quad , \tag{35}$$

where **m** is an eigenvector of **D**, (**m** is also called the "principal direction of stretch"). It is clear from the form of (35) that $\dot{\lambda} / \lambda$ (called the logarithmic rate of stretching) is an eigenvalue of **D**.

Example calculation

Suppose that at some instant at a particular point in the body, the deformation gradient tensor **F** is given by

$$[F_{iA}] = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad . \tag{36}$$

Notice that (36) yields a symmetric positive definite tensor. Since the polar decomposition of **F** is unique, it follows that **V** = **U** = **F** and **R** = **I** . The principal invariants of **F** = **U** are

$$\begin{aligned} I_1 &= \text{tr}\,\mathbf{U} = U_{AA} = 5 \\ I_2 &= \frac{1}{2}(\text{tr}^2\mathbf{U} - \text{tr}\mathbf{U}^2) = \frac{1}{2}(U_{AA}U_{BB} - U_{AB}U_{BA}) = 7 \\ I_3 &= \det \mathbf{U} = 3 \;. \end{aligned} \tag{37}$$

The characteristic equation of **F** is then

$$-\beta^3 + 5\beta^2 - 7\beta + 3 = 0 \;. \tag{38}$$

The roots of (38), i.e. the eigenvalues of **F**, are

$$\beta^{(1)} = 3$$
$$\beta^{(2)} = 1 \qquad (39)$$
$$\beta^{(3)} = 1 \; .$$

To solve for the eigenvectors of **F**, substitute each of the eigenvalues into equation (2), or

$$\mathbf{F}\mathbf{v} = \beta\mathbf{v} \; . \qquad (40)$$

For $\beta^{(1)} = 3$, equation (40) gives

$$(\mathbf{F} - 3\mathbf{I})\mathbf{M}^{(1)} = \mathbf{0} \; ,$$

or

$$-\, M_1^{(1)} + M_2^{(1)} = 0 \quad \text{and} \quad -2M_3^{(1)} = 0 \; .$$

Thus, the eigenvector $\mathbf{M}^{(1)}$ is given by

$$[M_i^{(1)}] = c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \; ,$$

where c_1 is an arbitrary constant. Similarly, for $\beta^{(2)} = \beta^{(3)} = 1$, equation (40) gives

$$(\mathbf{F} - \mathbf{I})\mathbf{M}^{(\alpha)} = \mathbf{0}$$

where $\alpha = 2,3$, or

$$M_1^{(\alpha)} = -M_2^{(\alpha)} \; .$$

Solving for $\mathbf{M}^{(\alpha)}$ gives

$$[M_i^{(2)}] = c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad [M_i^{(3)}] = c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \; .$$

We may take unit eigenvectors

$$[M_i^{(1)}] = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} , \quad [M_i^{(2)}] = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} , \quad [M_i^{(3)}] = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} ,$$

which form a right-handed triad. Note that every vector in the space spanned by $\mathbf{M}^{(2)}$ and $\mathbf{M}^{(3)}$ is also an eigenvector with eigenvalue 1.

One can better understand the physical significance of these results by looking at the effect of the deformation (36) upon a unit cube. The graph below gives the cross section of the cube before and after the deformation. As expected from the preceding calculations, the principal directions (i.e. the directions in which the stretch attains extremum values) are in the directions of the eigenvectors $\mathbf{M}^{(1)}$, $\mathbf{M}^{(2)}$, and $\mathbf{M}^{(3)}$. Line elements along $\mathbf{M}^{(1)}$ are stretched to three times the initial length. Thus, the stretch $\lambda = 3$ is equal to $\beta^{(1)}$, as expected. Line elements in the directions $\mathbf{M}^{(2)}$ and $\mathbf{M}^{(3)}$ retain their initial length. Therefore, the stretch λ in these directions equals unity, corresponding to $\beta^{(2)} = \beta^{(3)} = 1$. In this example, line elements along principal directions undergo pure stretch as a result of the deformation; however, line elements in other directions may undergo both stretch and rotation. This characteristic is true in general whenever the deformation $\mathbf{F}$ is of the pure stretch type (i.e. whenever $\mathbf{R} = \mathbf{I}$).

Supplement to Part I, Section 5

Material line elements, area elements, volume elements and their material time derivatives.

Preliminary results

Claim:

$$\det \mathbf{A} = \frac{1}{6}\,\varepsilon_{ijk}\varepsilon_{lmn}A_{li}A_{mj}A_{nk} \tag{1}$$

Proof: Consider an arbitrary tensor **A** with components A_{ij}. Let $a_m = A_{m1}$, $b_m = A_{m2}$, and $c_m = A_{m3}$. Recall that

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \varepsilon_{lmn}a_l b_m c_n = \varepsilon_{lmn}A_{l1}A_{m2}A_{n3} = \det\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33}\end{bmatrix} = \det \mathbf{A}^T \equiv A\ . \tag{2}$$

Now define T_{ijk} by

$$T_{ijk} = \varepsilon_{lmn}A_{li}A_{mj}A_{nk} \tag{3}$$

where by (2), $T_{123} = A$. Since l, m, and n are dummy indices, we may switch them and use the properties of ε_{lmn} to show that

$$T_{ijk} = T_{jki} = T_{kij} = -\,T_{ikj} = -\,T_{kji} = -\,T_{jik}\ . \tag{4}$$

Recalling the properties of ε_{ijk}, we can conclude from (4) that T_{ijk} must be a multiple of ε_{ijk}. >From (2) and (4), we see that

$$\begin{aligned} &T_{231} = T_{312} = T_{123} = A \\ &T_{321} = T_{213} = T_{132} = -\,A \\ &T_{ijk} = 0 \ \text{ if } \ i = j\ ,\ j = k\ ,\ \text{or}\ \ i = k\ . \end{aligned}$$

Hence, it is clear that

$$T_{ijk} = A\ \varepsilon_{ijk}\ . \tag{5}$$

Equate (3) and (5) and multiply the resulting equation by $\frac{1}{6}\,\varepsilon_{ijk}$, noting that $\varepsilon_{ijk}\varepsilon_{ijk} = 6$, to get

$$A = \det \mathbf{A}^T = \frac{1}{6}\,\varepsilon_{ijk}\varepsilon_{lmn}A_{li}A_{mj}A_{nk}\ . \tag{6}$$

Since $\det \mathbf{A}^T = \det \mathbf{A}$ for any tensor $\mathbf{A}$, we have proved equation (1).

Claim:

$$\frac{\partial(\det \mathbf{A})}{\partial \mathbf{A}} = (\det \mathbf{A})\,\mathbf{A}^{-T} \tag{7}$$

Proof: Assume that the components of **A** are independent, so that we can write

$$\frac{\partial A_{ij}}{\partial A_{rs}} = \delta_{ir}\delta_{js}\ . \tag{8}$$

Now differentiate (6) with respect to A_{rs} to obtain

$$\begin{aligned} \frac{\partial(\det \mathbf{A})}{\partial A_{rs}} = \frac{\partial A}{\partial A_{rs}} &= \frac{1}{6}\,\varepsilon_{ijk}\varepsilon_{lmn}\,\{\delta_{rl}\delta_{si}A_{mj}A_{nk} + A_{li}\delta_{rm}\delta_{sj}A_{nk} + A_{li}A_{mj}\delta_{rn}\delta_{sk}\} \\ &= \frac{1}{6}\{\varepsilon_{sjk}\varepsilon_{rmn}A_{mj}A_{nk} + \varepsilon_{isk}\varepsilon_{lrn}A_{li}A_{nk} + \varepsilon_{ijs}\varepsilon_{lmr}A_{li}A_{mj}\} \\ &= \frac{1}{6}\{3\varepsilon_{sjk}\varepsilon_{rmn}A_{mj}A_{nk}\} = \frac{1}{2}\varepsilon_{sjk}\varepsilon_{rmn}A_{mj}A_{nk}\ . \end{aligned} \tag{9}$$

Suppose now that **A** is non-singular ($\det \mathbf{A} \neq 0$), so that there exists a tensor **B** with components B_{ij} such that $A_{li}B_{ir} = \delta_{lr}$. If this is the case, then (9) may be rewritten using (3) and (5) as

$$\begin{aligned} \frac{\partial A}{\partial A_{rs}} &= \frac{1}{2}\,\varepsilon_{sjk}(\varepsilon_{lmn}\delta_{rl})A_{mj}A_{nk} \\ &= \frac{1}{2}\,\varepsilon_{sjk}\varepsilon_{lmn}(A_{li}B_{ir})A_{mj}A_{nk} \\ &= \frac{1}{2}\,\varepsilon_{sjk}(\varepsilon_{lmn}A_{li}A_{mj}A_{nk})B_{ir} \\ &= \frac{1}{2}\,\varepsilon_{sjk}\varepsilon_{ijk}AB_{ir} = \frac{1}{2}(2AB_{sr}) = AB_{sr}\ . \end{aligned} \tag{10}$$

Denoting $\mathbf{B}$ as $\mathbf{A}^{-T}$, equation (10) becomes

$$\frac{\partial(\det \mathbf{A})}{\partial \mathbf{A}} = \mathrm{A}\ \mathbf{A}^{-T} = (\det \mathbf{A})\ \mathbf{A}^{-T}\ ,$$

which is identical to (7).

Material line elements

Consider a motion of a body *B* given by $\mathbf{x} = \chi\,(\mathbf{X}\,, t)$, where $\mathbf{X}$ is the position vector of a certain particle in the reference configuration and $\mathbf{x}$ is the position vector of the same particle in the current configuration at time t. Two neighboring material particles in the reference configuration are located at $\mathbf{X}$ and $\mathbf{X} + d\mathbf{X}$. Consider a material line element $d\mathbf{X}$ in the reference configuration at the point $\mathbf{X}$ and having length dS and direction $\mathbf{M}$, so that $d\mathbf{X} = dS\ \mathbf{M}$. Under the motion $\chi(\mathbf{X}\,, t)$, the material particles $\mathbf{X}$ and $\mathbf{X} + d\mathbf{X}$ are taken to the positions $\mathbf{x}$ and $\mathbf{x} + d\mathbf{x}$. The corresponding material line element $d\mathbf{x}$ in the current configuration having length ds and direction $\mathbf{m}$ is related to $d\mathbf{X}$ by

$$d\mathbf{x} = \mathbf{F}\ d\mathbf{X}\ \ ,\quad \text{or}\quad dx_i = F_{iA}\ dX_A = x_{i,A}dX_A\ , \tag{11}$$

where $F_{iA} = x_{i,A}$ are the components of the deformation gradient $\mathbf{F}$. A diagram of this motion of the material line segment is given below. Recall from Section I/4 of the class notes that the square of the length of a material line element in the current configuration is given by

$$ds^2 = x_{i,A}x_{i,B}dX_AdX_B\ . \tag{12}$$

Recalling that $\dot{\overline{x_{i,A}}} = v_{i,j}x_{j,A}$, we can take the material derivative of (12) to get

$$\begin{aligned} \dot{\overline{ds^2}} &= \dot{\overline{x_{i,A}}}x_{i,B}dX_AdX_B + x_{i,A}\dot{\overline{x_{i,B}}}dX_AdX_B \\ &= v_{i,j}x_{j,A}x_{i,B}dX_AdX_B + x_{i,A}v_{i,j}x_{j,B}dX_AdX_B \\ &= (v_{i,j} + v_{j,i})x_{i,A}x_{j,B}dX_AdX_B \\ &= 2d_{ij}x_{i,A}x_{j,B}dX_AdX_B\ . \end{aligned} \tag{13}$$

This last step uses the definition of the rate of deformation tensor $\mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T)$. Equation (13) may be rewritten using (11) as

$$\dot{\overline{ds^2}} = 2d_{ij}dx_i dx_j \ . \tag{14}$$

Material area elements

Let ${}^1d\mathbf{X}$ and ${}^2d\mathbf{X}$ represent two material line elements in the reference configuration located at $\mathbf{X}$. Suppose that the motion $\chi(\mathbf{X}, t)$ takes ${}^1d\mathbf{X}$, ${}^2d\mathbf{X}$ into the elements ${}^1d\mathbf{x}$, ${}^2d\mathbf{x}$, respectively, at $\mathbf{x}$ in the current configuration. Define the material area element dA with orientation $\mathbf{N}$ in the reference configuration by

$$\mathbf{N}\, dA = {}^1d\mathbf{X} \times {}^2d\mathbf{X} \tag{15}$$

so that

$$dA_L = N_L dA = \mathbf{N}\, dA \cdot \mathbf{e}_L = ({}^1d\mathbf{X} \times {}^2d\mathbf{X}) \cdot \mathbf{e}_L = \varepsilon_{LMN}\, {}^1dX_M\, {}^2dX_N \ . \tag{16}$$

The corresponding material area element in the current configuration is da with orientation $\mathbf{n}$ (see Figure 2) defined by

$$\mathbf{n}\, da = {}^1d\mathbf{x} \times {}^2d\mathbf{x} \ , \tag{17}$$

so that

$$n_i\, da = \mathbf{n}\, da \cdot \mathbf{e}_i = ({}^1d\mathbf{x} \times {}^2d\mathbf{x}) \cdot \mathbf{e}_i \tag{18}$$

>From (18) and (11) and noting that $x_{m,D}X_{D,i} = \delta_{mi}$, we can write

$$n_i\, da = \varepsilon_{ijk} x_{j,A} x_{k,B}\, {}^1dX_A\, {}^2dX_B = \varepsilon_{mjk}\delta_{mi} x_{j,A} x_{k,B}\, {}^1dX_A\, {}^2dX_B$$

$$= \varepsilon_{mjk}(x_{m,D}X_{D,i}) x_{j,A} x_{k,B}\, {}^1dX_A\, {}^2dX_B$$

$$= \varepsilon_{mjk} x_{m,D} x_{j,A} x_{k,B} X_{D,i}\, {}^1dX_A\, {}^2dX_B \ .$$

Using (3), (5), and (16), this expression becomes

$$n_i\, da = \varepsilon_{DAB}(\det \mathbf{F}) X_{D,i}\, {}^1dX_A\, {}^2dX_B$$

or

$$n_i\, da = J\, X_{D,i}\, N_D\, dA\ . \qquad (19)$$

Assuming that **F** is non-singular and using (7) and (19), we can write

$$\overline{n_i\, da}^{\cdot} = \dot{J}\, X_{D,i}\, N_D\, dA + J\, \overline{X_{D,i}}^{\cdot}\, N_D\, dA\ . \qquad (20)$$

Notice that

$$\dot{J} = \frac{dJ}{dt} = \frac{\partial J}{\partial F_{jA}}\frac{\partial F_{jA}}{\partial t} = J\, (\mathbf{F}^{-1})_{Aj}\, v_{j,k} F_{kA} = J\, v_{j,k}\delta_{kj} = J\, v_{j,j}\ . \qquad (21)$$

Using (21), equation (20) becomes

$$\overline{n_i\, da}^{\cdot} = (J\, X_{D,i} v_{j,j} - J\, X_{D,j} v_{j,i}) N_D\, dA$$

or

$$\overline{n_i\, da}^{\cdot} = J(v_{j,j} X_{D,i} - v_{j,i} X_{D,j}) N_D\, dA\ . \qquad (22)$$

Material volume elements

Let ${}^1d\mathbf{X}$, ${}^2d\mathbf{X}$, and ${}^3d\mathbf{X}$ be three material line elements forming a right handed system located at **X** in the reference configuration. Suppose that the motion $\chi\,(\mathbf{X}, t)$ takes ${}^1d\mathbf{X}$, ${}^2d\mathbf{X}$, and ${}^3d\mathbf{X}$ into three line elements ${}^1d\mathbf{x}$, ${}^2d\mathbf{x}$, and ${}^3d\mathbf{x}$, respectively, at **x** in the current configuration (see Figure 3). Define the <u>material volume element</u> dV in the reference configuration by

$$dV = {}^1d\mathbf{X} \cdot ({}^2d\mathbf{X} \times {}^3d\mathbf{X}) = \varepsilon_{ABC}\, {}^1dX_A\, {}^2dX_B\, {}^3dX_C\ . \qquad (23)$$

The corresponding material volume element in the current configuration is then

$$dv = {}^1d\mathbf{x} \cdot ({}^2d\mathbf{x} \times {}^3d\mathbf{x}) = \varepsilon_{ijk}\, {}^1dx_i\, {}^2dx_j\, {}^3dx_k\ . \qquad (24)$$

Using equations (3), (5), (11), and (23), we can write (24) as

$$dv = \varepsilon_{ijk} x_{i,A} x_{j,B} x_{k,C}\, {}^1dX_A\, {}^2dX_B\, {}^3dX_C = \varepsilon_{ABC}(\det \mathbf{F})\, {}^1dX_A\, {}^2dX_B\, {}^3dX_C$$

$$= J(\varepsilon_{ABC}\ {}^{1}dX_A\ {}^{2}dX_B\ {}^{3}dX_C)$$

or

$$dv = J\, dV\, . \tag{25}$$

Taking the material derivative of (25) gives

$$\dot{\overline{dv}} = \dot{J}\, dV = J\, v_{j,j} dV = J(\text{div}\, \mathbf{v}) dV\, , \tag{26}$$

where equation (21) is used for the calculation of $\dot{J}$.

A special case of the above analysis occurs for <u>isochoric motions</u> (i.e. motions for which $dv = dV$ for all volumes occupied by any material region of the body). It is clear from this definition that $\dot{\overline{dv}} = 0$ for isochoric motions. We see from equations (25) and (26) that the necessary and sufficient condition for isochoric motion may be expressed either as

$$J = 1 \quad \text{or} \quad v_{i,i} = \text{div}\, \mathbf{v} = 0\, . \tag{27}$$

Summary

A summary of the basic equations derived in this appendix are given in the following:

Material line, area, and volume elements:

$$ds^2 = x_{i,A} x_{i,B} dX_A dX_B$$

$$n_i\, da = J\, X_{D,i}\, N_D\, dA$$

$$dv = J\, dV$$

Material derivatives of line, area, and volume elements:

$$\dot{\overline{ds^2}} = 2\, d_{ij} x_{i,A} x_{j,B} dX_A dX_B$$

$$\dot{\overline{da_i}} = J(v_{j,j} X_{D,i} - v_{j,i} X_{D,j}) dA_D$$

$$\dot{\overline{dv}} = J\, v_{j,j} dV\, .$$

Supplement to Part I, Section 7

Interpretation of infinitesimal strain measures.

Recall the following expressions:

$$ds^2 = dx_i dx_i = x_{i,A} x_{i,B} dX_A dX_B$$

$$dS^2 = dX_A dX_A = X_{A,i} X_{A,j} dx_i dx_j$$

$$E_{AB} = \frac{1}{2}(C_{AB} - \delta_{AB}) = \frac{1}{2}(U_{A,B} + U_{B,A} + U_{M,A} U_{M,B})$$

$$e_{ij} = \frac{1}{2}(\delta_{ij} - c_{ij}) = \frac{1}{2}(u_{i,j} + u_{j,i} - u_{m,i} u_{m,j}) \quad . \tag{1}$$

Also recall that the stretch λ and the extension E are defined by

$$\lambda = \frac{ds}{dS} = \sqrt{C_{AB} M_A M_B} = \sqrt{1 + 2\, E_{AB} M_A M_B} \tag{2a}$$

$$= 1/\sqrt{c_{ij} m_i m_j} = 1/\sqrt{1 - 2 e_{ij} m_i m_j}$$

and

$$E = \frac{ds - dS}{dS} = \lambda - 1 \quad , \tag{2b}$$

where

$$M_A M_A = m_i m_i = 1 \;\text{ and }\; dX_A = M_A dS \quad , \quad dx_i = m_i ds \quad . \tag{3}$$

For an infinitesimal deformation (see class notes Part I/7), we may approximate E_{AB} by neglecting term of $O(\varepsilon^2)$ as $\varepsilon \to 0$ as

$$E_{AB} = \frac{1}{2}(U_{A,B} + U_{B,A}) = O(\varepsilon) \;\text{ as }\; \varepsilon \to 0 \quad . \tag{4}$$

The distinction between Lagrangian and Eulerian strain disappears for an infinitesimal deformation. Thus,

$$e_{ij} = E_{AB}\delta_{iA}\delta_{jB} = O(\varepsilon) \quad \text{as} \quad \varepsilon \to 0 \quad . \tag{5}$$

Hence, for an infinitesimal deformation the extension and the stretch given in (2) may be written as

$$E = \lambda - 1 = E_{AB}M_AM_B + O(\varepsilon^2) = e_{ij}m_im_j + O(\varepsilon^2) \tag{6}$$

and

$$\lambda = \sqrt{1 + 2E_{AB}M_AM_B} = 1 + E_{AB}M_AM_B + O(\varepsilon^2) \tag{7a}$$

or, alternatively,

$$\lambda = 1/\sqrt{1 - 2e_{ij}m_im_j} = 1 + e_{ij}m_im_j + O(\varepsilon^2) \quad . \tag{7b}$$

In the following discussion, we will use e_{11}, e_{22}, ..., to denote the components of the infinitesimal strain tensor, since from (5) no distinction needs to be made between Lagrangian and Eulerian strain as $\varepsilon \to 0$. Consider a line element which lies along the X_1 axis, i.e.

$$dX_1 = dS \;\;,\;\; dX_2 = dX_3 = 0 \;\;,\;\; \text{and } M_A = (1, 0, 0) \;\;,$$

so that from (6)

$$E = \frac{ds - dS}{dS} = e_{11} \quad . \tag{8}$$

That is, e_{11} represents the extension (the change in length divided by the original length) of a line element lying along the X_1 axis. We note that to within $O(\varepsilon^2)$, the direction m_i in the current configuration coincides with the direction M_A in the reference configuration. Similar results hold for e_{22} and e_{33}. In summary: the diagonal elements of the infinitesimal strain tensor represent the extension of line elements in the corresponding coordinate directions.

Consider two orthogonal elements which initially lie along the X_1 and X_2 axes, so that

$$\begin{aligned} &dX_1 = dS \;\;,\;\; dX_2 = dX_3 = 0 \;\;,\;\; M_A = (1, 0, 0) \\ &d\bar{X}_2 = d\bar{S} \;\;,\;\; d\bar{X}_1 = d\bar{X}_3 = 0 \;\;,\;\; \bar{M}_A = (0, 1, 0) \;\;. \end{aligned} \tag{9}$$

Let θ denote the angle between the corresponding deformed line elements $d\mathbf{x}$ and $d\bar{\mathbf{x}}$, whose lengths are ds and $d\bar{s}$. >From (8), neglecting terms of $O(\varepsilon^2)$ as $\varepsilon \to 0$, we have

$$ds = (1 + e_{11})dS \quad \text{and} \quad d\bar{s} = (1 + e_{22})d\bar{S} \ . \tag{10}$$

Now observe that

$$\cos\theta = \frac{d\mathbf{x} \cdot d\bar{\mathbf{x}}}{|d\mathbf{x}|\ |d\bar{\mathbf{x}}|} = \frac{dx_i\, d\bar{x}_i}{ds\, d\bar{s}} = x_{i,A}x_{i,B}\, \frac{dX_A d\bar{X}_B}{ds\, d\bar{s}}$$

$$= C_{AB}\, \frac{dX_A d\bar{X}_B}{ds\, d\bar{s}} = C_{AB}\, \frac{M_A \bar{M}_B}{(1 + e_{11})(1 + e_{22})}$$

$$= C_{12}/(1 + e_{11})(1 + e_{22}) = 2e_{12}/(1 + e_{11} + e_{22}) + O(\varepsilon^2)$$

$$= 2e_{12} + O(\varepsilon^2) \ , \tag{11}$$

where we have used $(1)_3$, (5), (9), and (10) in deriving (11).

Let ϕ be the amount θ differs from a right angle (i.e. $\phi = -\theta + \pi/2$), so that

$$\cos\theta = \sin\phi = \phi + O(\phi^3) \ . \tag{12}$$

>From (11) and (12), we see, after neglecting terms of $O(\varepsilon^2)$, that

$$\phi = 2e_{12} \quad \text{or} \quad e_{12} = \frac{\phi}{2} \ . \tag{13}$$

The angle ϕ is known as the <u>shear</u> <u>angle</u>. Similar results hold for e_{13} and e_{23}. Hence: the off-diagonal components of the infinitesimal strain tensor are seen to represent half of the change in angle between two line elements initially along the corresponding coordinate axes.

Recall that dv=JdV, where $J = \det(x_{i,A})$. In the class notes Section I/7, it was shown that neglecting terms of $O(\varepsilon^2)$ as $\varepsilon \to 0$ gives

$$J = 1 + e_{kk} \ . \tag{14}$$

Thus, $dv = JdV = dV + e_{kk}dV$, or

$$\frac{dv - dV}{dV} = e_{kk} \quad . \tag{15}$$

The invariant e_{kk} is known as the dilation. Equation (15) implies that the trace of the infinitesimal strain tensor measures the local change in volume per unit volume.

Supplement to Part III, Section 2

Isotropic tensors.

Under a rotation of coordinate system, recall that the components of a vector change according to the rule

$$x_i = a_{im}x'_m \quad \text{or} \quad x'_m = a_{jm}x_j \;, \tag{1}$$

where x_i are the components of a vector **x** with respect to one coordinate system and x'_m are the components of the same vector with respect to a different coordinate system. The coefficients a_{im} are components of an orthogonal tensor such that

$$a_{ij}a_{ik} = a_{ji}a_{ki} = \delta_{jk} \text{ and } \quad \det(a_{ij}) = \pm 1 \;. \tag{2}$$

Also recall that the components $T_{ij...k}$ of a tensor **T** transform according to the rule

$$T_{ij...k} = a_{ip}a_{jq} \cdots a_{kr}T'_{pq...r} \;, \tag{3}$$

where a_{ij}'s obey equation (2).

Definition: A tensor **T** is called <u>isotropic</u> if its components are invariant to rotation of the coordinate system. Thus, if **T** is isotropic, then

$$T_{ij...k} = T'_{ij...k} \;,$$

or from (3),

$$T_{ij...k} = a_{ip}a_{jq} \cdots a_{kr}T_{pq...r} \tag{4}$$

for all orthogonal a_{ij}.

Theorem 1: A scalar invariant is an isotropic tensor of order zero.

Proof: Let $\phi = \hat{\phi}(x_i)$ be a scalar invariant, so that from (3) $\hat{\phi}(x_i) = \hat{\phi}(x'_i) = \phi'$. Since $\phi = \phi'$, the scalar invariant ϕ satisfies the definition of an isotropic tensor given by (4).

Theorem 2: The only isotropic tensor of order one is the zero vector.

Proof: If t_i are the components of an isotropic vector **t**, then from (4)

$$t_i = a_{ji} t_j \tag{5}$$

for all components a_{ji} of orthogonal tensors. Consider, for instance, the choice

$$\begin{bmatrix} a_{ji} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

With this choice for a_{ji}, (5) becomes $(t_1 , t_2 , t_3) = (-t_1 , -t_2 , t_3)$, which implies that $t_1 = 0$ and $t_2 = 0$. Now consider the choice

$$\begin{bmatrix} a_{ji} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} .$$

for which equation (5) becomes $(t_1 , t_2 , t_3) = (t_1 , -t_2 , -t_3)$, and so we must have $t_3 = 0$ also. Thus, if **t** is an isotropic tensor of order one, then **t** = **0**.

Theorem 3: Every isotropic tensor of order two is a scalar multiple of δ_{ij}.

Proof: Let T_{ij} be the components of an isotropic tensor **T**, so that from (4)

$$T_{ij} = a_{ip} a_{jq} T_{pq} \tag{6}$$

for all orthogonal tensor components a_{ij}. Consider now the following choices for a_{ij} and the resulting forms of equation (6):

$$\begin{bmatrix} a_{ij} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{ so that } \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} = \begin{bmatrix} T_{33} & T_{31} & -T_{32} \\ T_{13} & T_{11} & -T_{12} \\ -T_{23} & -T_{21} & T_{22} \end{bmatrix} ,$$

and

$$\begin{bmatrix} a_{ij} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \text{ so that } \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} = \begin{bmatrix} T_{33} & -T_{31} & T_{32} \\ -T_{13} & T_{11} & -T_{12} \\ T_{23} & -T_{21} & T_{22} \end{bmatrix} . \tag{7}$$

In order for $(7)_1$ and $(7)_2$ to both be satisfied, we must have

$$T_{11} = T_{22} = T_{33} \text{ and } T_{12} = T_{21} = T_{13} = T_{31} = T_{23} = T_{32} = 0 \ ,$$

or

$$T_{ij} = \lambda\delta_{ij} \ , \tag{8}$$

where

$$\lambda = T_{11} = T_{22} = T_{33} \ .$$

Theorem 4: Every isotropic tensor of order three is a scalar multiple of the permutation tensor ε_{ijk}.

Proof: Let T_{ijk} be an isotropic third order tensor, so that from (4) we can write

$$T_{ijk} = a_{ip}a_{jq}a_{kr}T_{pqr} \tag{9}$$

for all orthogonal a_{ij}. We can represent T_{ijk} as

$$\left[T_{ijk}\right] = \left[\begin{array}{ccc|ccc|ccc} T_{111} & T_{112} & T_{113} & T_{121} & T_{122} & T_{123} & T_{131} & T_{132} & T_{133} \\ T_{211} & T_{212} & T_{213} & T_{221} & T_{222} & T_{223} & T_{231} & T_{232} & T_{233} \\ T_{311} & T_{312} & T_{313} & T_{321} & T_{322} & T_{323} & T_{331} & T_{332} & T_{333} \end{array}\right] .$$

Consider the following choices for a_{ij} and the resulting forms of (9):

$$\left[a_{ij}\right] = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{ then } \left[T_{ijk}\right] = \left[\begin{array}{ccc|ccc|ccc} -T_{333} & -T_{331} & T_{332} & -T_{313} & -T_{311} & T_{312} & T_{323} & T_{321} & -T_{322} \\ -T_{133} & -T_{131} & T_{132} & -T_{113} & -T_{111} & T_{112} & T_{123} & T_{121} & -T_{122} \\ T_{233} & T_{231} & -T_{232} & T_{213} & T_{211} & -T_{212} & -T_{223} & -T_{221} & T_{222} \end{array}\right]$$

$$\left[a_{ij}\right] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{ then } \left[T_{ijk}\right] = \left[\begin{array}{ccc|ccc|ccc} T_{333} & T_{331} & T_{332} & T_{313} & T_{311} & T_{312} & T_{323} & T_{321} & T_{322} \\ T_{133} & T_{131} & T_{132} & T_{113} & T_{111} & T_{112} & T_{123} & T_{121} & T_{122} \\ T_{233} & T_{231} & T_{232} & T_{213} & T_{211} & T_{212} & T_{223} & T_{221} & T_{222} \end{array}\right]$$

$$\left[a_{ij}\right] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \text{ then } \left[T_{ijk}\right] = \left[\begin{array}{ccc|ccc|ccc} T_{333} & T_{332} & -T_{331} & T_{323} & T_{322} & -T_{321} & -T_{313} & -T_{312} & T_{311} \\ T_{233} & T_{232} & -T_{231} & T_{223} & T_{222} & -T_{221} & -T_{213} & -T_{212} & T_{211} \\ -T_{133} & -T_{132} & T_{131} & -T_{123} & -T_{122} & T_{121} & T_{113} & T_{112} & -T_{111} \end{array}\right] . \tag{10}$$

>From equation (10), we must have

$$\begin{aligned} & T_{333} = T_{331} = T_{332} = T_{111} = T_{112} \\ &= T_{122} = T_{133} = T_{211} = T_{233} = T_{311} = T_{322} = T_{121} = T_{131} = T_{212} = \\ &= T_{232} = T_{313} = T_{323} = 0 \end{aligned}$$

and

$$T_{123} = T_{231} = T_{312} = -T_{321} = -T_{132} = -T_{213} \equiv \lambda \ ,$$

thus we can write

$$T_{ijk} = \lambda \varepsilon_{ijk} \ . \tag{11}$$

Sufficiency may also be shown by substituting equation (11) into equation (3), giving

$$a_{ip}a_{jq}a_{kr}T_{pqr} = a_{ip}a_{jq}a_{kr}\lambda\varepsilon_{pqr} = \lambda(\det a_{pq})\varepsilon_{ijk}$$

which is equivalent to equation (9).

Theorem 5: The most general fourth order isotropic tensor has the form

$$T_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk} \ .$$

Proof: Let T_{ijkl} be the components of an isotropic tensor, so that from (4)

$$T_{ijkl} = a_{ip}a_{jq}a_{kr}a_{ls}T_{pqrs} \tag{12}$$

for all orthogonal a_{ij}. We proceed as in the previous proofs, choosing a_{ij} to be

$$\left[a_{ij}\right] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} , \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} , \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} , \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} .$$

Using the above choices for a_{ij} in equation (12), we can show that the non-zero values of T_{ijkl} are related as follows:

$$\begin{aligned} & T_{1111} = T_{2222} = T_{3333} \\ & T_{1122} = T_{2211} = T_{1133} = T_{3311} = T_{3322} = T_{2233} = \lambda \\ & T_{1212} = T_{2121} = T_{1313} = T_{3131} = T_{2323} = T_{3232} = \mu \\ & T_{1221} = T_{2112} = T_{1331} = T_{3113} = T_{2332} = T_{3223} = \gamma \ . \end{aligned} \tag{13}$$

Now consider the choice

$$\left[a_{ij}\right] = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

If we substitute this value for a_{ij} into (12) and set i = j = k = l = 1, we find that

$$T_{1111} = \frac{1}{4}(T_{1111} + T_{2211} + T_{2121} + T_{1221} + T_{2112} + T_{1212} + T_{1122} + T_{2222}) \ ,$$

or using (13),

$$T_{1111} = \lambda + \mu + \gamma \, . \tag{14}$$

>From the restriction (13) and (14), we see that T_{ijkl} can be represented as

$$T_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk} \, . \tag{15}$$

Sufficiency may be shown by substituting the expression for T_{ijkl} given by equation (15) into equation (4), yielding

$$\begin{aligned} &a_{ip}a_{jq}a_{kr}a_{ls}T_{pqrs} = a_{ip}a_{jq}a_{kr}a_{ls}(\lambda\delta_{pq}\delta_{rs} + \mu\delta_{pr}\delta_{qs} + \gamma\delta_{ps}\delta_{qr}) \\ &= \lambda a_{ip}a_{jp}a_{kr}a_{lr} + \mu a_{ip}a_{jq}a_{kp}a_{lq} + \gamma a_{ip}a_{jq}a_{kq}a_{lp} \\ &= \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk} = T_{ijkl} \end{aligned}$$

which is identical to equation (12). If an isotropic fourth order tensor is restricted to have the symmetry

$$T_{ijkl} = T_{jikl} \tag{16}$$

(i.e. symmetry in the first two indices), then from equation (13) we see that $\gamma = \mu$. Equation (15) may thus be written for this special case as

$$T_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \, . \tag{17}$$

Equation (17) is the most general form for an isotropic fourth order tensor symmetric in i and j. Notice that the representation given in equation (17) also possesses the additional symmetries given by

$$T_{ijkl} = T_{ijlk} = T_{klij} \, . \tag{18}$$

Supplement to Part III, Section 4

Material symmetry of elastic solids (general considerations).

For a homogeneous elastic solid, the constitutive equation of interest expresses the strain energy per unit mass as a function of some measure of strain. Thus, let the strain energy per unit mass ψ be given by

$$\psi = \hat{\psi}(E_{AB}) \quad . \tag{1}$$

We examine the behavior of (1) under a change of coordinates in the reference configuration given by

$$X_P = A_{PQ} X'_Q \quad ,$$

where

$$A_{PQ} = \mathbf{E}_P \cdot \mathbf{E}'_Q \quad \text{and} \quad A_{PM}\, A_{QM} = A_{MP}\, A_{MQ} = \delta_{PQ} \quad . \tag{2}$$

Under the transformation (2), E_{AB} becomes

$$E_{PQ} = A_{PM}\, A_{QN}\, E'_{MN} \quad , \tag{3}$$

where E'_{MN} are the components of the strain tensor referred to the base vectors $\mathbf{e}'_A$. Since ψ is a scalar invariant, we know that

$$\psi' = \psi \quad . \tag{4}$$

With the use of (1) and (3), the strain energy function after the transformation (2) can be written as

$$\psi' = \hat{\psi}(A_{PM}\, A_{QN}\, E'_{MN}) \quad , \tag{5a}$$

where as before

$$\psi = \hat{\psi}(E_{PQ}) \quad . \tag{5b}$$

Substituting (5) into (4), we conclude that

$$\hat{\psi}(E_{PQ}) = \hat{\psi}(A_{PM}\, A_{QN}\, E'_{MN}) \tag{6}$$

where $\bar{\psi}$ is in general a different function than $\hat{\psi}$. It is clear from equation (6) that the response of an elastic solid depends, in general, on the coordinates used to describe the reference configuration. The symmetry of a homogeneous elastic solid is characterized by the set of orthogonal coordinate transformations which leave the strain energy function form-invariant; that is, the set of A_{PQ} such that $\bar{\psi}$ is the same as $\hat{\psi}$, or such that

$$\hat{\psi}(E_{PQ}) = \hat{\psi}(E'_{PQ}) \quad . \tag{7}$$

If equation (7) holds for all possible orthogonal coordinate transformations (i.e. for all orthogonal tensors A_{PQ}), then the material is said to be isotropic.

Linear elastic solids

It was shown in the class notes that, in the linear theory, the strain energy function for a homogeneous elastic solid may be expressed in the form

$$\rho_o \psi = \frac{1}{2} C_{ABMN} E_{AB} E_{MN} \quad , \tag{8}$$

where C_{ABMN} is a fourth order tensor of material constants which has, in general, 81 independent components. Since E_{AB} is a symmetric tensor, the coefficients C_{ABMN} in (8) will have the obvious symmetries

$$C_{ABMN} = C_{BAMN} = C_{ABNM} \quad . \tag{9}$$

In addition, recalling that the strain energy is a scalar invariant we may write (after changing the dummies AB into MN and MN into AB)

$$\rho_o \psi = \frac{1}{2} C_{ABMN} E_{AB} E_{MN} = \frac{1}{2} C_{MNAB} E_{MN} E_{AB} \quad . \tag{10}$$

It then follows that C_{ABMN} must by symmetric in the pair of indices (AB) and (MN), i.e.,

$$C_{ABMN} = C_{MNAB} \quad . \tag{11}$$

In summary, the fourth order tensor in (8) must satisfy the symmetries

$$C_{ABMN} = C_{BAMN} = C_{ABNM} = C_{MNAB} \quad . \tag{12}$$

The symmetry of C_{ABMN} shown in (12) allows us to reduce the number of independent components of C_{ABMN} from 81 to 21. The remaining independent components are

$$\begin{array}{cccccc} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112} \\ & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{2212} \\ & & C_{3333} & C_{3323} & C_{3313} & C_{3312} \\ & & & C_{2323} & C_{2313} & C_{2312} \\ & & & & C_{1313} & C_{1312} \\ & & & & & C_{1212} \ . \end{array} \tag{13}$$

Linear elastic solids with symmetry

As mentioned previously, the symmetry of an elastic solid is characterized by the set of orthogonal tensors A_{PQ} such that (7) holds. Substituting (8) into (7), we find

$$\rho_o\psi = \frac{1}{2}\, C_{ABMN}E_{AB}E_{MN} = \frac{1}{2}\, C_{ABMN}E'_{AB}E'_{MN} \ . \tag{14}$$

Using (3), (14) can be written as

$$C_{ABMN}A_{AP}A_{BQ}A_{MR}A_{NS}E'_{PQ}E'_{RS} = C_{ABMN}E'_{AB}E'_{MN} \ . \tag{15}$$

Switching dummy indices, (15) becomes

$$(C_{ABMN} - C_{PQRS}A_{PA}A_{QB}A_{RM}A_{SN})E'_{AB}E'_{MN} = 0 \ . \tag{16}$$

Since (16) must hold for all E'_{AB} and since the quantity in parenthesis in (16) is independent of E'_{AB}, we may conclude that

$$C_{ABMN} = C_{PQRS}A_{PA}A_{QB}A_{RM}A_{SN} \tag{17}$$

for all orthogonal A_{PQ} which lie in the set which characterizes the symmetry of the material.

Isotropic linear elastic solids

If the material is isotropic, then equation (17) must hold for all orthogonal A_{PQ}. Hence, C_{ABMN} in an isotropic tensor which from (12) is symmetric in the first two indices. >From the analysis given in the Appendix to Section 2 of Part III, we can represent C_{ABMN} as

$$C_{ABMN} = \lambda\delta_{AB}\delta_{MN} + \mu(\delta_{AM}\delta_{BN} + \delta_{AN}\delta_{BM}) \ . \tag{18}$$

Eulerian strain

In the linear theory, there is no distinction between Lagrangian and Eulerian strain as is clear from the expression

$$E_{AB} = \delta_{Ai}\delta_{Bj}e_{ij} + O(\varepsilon^2) \ .$$

Substituting this into (8), we may write

$$\rho_o\psi = \frac{1}{2}\,(C_{ABMN}\delta_{Ai}\delta_{Bj}\delta_{Mk}\delta_{Nl})e_{ij}e_{kl} + O(\varepsilon^2) \ . \tag{19}$$

Now define

$$C_{ijkl} = C_{ABMN}\delta_{Ai}\delta_{Bj}\delta_{Mk}\delta_{Nl} \ , \tag{20}$$

so that (19) becomes

$$\rho_o\psi = \frac{1}{2}\,C_{ijkl}\,e_{ij}\,e_{kl} \tag{21}$$

If the material is isotropic, then from (18)

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\,\delta_{jk}) \ . \tag{22}$$

We will now consider materials with various symmetries and analyze the effect of the different symmetries on C_{ABMN}.

Case I(a) - Symmetry with respect to the X_1–X_2 plane

Consider a homogeneous elastic solid which in the reference configuration has material symmetry only with respect to the X_1–X_2 plane. For a material with such a symmetry, the mechanical behavior in the X_3 direction is the same as that in the $-X_3$ direction (dentoed by X'_3 in Figure 1). The coordinate transformation characterizing this particular type of symmetry is represented schematically in Figure 1 and can be expressed algebraically as

$$X_1 = X'_1 \ , \ X_2 = X'_2 \ , \ X_3$$

for which A_{PQ} is given by

$$\left[A_{PQ}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad . \tag{23}$$

Thus, a coordinate transformation which reverses the direction of the X_3 axis does not alter the strain energy function. We will show that the number of material constants needed to describe the behavior of a material with the above symmetry is reduced from 21 to 13.

For the symmetry under consideration, equation (17) must hold for the value of A_{PQ} given in (23), but necessarily for other choices of A_{PQ}. Substituting (23) into (17) gives the relationships

$$\begin{array}{llll}
C_{1111} = C_{1111} & C_{1122} = C_{1122} & C_{1133} = C_{1133} & C_{1123} = -C_{1123} \\
C_{1113} = -C_{1113} & C_{1112} = C_{1112} & C_{2222} = C_{2222} & C_{2233} = C_{2233} \\
C_{2223} = -C_{2223} & C_{2213} = -C_{2213} & C_{2212} = C_{2212} & C_{3333} = C_{3333} \\
C_{3323} = -C_{3323} & C_{3313} = -C_{3313} & C_{3312} = C_{3312} & C_{2323} = C_{2323} \\
C_{2313} = C_{2313} & C_{2312} = -C_{2312} & C_{1313} = C_{1313} & C_{1312} = -C_{1312} \\
C_{1212} = C_{1212} & & &
\end{array}$$

which implies that

$$0 = C_{1123} = C_{1113} = C_{2223} = C_{2213} = C_{3323} = C_{3313} = C_{1312} = C_{2312} \quad .$$

The remaining material constants from the set (13) are

$$\begin{array}{llllll}
C_{1111} & C_{1122} & C_{1133} & 0 & 0 & C_{1112} \\
 & C_{2222} & C_{2233} & 0 & 0 & C_{2212} \\
 & & C_{3333} & 0 & 0 & C_{3312} \\
 & & & C_{2323} & C_{2313} & 0 \\
 & & & & C_{1313} & 0 \\
 & & & & & C_{1212} \quad .
\end{array} \tag{24}$$

The material properties of an elastic solid with material symmetry with respect to the $X_1 - X_2$ plane are characterized by the 13 constants in (24).

Case I(b) - Symmetry with respect to the X_2–X_3 plane

Consider now a solid that has material symmetry with respect to the X_2–X_3 plane. For such a material, the mechanical behavior in the X_1 direction is the same as the corresponding behavior in the $-X_1$ direction (denoted by X_1 in Figure 2). The coordinate transformation for which (17) holds, represented schematically in Figure 2, is given by

$$X_1 = -X'_1 \ , \ X_2 = X'_2 \ , \ X_3 = X'_3 \ . \qquad (25)$$

Since the material has symmetry with respect to the X_2–X_3 plane, a coordinate transformation which reverses the direction of the X_1 axis (such as that in (25)) will not alter the strain energy function.

Case II - Orthotropy

A material which is symmetric with respect to two orthogonal planes is called orthotropic. Consider a homogenous elastic solid which has the material symmetry properties discussed in I(a) as well as the symmetry properties discussed in I(b) (i.e. a material which is symmetric with respect to both the X_1–X_2 plane and the X_2–X_3 plane). We will show that the number of material constants needed to describe the mechanical behavior of a material with the above symmetries is 9. For the coordinate transformation given in equation (25), A_{PQ} is given by

$$\left[A_{PQ}\right] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \ .$$

Substituting this choice for A_{PQ} into (17), we see that the 13 constants in (24) are related by

$$\begin{aligned}
C_{1111} = C_{1111} \ , \ C_{1122} = C_{1122} \ , \ C_{1133} = C_{1133} \ , \ C_{1112} &= -C_{1112} \\
C_{2222} = C_{2222} \ , \ C_{2233} = C_{2233} \ , \ C_{2212} &= -C_{2212} \\
C_{3333} = C_{3333} \ , \ C_{3312} &= -C_{3312} \\
C_{2323} = C_{2323} \ , \ C_{2313} &= -C_{2313} \\
C_{1313} = C_{1313} \ , \ C_{1212} &= C_{1212} \ ,
\end{aligned}$$

so that

$$0 = C_{1112} = C_{2212} = C_{3312} = C_{2313} \ .$$

The remaining independent constants are

$$\begin{matrix} C_{1111} & C_{1122} & C_{1133} & 0 & 0 & 0 \\ & C_{2222} & C_{2233} & 0 & 0 & 0 \\ & & C_{3333} & 0 & 0 & 0 \\ & & & C_{2323} & 0 & 0 \\ & & & & C_{1313} & 0 \\ & & & & & C_{1212} \ . \end{matrix} \tag{26}$$

The material properties of a linear elastic solid with material symmetry with respect to the X_1–X_2 plane and with respect to the X_2–X_3 plane are thus characterized by the 9 constants given in (26). Notice that by comparing (17) and (26), one can see that a material with the two symmetries described above is also symmetric with respect to a third plane which is orthogonal with respect to both the original planes, i.e. the X_1–X_3 plane.

Case III - Transverse isotropy

A homogeneous orthotropic solid that also has material symmetry in every direction lying in a certain plane (say, the X_1–X_2 plane) is said to be transversely isotropic. For such a material, a coordinate transformation which alters the X_1 and X_2 axes by an arbitrary rotation about the X_3 axis will not alter the strain energy function. This property implies that equation (17) must hold for a transformation of the type (also represented schematically in Figure 3)

$$\begin{aligned} X'_1 &= X_1 \cos\alpha + X_2 \sin\alpha \\ X'_2 &= -X_1 \sin\alpha + X_2 \cos\alpha \\ X'_3 &= X_3 \ , \end{aligned} \tag{27}$$

where the angle α is arbitrary. We will show that the number of material constants needed to describe the mechanical behavior of a material with the above symmetry is 5. The components of the orthogonal tensor A_{PQ} corresponding to (27) are

$$\left[A_{PQ}\right] = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad . \tag{28}$$

Substituting (28) and (26), since the material is also orthotropic, into (17) gives the relationships

$$C_{1111} = C_{2222} \ , \quad C_{1133} = C_{2233} \ , \quad C_{1313} = C_{2323}$$
$$C_{1212} = \frac{1}{2}(C_{1111} - C_{1122}) \ .$$

The remaining independent compenents are

$$\begin{matrix} C_{1111} & C_{1122} & C_{1133} & 0 & 0 & 0 \\ & C_{1111} & C_{1133} & 0 & 0 & 0 \\ & & C_{3333} & 0 & 0 & 0 \\ & & & C_{1313} & 0 & 0 \\ & & & & C_{1313} & 0 \\ & & & & & \frac{1}{2}(C_{1111} - C_{1122}) \ . \end{matrix} \tag{29}$$

The material properties of a linear elastic solid which is transversely isotropic are thus characterized by the five constants given in (29).

Case IV - Isotropic material

A homogeneous elastic solid which has material symmetry in the reference configuration with respect to all directions is said to be isotropic. Thus, equation (17) will remain valid for any proper orthogonal choice of A_{PQ}. We will now derive equation (18) using the symmetries which characterize an isotropic material. We will thus show that only two material constants are needed to describe the mechanical behavior of a linear elastic isotropic solid.

An isotropic material can be considered as a material transversely isotropic in three mutually orthogonal planes (such as the X_1–X_2, X_1–X_3, and X_2–X_3 planes). Thus, a coordinate transformation which alters the X_2 and X_3 axes by an arbitrary rotation about the X_1 axis will not alter the strain energy function (see Figure 4a). The same is true with respect to an arbitrary rotation of the X_1 and X_3 axes about the X_2 axes (see Figure 4b).

Thus, equation (17) must remain valid under the transformations

$$
\begin{aligned}
X'_1 &= X_1 \\
X'_2 &= X_2 \cos\phi + X_3 \sin\phi \\
X'_3 &= -X_2 \sin\phi + X_3 \cos\phi
\end{aligned}
\tag{30}
$$

and

$$
\begin{aligned}
X'_1 &= X_1 \cos\beta - X_3 \sin\beta \\
X'_2 &= X_2 \\
X'_3 &= X_3 \cos\beta + X_1 \sin\beta \ ,
\end{aligned}
\tag{31}
$$

where ϕ and β are arbitrary angles, in addition to the transformation (27). For the coordinate transformation in (30), A_{PQ} is given by

$$
\left[A_{PQ} \right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}
\tag{32}
$$

and for the coordinate transformation in (31), A_{PQ} is given by

$$
\left[A_{PQ} \right] = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \ .
\tag{33}
$$

>From (17), (29), (32), and (33), we can show the relationships

$$
\begin{aligned}
&C_{1122} = C_{1133} \ , \quad C_{1111} = C_{3333} \\
&C_{1212} = C_{1313} = \frac{1}{2}(C_{1111} - C_{1122}) \ .
\end{aligned}
$$

The remaining independent constants are given by

$$
\begin{matrix}
C_{1111} & C_{1122} & C_{1122} & 0 & 0 & 0 \\
 & C_{1111} & C_{1122} & 0 & 0 & 0 \\
 & & C_{1111} & 0 & 0 & 0 \\
 & & & \mu & 0 & 0 \\
 & & & & \mu & 0 \\
 & & & & & \mu
\end{matrix}
\tag{34}
$$

where $\mu = \frac{1}{2}(C_{1111} - C_{1122})$. Letting $\lambda = C_{1122}$, equation (34) may be expressed as

$$\begin{matrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ & & \lambda + 2\mu & 0 & 0 & 0 \\ & & & \mu & 0 & 0 \\ & & & & \mu & 0 \\ & & & & & \mu \end{matrix} \quad . \tag{35}$$

An equivalent way of writing (35) is given simply

$$C_{ABMN} = \lambda\delta_{AB}\delta_{MN} + \mu(\delta_{AM}\delta_{BN} + \delta_{AN}\delta_{BM}) \quad . \tag{36}$$

Notice that (36) is the same as equation (18) obtained previously.

Appendix L: Some Results from Linear Algebra

The object of this appendix is to introduce the concept of a tensor in a form which is especially useful in continuum mechanics. Thus, after a rapid review and summary of basic results from linear algebra, a tensor is regarded as a linear transformation. Many aspects of the algebra and the calculus of tensors are subsequently discussed in coordinate-free form and frequently are represented with reference to rectangular Cartesian basis vectors.

1. Sets.

We use the symbol X to represent a set, class, collection or family which contains elements or objects $x_1, x_2, \ldots$. We write this as

$$X = \{x_1, x_2, \ldots\}. \tag{1.1}$$

The element x is a member of X, or belongs to X, or is in X, and this is denoted by

$$x \in X. \tag{1.2}$$

Let X and Y be two sets. The set X is a *subset* of Y if every x in X is also in Y. This relation is designated by

$$X \subseteq Y. \tag{1.3}$$

Also Y is said to *include* or contain X, and we write $Y \supseteq X$. The *empty* set is contained in every set and is designated by $\emptyset$. If

$$X \subseteq Y \quad \text{and} \quad Y \subseteq X, \tag{1.4}$$

then the set Y and X are said to *coincide* or be equal and we write

$$Y = X. \tag{1.5}$$

If X is a subset of Y but without X being equal to Y, i.e., $X \subset Y$, then X is a proper subset of Y and we write[1]

$$X \subset Y \ , \ Y \supset X. \tag{1.6}$$

It should be clear that $X \subset Y$ means that (i) if $x \in X$, then $x \in Y$ and that (ii) there exists a $y \in Y$ such that $y \in X$.

The *union* of two sets X and Y comprises all the elements of the two sets and is denoted by

$$X \cup Y. \tag{1.7}$$

The *intersection* of two sets consists only of those elements which belong to both X and Y and is denoted by

$$X \cap Y. \tag{1.8}$$

The Cartesian product of two sets X and Y is the set

$$X \times Y = \{(x,y) : x \in X, \ y \in Y\} \tag{1.9}$$

of ordered pairs. For example, if $X = \{x_1,x_2,x_3\}$ and $Y = \{y_1,y_2\}$, then $X \times Y = \{(x_1,y_1),(x_1,y_2),(x_2,y_1),(x_2,y_2),(x_3,y_1),(x_3,y_2)\}$.

For completeness, we introduce now the following definition of a function: A *function* f from the set X to the set Y is a *rule* which assigns to every element of X a unique element of Y. It is often displayed as

$$f : X \to Y \quad , \quad x \mapsto y \ . \tag{1.10}$$

---

[1] Our use of the symbols $\subseteq$ and $\subset$ is analogous to the use of the symbols $\leq$ and $<$ to denote weak and strict inequalities.

The first of (1.10) indicates that the set X is transformed into Y, while the second of (1.10) indicates that the element x goes to the element y. The function f, or the rule defined by $(1.10)_1$, is sometimes called an operation or a transformation or a mapping. This definition of a function rules out multivalued functions.

2. Vector spaces.

Consider a set V of arbitrary elements **u**,**v**,**w**,...,etc., and admit the following two operations labeled as (1) and (2):

(1) Let there exist an operation, denoted by "+", which to every pair **u**,**v** assigns a unique element **u** + **v** in V and which has the properties:

(A_1) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (commutative property).

(A_2) $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ (associative property).

(A_3) There exists a zero vector **o** such that

$$\mathbf{v} + \mathbf{o} = \mathbf{v} .$$

(A_4) For every vector **v** there is a corresponding negative vector (–**v**) such that

$$\mathbf{v} + (-\mathbf{v}) = \mathbf{o}.$$

It is customary to write $\mathbf{v} + (-\mathbf{w}) = \mathbf{v} - \mathbf{w}$.

(2) Let there exist another operation, indicated by placing a juxtaposition of an element of V and a real number α , which to every element **v** assigns a unique element $\alpha\mathbf{v}$ in V and which has the following properties:

(S_1) $1\mathbf{v} = \mathbf{v}$.

(S_2) $\alpha(\beta\mathbf{v}) = (\alpha\beta)\mathbf{v}$ (associative property).

(S_3) $(\alpha+\beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$ (distributive property for scalar addition).

(S_4) $\alpha(\mathbf{v}+\mathbf{w}) = \alpha\mathbf{v} + \alpha\mathbf{w}$ (distributive property for vector addition).

We then say that V is a *vector space* over the field of real numbers and that **u**,**v**,**w**,... are vectors in V. The first operation defined under (1) above is callled *vector addition* with elements **u** + **v** called the *sum* of **u** and **v**. The second operation defined under (2) is called *multiplication* of vectors by a real number.

All the usual algebraic results can be derived from the above axioms. For example, a vector equation of the form $\alpha\mathbf{u} = \mathbf{o}$ has the solution $\alpha = 0$ or $\mathbf{u} = \mathbf{o}$.

Vector sub-spaces. A sub-space of a vector space V is any nonempty subset U of V which is such that if **u** and **v** belong to U and α is any real number, then the vectors **u** + **v** and $\alpha\mathbf{v}$ also belong to U.

3. n-Dimensional vector spaces.

Let $\mathbf{v}_1,\mathbf{v}_2,..,\mathbf{v}_p$ be any p vectors in a vector space V, p being some integer. The sum $\sum_{i=1}^{p} \alpha_i \mathbf{v}_i$ is called a linear combination of the p vectors $\mathbf{v}_i$ with coefficients α_i as real numbers. These vectors are said to form a *linearly independent* set of order p if the only coefficients that satisfy the equation

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_p\mathbf{v}_p = \mathbf{o} \tag{3.1}$$

are $\alpha_1 = \alpha_2 = \alpha_3 = \cdots = \alpha_p = 0$. A *linearly dependent* set is one which is *not* linearly independent.

Note that every subset of vectors containing the zero vector is linearly dependent.

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + 1\,\mathbf{o} + \alpha_p\mathbf{v}_p = 0.$$

Consider the set of all systems of linearly independent vectors in the vector space V. There are two possibilities: *Either* (1) there exist linearly independent systems of arbitrarily large order p; then the vector space is said to be an *infinite-dimensional* space, *or* (2) the order of the linearly independent sytstem is bounded. In the second case, there exists an integer n such that the order $p \leq n$ and there exist linearly independent systems of order n but not of n + 1. The vector space V is said to be a *finite-dimensional* vector space. The number n is termed the dimension of the vector space and we shall use V^n to designate finite n-dimensional vector space; and henceforth in this Appendix we shall be concerned with finite dimensional vector spaces only. Let $\{\mathbf{g}_1,\mathbf{g}_2,..,\mathbf{g}_n\}$ be any such system of order n in a finite dimensional vector space V^n; it will be called a basis of V^n. Thus, we have the following definition:

A *basis* in a vector space V^n is any linearly independent set of n vectors.

Let $\mathbf{v}$ be any vector in V^n. The set of n + 1 vectors $\{\mathbf{v},\mathbf{g}_1,\mathbf{g}_2,..,\mathbf{g}_n\}$ is necessarily linearly dependent, so there exists n + 1 numbers $\lambda,\alpha_1,\alpha_2,..,\alpha_n$ not all zero such that

$$\lambda \mathbf{v} + \alpha_1 \mathbf{g}_1 + \alpha_2 \mathbf{g}_2 + \cdots + \alpha_n \mathbf{g}_n = \mathbf{o}. \tag{3.2}$$

If $\mathbf{v} = \mathbf{o}$, then $\alpha_i \mathbf{g}_i = \mathbf{o}$, (no sum on i), and hence $\alpha_i = 0$. For $\mathbf{v} \neq \mathbf{o}$, the number λ must be different from zero, for otherwise the system $\mathbf{g}_i$ (i = 1,2,..,n) will not be linearly independent. Equation (3.2) can, therefore, be solved for $\mathbf{v}$, and there exist numbers $v^1, v^2, ..., v^n$ such that

$$\mathbf{v} = v^1 \mathbf{g}_1 + v^2 \mathbf{g}_2 + \cdots + v^n \mathbf{g}_n \ . \tag{3.3}$$

Thus, the vector $\mathbf{v}$ is expressible as a linear combination of the $\mathbf{g}_i$. This combination is unique; for otherwise, there would exist another, and their difference would constitute a linear combination of the $\mathbf{g}_i$ equal to the null vector $\mathbf{o}$ and with coefficients not all zero. The numbers $(v^1, v^2, .., v^n)$ are called the components of $\mathbf{v}$ with respect to the basis $\{\mathbf{g}_1, \mathbf{g}_2, .., \mathbf{g}_n\}$. It is convenient to replace (3.3) by the shorter notation

$$\mathbf{v} = v^k \mathbf{g}_k \tag{3.4}$$

in which the summation convention is used. Whenever an index appears twice in the same term, a summation is implied over all terms by letting that index assume all its possible values, unless the contrary is stated. Normally the convention applies to an index which appears once as a subscript and once as a superscript; but, as will become apparent later, in a special case it will suffice to adopt the convention for repeated subscripts. It is convenient to recall here a theorem the proof of which can be easily found in a book of linear algebra[2]. A statement of the theorem is as follows:

For a set of vectors to constitute a basis in V^n it is necessary and sufficient that every vector in V^n can be expressed in one, and only one, way as a linear combination of the vectors of that set.

---

[2] See a standard book on linear algebra.

4. Euclidean vector spaces.

Consider a vector space of dimension n defined over the field of real numbers. Suppose there is an operation, denoted by " · ", which to every pair of vectors **u** and **v** assigns a real number, denoted by $\mathbf{u} \cdot \mathbf{v}$, and which has the following properties:

(I_1) $\quad \mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.

(I_2) $\quad \mathbf{u} \cdot (\mathbf{v}+\mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$.

(I_3) $\quad (\alpha\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (\alpha\mathbf{v}) = \alpha(\mathbf{u} \cdot \mathbf{v})$.

(I_4) $\quad \mathbf{u} \cdot \mathbf{u} \geq 0$ and $\mathbf{u} \cdot \mathbf{u} = 0 \Rightarrow \mathbf{u} = \mathbf{o}$.

The above operation (known as a rule of composition) is called the dot product or the scalar product or the inner product of two vectors. A vector space V^n obeying the rules of composition (I_1) to (I_4) is a real inner product space and is called a Euclidean vector space[3] E^n .

The magnitude or the norm of the vector **v** is defined by

$$||\mathbf{v}|| = (\mathbf{v}\cdot\mathbf{v})^{1/2} . \tag{4.1}$$

If $||v|| = 1$, the vector is said to be normalized. By (I_4) $||v||^2$ is always positive if $\mathbf{v} \neq \mathbf{o}$ and it vanishes when $\mathbf{v} = \mathbf{o}$, so that its square root is real.

The norms of two vectors and the magnitude of their scalar product satisfy the *Schwarz inequality*

[3] This terminology stems from the fact that once a scalar product is admitted, all the standard theorems of Euclid's geometry can be established by an appeal to the properties of an inner product space.

$$|\mathbf{u} \cdot \mathbf{v}| \le ||\mathbf{u}|| \ ||\mathbf{v}|| \ . \tag{4.2}$$

To prove this, consider the vector $\alpha\mathbf{u} + \mathbf{v}$ where α is an arbitrary real number. Then,

$$||(\alpha\mathbf{u} + \mathbf{v})||^2 = \alpha^2 ||\mathbf{u}||^2 + 2\alpha\mathbf{u} \cdot \mathbf{v} + ||\mathbf{v}||^2 \ .$$

The left-hand side of the last expression is positive or zero for all real α and this implies

$$(\mathbf{u} \cdot \mathbf{v})^2 \le ||\mathbf{u}||^2 \ ||\mathbf{v}||^2 \ ,$$

which is equivalent to (4.2). In view of (4.2), we define the angle between two nonzero vectors **u** and **v** by

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}|| \ ||\mathbf{v}||} \ , \tag{4.3}$$

where $0 \le \theta \le \pi$.

Vectors **u** and **v** are said to be orthogonal if

$$\mathbf{u} \cdot \mathbf{v} = 0 \ . \tag{4.4}$$

A set of vectors in E^n is said to be *orthonormal* if all the vectors in the set are normalized and mutually orthogonal. Let $\mathbf{e}_i$ be a system of orthonormal basis in E^n . It follows that $\mathbf{e}_i$ must satisfy the conditions

$$\mathbf{e}_i \cdot \mathbf{e}_k = \delta_{ik} \ , \tag{4.5}$$

where δ_{ik} (i,k = 1,2,...,n) is the Kronecker delta defined by

$$\delta_{ik} = \begin{cases} 0 \text{ if } i \neq k, \\ 1 \text{ if } i = k. \end{cases} \tag{4.6}$$

Every orthonormal set is necessarily linearly independent. An infinity of bases exist which are orthonormal; and in this appendix a typical orthonormal basis will be denoted by $\mathbf{e}_i$. Any vector

$\mathbf{v}$ in E^n, when referred to the basis $\mathbf{e}_i$, can be expressed in the form

$$\mathbf{v} = \sum_{i=1}^{n} v_i \mathbf{e}_i \quad . \tag{4.7}$$

Then, by taking the scalar product of both sides of (4.7) with $\mathbf{e}_k$, we have

$$v_k = \mathbf{v} \cdot \mathbf{e}_k \quad . \tag{4.8}$$

The numbers v_k are the components of $\mathbf{v}$ with respect to the orthonormal basis $\mathbf{e}_k$ and the square of the magnitude of $\mathbf{v}$ is

$$||\mathbf{v}||^2 = \mathbf{v} \cdot \mathbf{v} = \sum_{i=1}^{n} v_i v_i \quad . \tag{4.9}$$

5. Points.

We denote points by boldface letters $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, etc. Given any two points $\mathbf{x}$ and $\mathbf{y}$, we associate with them a point difference $\mathbf{y} - \mathbf{x}$ which is a vector $\mathbf{v}$ in E^n and which we denote by

$$\mathbf{v} = \mathbf{y} - \mathbf{x} \tag{5.1}$$

satisfying the following rules:

(P_1) $(\mathbf{y} - \mathbf{x}) + (\mathbf{x} - \mathbf{z}) = (\mathbf{y} - \mathbf{z})$.

(P_2) Given any point $\mathbf{x}$ and any vector $\mathbf{v}$, there is a unique point $\mathbf{y}$ such that $\mathbf{y} - \mathbf{x} = \mathbf{v}$.

The point $\mathbf{y}$ determined in rule (P_2) is denoted by

$$\mathbf{y} = \mathbf{x} + \mathbf{v} = \mathbf{v} + \mathbf{x} . \tag{5.2}$$

In view of (5.1) and (5.2), it is meaningful to speak of the *difference of two points* which is a vector, and of the *sum of a point and a vector* , which is a point. In particular,

$$\mathbf{x} - \mathbf{x} = \mathbf{o} \tag{5.3}$$

holds for every point $\mathbf{x}$.

The distance between two points $\mathbf{x}, \mathbf{y}$ is

$$|\mathbf{x} - \mathbf{y}| = \{(\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y})\}^{1/2} . \tag{5.4}$$

The set of all points associated with a Euclidean vector space E^n is called *Euclidean Point Space.*[4]

[4] Euclidean vector and point spaces are abstract concepts which at this point in our discussion are unrelated to ordinary vectors and points of elementary geometry. Geometrical representations of these abstract concepts will be discussed in Section 6.

6. Geometric space.

Consider the space of elementary geometry. This space is composed of elements P,Q,R, called points. Choose O to be a reference point or origin. The directed line segments $\vec{OP}$, $\vec{OQ}$,... may be associated with points **x**,**y**,... of Euclidean point space (Section 5) and the directed line segments $\vec{PQ}$, $\vec{QR}$ may be associated with vectors **u**,**v**,... in Euclidean vector space (Section 4). Thus we set

$$\mathbf{x} = \vec{OP} \quad , \quad \mathbf{y} = \vec{OQ} \quad , \quad \mathbf{z} = \vec{OR} \quad ,... \tag{6.1}$$

and

$$\mathbf{u} = \vec{PQ} \quad , \quad \mathbf{v} = \vec{QR} \quad ,... \quad \mathbf{o} = \vec{PP} \quad . \tag{6.2}$$

The properties (A_1)–(A_4) of abstract vectors and the properties (P_1),(P_2) of points have their immediate geometrical representation in our geometric space. Thus, the class of all line segments with the same direction and length as $\vec{PQ}$, called a geometrical vector or free vector, represent vectors, and directed segments such as $\vec{OP}$, called position vectors, represent points. For example, we have the properties

$$\vec{PQ} = -\vec{QP} \quad , \quad \vec{PQ} + \vec{QR} = \vec{PR} \quad .$$

Next, the properties (S_1)–(S_4) have their geometric representation in our space of elementary geometry. Multiplication of a vector by a scalar corresponds to multiplication of a line element $\vec{PQ}$ by a scalar α and is represented by a line element of length $|\alpha|\ |PQ|$, where $|PQ|$ is the length of $\vec{PQ}$ in the same or opposite direction as $\vec{PQ}$ according to whether α is positive or negative.

The rules (I_1)–(I_4) of Section 4 have their representation in the geometrical space. Recalling also (4.3), these rules are represented by the scalar product of two geometrical vectors as

$$\vec{PQ} \cdot \vec{RS} = |PQ|\,|RS| \cos\ \theta \ , \tag{6.3}$$

where θ is the angle between $\vec{PQ}$ and $\vec{RS}$. Thus, geometrical vectors give a geometrical representation of a Euclidean point space with its associated Euclidean vector space. Geometrical space has three dimensions.

Given a geometrical space, we select an origin O and a (constant) basis $\{\mathbf{f}_i\}$ of the associated vector space. These are said to constitute a coordinate system consisting of origin and a basis $(O,\mathbf{f}_i)$ for our geometrical space. Any point $\mathbf{x}$ referred to the basis $\mathbf{f}_i$ can be expressed in terms of its components $\bar{x}_i$ by the formula

$$\mathbf{x} = \sum_{i=1}^{3} \bar{x}^i \mathbf{f}_i \ , \tag{6.4}$$

where now summation is over i = 1,2,3. The components $\bar{x}^i$ are called *rectilinear coordinates* of $\mathbf{x}$. If $(\mathbf{f}_i)$ is identified with a constant orthonormal basis $\mathbf{e}_i$, then referred to $\mathbf{e}_i$ the point $\mathbf{x}$ can be represented as

$$\mathbf{x} = \sum_{i=1}^{3} x_i \mathbf{e}_i \tag{6.5}$$

and x_i are called *Cartesian rectilinear coordinates* of $\mathbf{x}$.

We may select a different basis $\mathbf{g}_i$ (say) at each point $\mathbf{x}$ in our geometric space so that $\mathbf{g}_i = \mathbf{g}_i(x)$. This situation arises when we consider curvilinear coordinates in geometric space.

7. Indicial notation.

8. Linear transformation. Tensors.

A linear transformation or tensor **T** is an operation which assigns to each vector **v** in V^n another vector **Tv** in V^m such that[5]

$$\mathbf{T}(\mathbf{v}+\mathbf{w}) = \mathbf{Tv} + \mathbf{Tw} \quad , \quad \mathbf{T}(\alpha\mathbf{v}) = \alpha\mathbf{Tv} \tag{8.1}$$

for all **v**,**w** in V^n. Thus, we have $\mathbf{T}: \quad V^n \rightarrow V^m$.

We denote the set of all possible tensors which transform vectors in V^n linearly into vectors in V^m by $L(V^n, V^m)$ and we define the *sums* of two tensors and the *scalar multiples* of tensors by

$$(\mathbf{T}+\mathbf{S})\mathbf{v} = \mathbf{Tv} + \mathbf{Sv} \quad , \quad (\alpha\mathbf{T})\mathbf{v} = \alpha(\mathbf{Tv}) \ . \tag{8.2}$$

The transformations **T** + **S** and α**T** defined in (8.2) obey the rules (8.1), i.e., they are *tensors.* Also the sums and scalar multiples defined in $L(V^n, V^m)$ obey the rules (A_1)–(A_4) and (S_1)–(S_4). Hence the set of all tensors constitutes a vector space. The zero element in $L(V^n, V^m)$, i.e., the element which is to be substituted for **o** in rules (A_3) and (A_4) is the transformation which assigns the zero vector to every vector. We call this transformation the *zero tensor* and denote it by **O**. In other words, the zero tensor **O** is such that

$$\mathbf{Ov} = \mathbf{o} \tag{8.3}$$

for all vectors **v**.

The *identity tensor* can only be defined for transformations from V^n to V^n. Thus, when the dimension m = n, we define the identity or the unit tensor **I** by

$$\mathbf{Iv} = \mathbf{v} \tag{8.4}$$

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[5] This special definition is particularly useful in the present Appendix. A more general definition of a tensor is given in Appendix C.

for all vectors $\mathbf{v}$ in V^n.

We associate with Euclidean vector spaces E^n and E^m a vector space of dimensions nm denoted by $E^n \otimes E^m$ called the tensor product space of E^n and E^m, and defined in the following way: If $\mathbf{a} \in E^n$ and $\mathbf{b} \in E^m$, then $\mathbf{a} \otimes \mathbf{b}$ is an element of $E^n \otimes E^m$ such that

$$(\mathbf{a} \otimes \mathbf{b})\mathbf{v} = \mathbf{a}(\mathbf{b} \cdot \mathbf{v}) \tag{8.5}$$

for all vectors $\mathbf{v} \in E^m$. It is seen that $\mathbf{a} \otimes \mathbf{b}$, which sometimes is denoted as $\mathbf{ab}$, obeys the rules (8.1) and hence is a tensor, i.e., it assigns to each vector $\mathbf{v} \in E^m$ the vector $\mathbf{a}(\mathbf{b} \cdot \mathbf{v}) \in E^n$. Sums and scalar multiples of such tensors can then be calculated by the rules laid down for tensors in (8.2). Not all tensors generated in this way can be expressed as an element of space of the form $\mathbf{a} \otimes \mathbf{b}$. The zero tensor in $E^n \otimes E^n$ is defined by the tensor $\mathbf{o} \otimes \mathbf{o}$.

Consider now the tensor product space $E^n \otimes E^n$ of dimension n^2. Let $\mathbf{e}_i$ be an orthonormal basis in Euclidean vector space E^n. Then,

$$\mathbf{e}_i \otimes \mathbf{e}_k \tag{8.6}$$

is a basis in $E^n \otimes E^n$. To prove this, suppose α_{ik} are a set of n^2 numbers such that

$$\alpha_{ik}\mathbf{e}_i \otimes \mathbf{e}_k = \mathbf{O} \ .$$

Then,

$$(\alpha_{ik}\mathbf{e}_i \otimes \mathbf{e}_k)\mathbf{e}_r = \mathbf{O}\mathbf{e}_r = \mathbf{o} \ .$$

With the help of (8.5) this last equation becomes

$$\alpha_{ik}\mathbf{e}_i\delta_{kr} = \alpha_{ir}\mathbf{e}_i = \mathbf{o} \ .$$

Since $\mathbf{e}_i$ is a basis in E^n it follows that

$$\alpha_{ik} = 0 \ .$$

Hence $\mathbf{e}_i \otimes \mathbf{e}_k$ are linearly independent and form a basis in $E^n \otimes E^n$. When **u** and **v** are vectors in E^n so that

$$\mathbf{u} = u_i\mathbf{e}_i \quad , \quad \mathbf{v} = v_i\mathbf{e}_i \ ,$$

then

$$\mathbf{uv} = \mathbf{u} \otimes \mathbf{v} = u_i v_k \mathbf{e}_i \otimes \mathbf{e}_k \ . \tag{8.7}$$

Let **T** be a tensor in $L(E^n, E^n)$ and let t_{ik} denote the components of the vector $\mathbf{Te}_k$ with respect to the basis $\mathbf{e}_i$ in E^n so that

$$\mathbf{Te}_k = t_{ik}\mathbf{e}_i \quad , \quad t_{ik} = \mathbf{Te}_k \cdot \mathbf{e}_i \ . \tag{8.8}$$

Associated with each tensor **T** in $L(E^n, E^n)$ we have the ordered array of n^2 scalars t_{ik} which combine by the rules (A_1)–(A_4) and (S_1)–(S_4) for vector spaces. Hence, $L(E^n, E^n)$ is a vector space of dimension n^2. But, $\mathbf{e}_i \otimes \mathbf{e}_k$ is a linear transformation in the space $L(E^n, E^n)$ and hence also forms a basis in $L(E^n, E^n)$ which is identical to $E^n \otimes E^n$. Moreover, with the use of (8.8), for any vector $\mathbf{v} \in E^n$ we have

$$(\mathbf{T} - t_{ik}\mathbf{e}_i \otimes \mathbf{e}_k)\mathbf{v} = \mathbf{Te}_k v_k - t_{ik}\mathbf{e}_i v_k = \mathbf{o} \ .$$

Hence, we conclude that

$$\mathbf{T} = t_{ik}\mathbf{e}_i \otimes \mathbf{e}_k \ . \tag{8.9}$$

The n^2 numbers t_{ik} in (8.9) are called the components (or Cartesian components) of **T** with respect to the basis $\mathbf{e}_i \otimes \mathbf{e}_k$.

Similarly, we write

$$\mathbf{I} = \delta_{ik}\mathbf{e}_i \otimes \mathbf{e}_k \tag{8.10}$$

so that

$$\mathbf{Iv} = \mathbf{v} \tag{8.11}$$

for all vectors $\mathbf{v}$ in E^n. The tensor **I** with components δ_{ik} is the unit tensor in $E^n \otimes E^n$ or $L(E^n, E^n)$.

The system of numbers t_{ik} in (8.8) is usually regarded as an array of numbers, *a matrix,* having n rows and n columns and denoted by $[t_{ik}]$ or $\{t_{ik}\}$.

Let $\mathbf{T}$ be a tensor in $E^n \otimes E^n$. The *transpose* of a tensor $\mathbf{T}$ is a tensor $\mathbf{T}^T$ defined by

$$\mathbf{w} \cdot \mathbf{Tv} = \mathbf{v} \cdot \mathbf{T}^T\mathbf{w} \tag{8.12}$$

for all vectors $\mathbf{v}$ and $\mathbf{w}$ in E^n. The transpose $\mathbf{T}^T$ satisfies (8.1) and is a tensor, and the operation which associates $\mathbf{T}^T$ with $\mathbf{T}$ is called transposition. From (8.12) it may be shown that

$$(\mathbf{T}^T)^T = \mathbf{T} \quad , \quad (\mathbf{T}+\mathbf{S})^T = \mathbf{T}^T + \mathbf{S}^T \quad , \quad (\alpha\mathbf{T})^T = \alpha\mathbf{T}^T \ , \tag{8.13}$$

so that transposition is a linear operation. Further, it follows from (8.9) and (8.12) that

$$[t_{ik}] = \mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_k = \mathbf{e}_k \cdot \mathbf{T}^T\mathbf{e}_i = [t_{ik}]^T \ ,$$

where $[t_{ik}]^T$ are components of the tensor $\mathbf{T}^T$.

A tensor $\mathbf{T}$ is said to be *symmetric* if

$$\mathbf{T}^T = \mathbf{T} \ . \tag{8.14}$$

Also, by (8.9) and (8.14) we have

$$t_{ki} = t_{ik} \ . \tag{8.15}$$

Also, from (8.12) and (8.14) it follows that if $\mathbf{T}$ is symmetric, then

$$\mathbf{w} \cdot \mathbf{Tv} = \mathbf{v} \cdot \mathbf{Tw} \tag{8.16}$$

for every $\mathbf{v},\mathbf{w}$ in E^n and conversely.

A tensor $\mathbf{T}$ is said to be *skew* or anti-symmetric *(skew*- symmetric) if

$$\mathbf{T}^T = -\mathbf{T} \tag{8.17}$$

or

$$t_{ki} = -t_{ik} \ . \tag{8.18}$$

Every tensor $\mathbf{T}$ in $E^n \otimes E^n$ can be written as

$$\mathbf{T} = \mathbf{T}^S + \mathbf{T}^A \ , \tag{8.19}$$

where

$$\mathbf{T}^S = \frac{1}{2}(\mathbf{T}+\mathbf{T}^T) - (\mathbf{T}^S)^T \quad , \quad \mathbf{T}^A = \frac{1}{2}(\mathbf{T}-\mathbf{T}^T) = -(\mathbf{T}^A)^T \ . \tag{8.20}$$

9. Multiplication of tensors.

Let **T** and **S** be two tensors in $L(V^n,V^n)$. The product **TS** of two tensors **T** and **S** is the composition of **T** and **S** , i.e., **TS** defined by the requirement that

$$(\mathbf{TS})\mathbf{v} = \mathbf{T}(\mathbf{Sv}) \tag{9.1}$$

hold for all vectors **v** in V^n. It is seen that **TS** satisfies (8.1) and is a tensor in $L(V^n,V^n)$. The following rules may be established

(M_1) $(\mathbf{TS})\mathbf{R} = \mathbf{T}(\mathbf{SR})$,

(M_2) $\mathbf{T}(\mathbf{R}+\mathbf{S}) = \mathbf{TR} + \mathbf{TS}$,

(M_3) $(\mathbf{R}+\mathbf{S})\mathbf{T} = \mathbf{RT}+\mathbf{ST}$,

(M_4) $\alpha(\mathbf{TS}) = (\alpha\mathbf{T})\mathbf{S} = \mathbf{T}(\alpha\mathbf{S})$,

(M_5) $\mathbf{IT} = \mathbf{TI} = \mathbf{T}$,

where **T**,**S**,**R** are tensors in $L(V^n,V^n)$.

For tensors **T**,**S** in $L(E^n,E^n)$ it follows from (6.9) and (8.1) that

$$(\mathbf{TS})\mathbf{v} = \mathbf{T}(s_{mk}v_k\mathbf{e}_m) = t_{im}s_{mk}v_k\mathbf{e}_i$$

for all vectors $\mathbf{v} \in E^n$ so that

$$\mathbf{TS} = t_{im}s_{mk}\mathbf{e}_i \otimes \mathbf{e}_k \tag{9.2}$$

and $t_{im}s_{mk}$ are the components of **TS** with respect to the basis $\mathbf{e}_i \otimes \mathbf{e}_j$. If $\{t_{ik}\}$ and $\{s_{ik}\}$ are matrix representations of the tensors **T** and **S**, respectively, then

$$\{t_{ik}\}\{s_{ik}\} = \{t_{im}s_{mk}\} \ . \tag{9.3}$$

Note that

$$(\mathbf{TS})^{\mathrm{T}} = \mathbf{S}^{\mathrm{T}}\mathbf{T}^{\mathrm{T}} \ . \tag{9.4}$$

The commutative law is *not*, in general, satisfied, i.e., **TS** is not the same as **ST**.

The truth of the following formulae can be readily verified:

$$\begin{aligned}
&(\mathbf{a} \otimes \mathbf{b})^{\mathrm{T}} = \mathbf{b} \otimes \mathbf{a} \ , \\
&\mathbf{T}(\mathbf{a} \otimes \mathbf{b}) = \mathbf{Ta} \otimes \mathbf{b} \ , \\
&\mathbf{a} \otimes (\mathbf{T}^{\mathrm{T}}\mathbf{b}) = (\mathbf{a} \otimes \mathbf{b})\mathbf{T} \ , \\
&(\mathbf{a} \otimes \mathbf{b})(\mathbf{c} \otimes \mathbf{d}) = (\mathbf{b} \cdot \mathbf{c})\mathbf{a} \otimes \mathbf{d} \ .
\end{aligned} \tag{9.5}$$

The tensor **T** is invertible if, for every choice of **w**, the equation $\mathbf{w} = \mathbf{Tv}$ can be solved uniquely for the vector **v**. If **T** is invertible, then **v** is unique and we write $\mathbf{v} = \mathbf{T}^{-1}\mathbf{w}$. The transformation $\mathbf{T}^{-1}$ obeys the rules (8.1) and is a tensor called the inverse of **T**. If the inverse $\mathbf{T}^{-1}$ exists, then

$$(\mathbf{TT}^{-1} - \mathbf{I})\mathbf{w} = \mathbf{o} \quad , \quad (\mathbf{T}^{-1}\mathbf{T} - \mathbf{I})\mathbf{v} = \mathbf{o}$$

for every choice of **w** and every choice of **v** so that

$$\mathbf{TT}^{-1} = \mathbf{T}^{-1}\mathbf{T} = \mathbf{I} \quad , \quad (\mathbf{T}^{-1})^{-1} = \mathbf{T} \ . \tag{9.6}$$

If **S** and **T** are invertible tensors in $\mathrm{L}(V^{\mathrm{n}}, V^{\mathrm{n}})$ and if $\mathbf{w} = (\mathbf{TS})\mathbf{v}$, it follows that

$$\mathbf{Sv} = \mathbf{T}^{-1}\mathbf{w} \quad , \quad \mathbf{v} = \mathbf{S}^{-1}\mathbf{T}^{-1}\mathbf{w} \ .$$

Hence, the product **TS** is invertible and

$$(\mathbf{TS})^{-1} = \mathbf{S}^{-1}\mathbf{T}^{-1} \ . \tag{9.7}$$

If the only solution of the equation $\mathbf{Tv} = \mathbf{o}$ is $\mathbf{v} = \mathbf{o}$, then $\mathbf{T}$ is *nonsingular*; otherwise it is singular. If $\mathbf{T}$ is invertible, then $\mathbf{T}$ is nonsingular. It can be shown that a tensor $\mathbf{T}$ is invertible or nonsingular if and only if $\det \mathbf{T} \neq 0$.

A tensor $\mathbf{Q}$ is *orthogonal* if it is invertible and if the inverse coincides with its transpose, i.e.,

$$\mathbf{Q}^{-1} = \mathbf{Q}^{T} \quad \text{or} \quad \mathbf{Q}\mathbf{Q}^{T} = \mathbf{Q}^{T}\mathbf{Q} = \mathbf{I} \ . \tag{9.8}$$

A tensor $\mathbf{Q}$ is orthogonal if and only if it preserves inner products in the sense that

$$(\mathbf{Qu}) \cdot (\mathbf{Qv}) = \mathbf{u} \cdot \mathbf{v} \tag{9.9}$$

for every vector $\mathbf{u},\mathbf{v}$ in E^{n}.

The tensor $\mathbf{T}$ is positive semidefinite if it is symmetric and if $\mathbf{v} \cdot \mathbf{Tv} \geq o$ for all vectors $\mathbf{v}$ in E^{n}. If $\mathbf{v} \cdot \mathbf{Tv} > o$ when $\mathbf{v} \neq \mathbf{o}$, then $\mathbf{T}$ is positive definite.

An operation tr which assigns to each tensor $\mathbf{T}$ in $E^{n} \otimes E^{n}$ a number $\operatorname{tr} \mathbf{T}$ is a *trace* if

$$\operatorname{tr}(\mathbf{S}+\mathbf{T}) = \operatorname{tr} \mathbf{S} + \operatorname{tr} \mathbf{T} \quad , \quad \operatorname{tr}(\alpha \mathbf{T}) = \alpha \operatorname{tr} \mathbf{T} \ , \tag{9.10}$$

$$\operatorname{tr} \mathbf{a} \otimes \mathbf{b} = \mathbf{a} \cdot \mathbf{b} \ , \tag{9.11}$$

for all vectors $\mathbf{a},\mathbf{b}$ in E^{n} and all scalars α. The tr operation is a linear function and the following consequences are clear:

$$\operatorname{tr} \mathbf{T} = t_{ik}\mathbf{e}_i \cdot \mathbf{e}_k = t_{ii} \ . \tag{9.12}$$

Also,

$$\begin{aligned} &\operatorname{tr} \mathbf{T}^{T} = \operatorname{tr} \mathbf{T} \quad , \quad \operatorname{tr}(\mathbf{TS}) = \operatorname{tr}(\mathbf{ST}) \ , \\ &\operatorname{tr} \mathbf{I} = n \quad , \quad \operatorname{tr} \mathbf{T}^{S} = \operatorname{tr} \mathbf{T} \quad , \quad \operatorname{tr} \mathbf{T}^{A} = 0 \ , \end{aligned} \tag{9.13}$$

where n in $(9.12)_3$ is the dimension of the space.

If one defines the *inner product* of two tensors $\mathbf{T}$,$\mathbf{S}$ to be the number

$$\mathbf{T} \cdot \mathbf{S} = \mathrm{tr}(\mathbf{T}\mathbf{S}^T) \ , \tag{9.14}$$

then the rules (I_1)–(I_5) are satisfied. Hence with $\mathrm{tr}(\mathbf{S}\mathbf{T}^T)$ as the inner product, the space of all tensors is an inner product space. We define the magnitude of $\mathbf{T}$ as

$$|\mathbf{T}| = \sqrt{\mathrm{tr}(\mathbf{T}\mathbf{T}^T)} \ . \tag{9.15}$$

10. Change of basis.

Let $\mathbf{e}_i$ and $\bar{\mathbf{e}}_i$ be two arbitrary orthonormal bases in E^n. Each vector in one basis may be expressed in terms of those of the other basis by a nonsingular transformation. Thus

$$\begin{aligned} \bar{\mathbf{e}}_i &= \mathbf{A}\mathbf{e}_i = a_{ki}\mathbf{e}_k \ , \\ \mathbf{e}_i &= \bar{\mathbf{A}}\,\bar{\mathbf{e}}_i = \bar{a}_{ki}\bar{\mathbf{e}}_k \quad , \quad \bar{\mathbf{A}} = \mathbf{A}^{-1} \ , \end{aligned} \tag{10.1}$$

where

$$\begin{aligned} \bar{\mathbf{e}}_k \cdot \mathbf{A}\mathbf{e}_i &= \delta_{ik} = \mathbf{e}_i \cdot \bar{\mathbf{A}}\bar{\mathbf{e}}_k \ , \\ \bar{a}_{ki} &= \mathbf{e}_i \cdot \bar{\mathbf{e}}_k = a_{ik} \quad , \quad \bar{\mathbf{A}} = \mathbf{A}^T \ . \end{aligned} \tag{10.2}$$

Hence, $\mathbf{A}^{-1} = \mathbf{A}^T$ and $\mathbf{A}$ is orthogonal.

If $\mathbf{v}$ is an arbitary vector and

$$\mathbf{v} = v_i\mathbf{e}_i = \bar{v}_i\bar{\mathbf{e}}_i \ , \tag{10.3}$$

then the components of $\mathbf{v}$ in the two bases $\mathbf{e}_i$ and $\bar{\mathbf{e}}_i$ are related by the equations

$$\bar{v}_i = a_{ki}v_k \quad , \quad v_i = a_{ik}\bar{v}_k \ . \tag{10.4}$$

Also

$$\mathbf{v} \cdot \mathbf{v} = v_iv_i = \bar{v}_i\bar{v}_i \quad , \quad u \cdot v = u_iv_i = \bar{u}_i\bar{v}_i \ , \tag{10.5}$$

and the scalar product has the same form in terms of the components u_i, v_i as it has in terms of the components $\bar{u}_i, \bar{v}_i$ and is called a scalar invariant.

Again, if $\mathbf{T}$ is an arbitrary tensor in $E^n \otimes E^n$ and

$$\mathbf{T} = t_{ik}\mathbf{e}_i \otimes \mathbf{e}_k = \bar{t}_{ik}\bar{\mathbf{e}}_i \otimes \bar{\mathbf{e}}_k \ , \tag{10.6}$$

then the components of **T** in the two bases $\mathbf{e}_i$ and $\bar{\mathbf{e}}_i$ are related by

$$\bar{t}_{ik} = a_{ri}a_{sk}t_{rs} \quad , \quad t_{ik} = a_{ir}a_{ks}\bar{t}_{rs} \ . \tag{10.7}$$

If **T** and **S** are two tensors in $E^n \otimes E^n$ and

$$\mathbf{S} = s_{ik}\mathbf{e}_i \otimes \mathbf{e}_k = \bar{s}_{ik}\bar{\mathbf{e}}_i \otimes \bar{\mathbf{e}}_k \ ,$$

then

$$\begin{aligned} \mathbf{T} \cdot \mathbf{S} &= t_{ir}s_{ir} = \bar{t}_{ir}\bar{s}_{ir} \ , \\ \mathbf{T} \cdot \mathbf{T} &= t_{ir}t_{ir} = \bar{t}_{ir}\bar{t}_{ir} \end{aligned} \tag{10.8}$$

are scalar invariant forms.

11. Point, vector and tensor functions.

The *distance* between two points **x**,**y** is defined in (6.4). The distance vanishes if and only if **x** = **y**. This idea of distance can be used to define limits, convergence, continuity, etc. For example, we say that

$$\lim_{n\to\infty} \mathbf{x}_n = \mathbf{x} \quad \text{if} \quad \lim_{n\to\infty} |\mathbf{x}_n - \mathbf{x}| = \mathbf{O} \ . \tag{11.1}$$

Suppose $\mathbf{z}(t)$ is a function of a real variable t called a *point function.* The derivative $\dot{\mathbf{z}}(t)$, if it exists, on an open subset of the real numbers is defined by

$$\dot{\mathbf{z}}(t) = \frac{d\mathbf{z}}{dt} = \lim_{\Delta t\to\infty} \frac{\mathbf{z}(t+\Delta t) - \mathbf{z}(t)}{\Delta t} \ . \tag{11.2}$$

Such a derivative is a vector-valued function.

In a similar way we may define limits, derivatives, etc., of vector- and tensor-valued functions **v**,**T** of t using the appropriate definitions (1.2) and (9.15) for distance. The derivative of a vector-valued function of t is a vector function and the derivative of a tensor-valued function is a tensor function. The usual rules of differential calculus are easily extended so as to apply to point, vector or tensor functions. For example, if $\mathbf{T} = \mathbf{T}(t)$ and $\mathbf{S} = \mathbf{S}(t)$ are tensors, then

$$\dot{\overline{\mathbf{TS}}} = \dot{\mathbf{T}}\mathbf{S} + \mathbf{T}\dot{\mathbf{S}} \ , \tag{11.3}$$

where, of course, the correct order must be maintained for the tensors.

If $\mathbf{Q} = \mathbf{Q}(t)$ is an orthogonal tensor function, then from (9.8) and (11.3),

$$\dot{\mathbf{Q}}\mathbf{Q}^T + \mathbf{Q}\dot{\overline{\mathbf{Q}^T}} = \mathbf{O} \ . \tag{11.4}$$

Since $\dot{\mathbf{Q}}^T = \overline{\dot{\mathbf{Q}}}^T$ it follows that

$$\Omega = \dot{\mathbf{Q}}\,\mathbf{Q}^T = -\,\mathbf{Q}\,\dot{\mathbf{Q}}^T \tag{11.5}$$

is a skew tensor.

If $\phi(\mathbf{x})$ is a scalar function of $\mathbf{x}$ defined on some open region U of Euclidean point space, then ϕ is differentiable on U if there is a vector field $\mathbf{w}(\mathbf{x})$ on U such that

$$\lim_{\mathbf{x}\to{}_o\mathbf{x}} \frac{|\phi(\mathbf{x}) - \phi({}_o\mathbf{x}) - \mathbf{w}({}_o\mathbf{x})\cdot(\mathbf{x} - {}_o\mathbf{x})|}{|\mathbf{x}-{}_o\mathbf{x}|} = 0 \tag{11.6}$$

for every ${}_o\mathbf{x}$ in U. If this is so, $\mathbf{w}$ is unique. We call it the gradient of ϕ and write

$$\mathbf{w} = \text{grad } \phi(\mathbf{x}) = \frac{\partial\phi}{\partial\mathbf{x}} . \tag{11.7}$$

Alternatively, the gradient of $\phi(\mathbf{x})$ may be defined by means of

$$\lim_{\beta\to 0} \frac{\phi(\mathbf{x}+\beta\mathbf{u}) - \phi(\mathbf{x})}{\beta} = \frac{d}{d\beta}\phi(\mathbf{x}+\beta\mathbf{u})\Big|_{\beta=0} = \frac{\partial\phi}{\partial\mathbf{x}}\cdot\mathbf{u} = \mathbf{w}\cdot\mathbf{u} \tag{11.8}$$

for all vectors $\mathbf{u} \in E^n$. It may be shown that this definition of gradient is equivalent to that given in (11.6).

Sometimes we use the symbol ∇ to denote the gradient operator:

$$\nabla = \frac{\partial}{\partial\mathbf{x}} = \mathbf{e}_r \frac{\partial}{\partial x_r} \tag{11.9}$$

if $\mathbf{x} = x_r\mathbf{e}_r$ and $\mathbf{e}_r$ is an orthonormal basis. From (11.6) it follows that

$$\mathbf{w} = \text{grad } \phi(\mathbf{x}) = \frac{\partial\phi}{\partial\mathbf{x}} = \nabla\phi = \mathbf{e}_i \frac{\partial\phi}{\partial x_i} . \tag{11.10}$$

If $\mathbf{v}(\mathbf{x})$ is a vector function of points $\mathbf{x}$ in some open region U of Euclidean space, then $\mathbf{v}(\mathbf{x})$ is said to be a vector field. The gradient of a vector field $\mathbf{v}(\mathbf{x})$ is a tensor field $\mathbf{T}(\mathbf{x})$ defined by

$$\lim_{\mathbf{x}\to{}_o\mathbf{x}} \frac{|\mathbf{v}(\mathbf{x}) - \mathbf{v}({}_o\mathbf{x}) - \mathbf{T}({}_o\mathbf{x})(\mathbf{x}-{}_o\mathbf{x})|}{|\mathbf{x}-{}_o\mathbf{x}|} = o \tag{11.11}$$

for every ${}_o\mathbf{x}$ in U. We write this as

$$\mathbf{T} = \operatorname{grad} \mathbf{v}(\mathbf{x}) = \frac{\partial \mathbf{v}}{\partial \mathbf{x}} \ . \tag{11.12}$$

If $\mathbf{v} = v_i\mathbf{e}_i$, then it follows from (11.12) that

$$\mathbf{T} = \frac{\partial v_i}{\partial x_k}\mathbf{e}_i \otimes \mathbf{e}_k \tag{11.13}$$

so that $\frac{\partial v_i}{\partial x_k}$ are the components of the grad $\mathbf{v}$ with respect to the orthonormal basis $\mathbf{e}_i$. It can be shown that

$$\{\operatorname{grad} \mathbf{v}(\mathbf{x})\}^T\mathbf{c} = \operatorname{grad}\{\mathbf{c}\cdot\mathbf{v}(\mathbf{x})\} \tag{11.14}$$

for every constant vector $\mathbf{c}$. Equation (11.14) could be taken as the defining equation for grad $\mathbf{v}(\mathbf{x})$.

The *divergence* of the vector field $\mathbf{v}(\mathbf{x})$ is a scalar field defined by

$$\operatorname{div} \mathbf{v}(\mathbf{x}) = \operatorname{tr}\{\operatorname{grad} \mathbf{v}(\mathbf{x})\} = \nabla \cdot \mathbf{v} = \mathbf{e}_i \cdot \frac{\partial \mathbf{v}}{\partial x_i} = \frac{\partial v_i}{\partial x_i} \ . \tag{11.15}$$

The *divergence* of a tensor field $\mathbf{T}(\mathbf{x})$ is the vector field div $\mathbf{T}(\mathbf{x})$ for which

$$\mathbf{c} \cdot \operatorname{div} \mathbf{T}(\mathbf{x}) = \operatorname{div}\{\mathbf{T}^T(\mathbf{x})\mathbf{c}\} \tag{11.16}$$

for every constant vector $\mathbf{c}$. When

$$\mathbf{T} = t_{ij}\mathbf{e}_i \otimes \mathbf{e}_j \ , \tag{11.17}$$

it follows that

$$\mathrm{div}\ \mathbf{T}(\mathbf{x}) = \frac{\partial t_{ij}}{\partial x_j}\,\mathbf{e}_i\ . \tag{11.18}$$

If P is a bounded region whose boundary ∂P is sufficiently well-behaved, then by application of the divergence theorem (Green's theorem) to a vector **v** and a tensor **T** we have

$$\int_P \mathrm{div}\,\mathbf{v}(\mathbf{x})dV(\mathbf{x}) = \int_{\partial P} \mathbf{v}(\mathbf{x})\cdot\mathbf{n}(\mathbf{x})dA(\mathbf{x})\ , \tag{11.19}$$

$$\int_P \mathrm{div}\,\mathbf{T}(\mathbf{x})dV(\mathbf{x}) = \int_{\partial P} \mathbf{T}(\mathbf{x})\mathbf{n}(\mathbf{x})dA(\mathbf{x})\ , \tag{11.20}$$

where **n** is the outward unit normal to ∂P, the integrations are over the volume P and the boundary surface ∂P and dV and dA represent elements of volume and area, respectively. In component form, (11.19) and (11.20) read:

$$\int_P v_{i,i}\,dv = \int_{\partial P} v_i\,n_i\,da\ , \tag{11.21}$$

$$\int_P t_{ij,j}\,dv = \int_{\partial P} t_{ij}\,n_j\,da\ . \tag{11.22}$$

12. Vector product. Axial vectors. The curl operator.

Consider a Euclidean vector space E^3 of three dimensions. If **u**,**v**,**w** are any vectors in E^3, the *vector product* of **u** and **v** is a vector in E^3 denoted by $\mathbf{u} \times \mathbf{v}$ and defined by the properties:

(V_1) $\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v}) = \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \mathbf{v}\cdot(\mathbf{w}\times\mathbf{u})$ and is a scalar denoted by

[**uvw**] and called the scalar triple product ,

(V_2) $\mathbf{u}\times\mathbf{v} = -\mathbf{v}\times\mathbf{u}$,

(V_3) $|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\sin\theta$,

where θ is defined in (4.3). From (V_1) and (V_2), it follows that

$$\mathbf{u}\times\mathbf{u} = \mathbf{o} \quad , \quad \mathbf{u}\cdot(\mathbf{u}\times\mathbf{v}) = 0 \quad , \quad \mathbf{v}\cdot(\mathbf{u}\times\mathbf{v}) = 0 \qquad (12.1)$$

so that $\mathbf{u}\times\mathbf{v}$ is orthogonal to both **u** and **v**. It can be shown that three nonzero vectors **u**,**v**,**w** are linearly independent if and only if their scalar triple product $[\mathbf{u},\mathbf{v},\mathbf{w}] = \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}$.

The following properties may be deduced from the definitions:

$$(\alpha\mathbf{u})\times\mathbf{v} = \alpha(\mathbf{u}\times\mathbf{v}) = \mathbf{u}\times(\alpha\mathbf{v}) \ ,$$

$$\mathbf{u}\times(\mathbf{v}+\mathbf{w}) = \mathbf{u}\times\mathbf{v}+\mathbf{u}\times\mathbf{w} \ ,$$

$$(\mathbf{u}+\mathbf{v})\times\mathbf{w} = \mathbf{u}\times\mathbf{w}+\mathbf{v}\times\mathbf{w} \ , \qquad (12.2)$$

$$\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = \mathbf{v}(\mathbf{u}\cdot\mathbf{w}) - \mathbf{w}(\mathbf{u}\cdot\mathbf{v}) = (\mathbf{v}\otimes\mathbf{w} - \mathbf{w}\otimes\mathbf{v})\mathbf{u} \ ,$$

$$[\mathbf{u}+\mathbf{x},\mathbf{v},\mathbf{w}] = [\mathbf{uvw}] + [\mathbf{xvw}] \ .$$

The above definition of vector product does not restrict $\mathbf{e}_i$ to be either right-handed or left-handed basis.

Let $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$ be an orthonormal system. Then, in view of (V_3) and (12.1),

$$\mathbf{e}_1 \times \mathbf{e}_2 = \pm\, \mathbf{e}_3 \; . \tag{12.3}$$

A right-handed orthonormal sytem is one for which

$$\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3 \tag{12.4}$$

and a left-handed orthonormal system corresponds to the choice of the - sign in (12.3). For a right-handed orthonormal system it follows that

$$[\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3] = 1 \quad , \quad \mathbf{e}_i \times \mathbf{e}_j = e_{ijk}\mathbf{e}_k \quad , \quad e_{ijk} = [\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k] \; , \tag{12.5}$$

where the components e_{ijk} are known as the permutation symbol or the components of the alternating tensor. Also, for any two vectors

$$\begin{aligned} &\mathbf{u} = u_i\mathbf{e}_i \quad , \quad \mathbf{v} = v_j\mathbf{e}_j \; , \\ &\mathbf{u} \times \mathbf{v} = e_{ijk}u_iv_j\mathbf{e}_k \; . \end{aligned} \tag{12.6}$$

Let $\mathbf{T}^A$ be a skew tensor in E^3 so that

$$\mathbf{T}^A = t_{ik}\mathbf{e}_i \otimes \mathbf{e}_k = \frac{1}{2}\, t_{ik}[\mathbf{e}_i \otimes \mathbf{e}_k - \mathbf{e}_k \otimes \mathbf{e}_i] \; . \tag{12.7}$$

Hence, if $\mathbf{u}$ is an arbitrary vector in E^3

$$\mathbf{T}^A\mathbf{u} = \frac{1}{2}\, t_{ik}[\mathbf{e}_i(\mathbf{u}\cdot\mathbf{e}_k) - \mathbf{e}_k(\mathbf{u}\cdot\mathbf{e}_i)] = \frac{1}{2}\, \mathbf{u} \times [t_{ik}\mathbf{e}_i \times \mathbf{e}_k] \; , \tag{12.8}$$

if we use $(12.2)_4$. Hence

$$\mathbf{T}^A\mathbf{u} = \mathbf{t}^A \times \mathbf{u} \; , \tag{12.9}$$

where the vector

$$\mathbf{t}^A = \frac{1}{2} t_{ki}\mathbf{e}_i \times \mathbf{e}_k \ . \tag{12.10}$$

Let $\mathbf{e}_i$ be a right-handed orthonormal basis. Then, referred to $\mathbf{e}_i$, $\mathbf{T}^A$ reads

$$\mathbf{T}^A = t_{ij}\mathbf{e}_i \otimes \mathbf{e}_j \tag{12.11}$$

and the axial vector $\mathbf{t}^A$ with components t_k^A assumes the form

$$\mathbf{t}^A = \frac{1}{2} t_{ji}e_{ijk}\mathbf{e}_k = t_k^A \, \mathbf{e}_k \ . \tag{12.12}$$

Alternatively

$$t_{ji} = e_{ijk}t_k^A \ . \tag{12.13}$$

If

$$\mathbf{T}^A = \frac{\partial \mathbf{v}}{\partial \mathbf{x}} - (\frac{\partial \mathbf{v}}{\partial \mathbf{x}})^T \ , \tag{12.14}$$

the corresponding axial vector $\mathbf{t}^A$ is called the *curl* of $\mathbf{v}$ and written as

$$\mathbf{t}^A = \text{curl}\, \mathbf{v} \ . \tag{12.15}$$

In a right-handed orthonormal system

$$\text{curl}\, \mathbf{v} = e_{ijk} \frac{\partial v_j}{\partial x_i} \mathbf{e}_k \ . \tag{12.16}$$

References for Appendix L

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