Introduction

Structural estimation of parametric auction models is of interest since it recovers the parameters of the distribution of the private signals of bidders. Using these parameters, other auction forms can be simulated; comapring the expected revenue from these alternative auction forms would aid the seller in designing an optimal auction. Besides mechanism design, the impact of policy changes can be studied by estimating a structural model. Parametric structural auction models are, however, difficult to estimate because the density function of bids is a complicated function of the **unobserved** private values of the participants in the auction. Since the equilibrium bid of a bidder is a function of her unobserved private value, this private value is obtained by inverting the equilibrium bid function. In most cases this is a difficult exercise. Further, with the support of the data depending on the parameters to be estimated, standard regularity conditions for maximum likelihood estimation are violated (Poirier, p. 259, 1995),

This paper proposes a likelihood-based approach for estimating parametric structural auction models. Since the private values of the bidders and the parameters are all elements of the model that are not observed, the method I propose obtains draws from the posterior distribution of the private values and the parameters of the distribution of private values using data augmentation. I draw samples from the non-standard posterior distribution using the the Metropolis-Hastings algorithm (Chib and Greenberg, 1995). Inference about the parameters is based on this posterior distribution. This method is unaffected by the dependence of the support on the parameters. Further, data augemntation avoids direct evaluation of the likelihood function since the proposed method draws from the joint distribution of private values and the parameters instead of inverting the equilibrium bid function to obtain the private values of the bidders.

By treating the number of potential bidders as a parameter of the distribution of private values of the bidders, I am able to obtain the posterior distribution for the number of potential bidders. Inference about the number of potential bidders may be of interest since the number of potential bidders affects the average revenue of the seller in a descending-price auction under the symmetric independent private-values model (McAfee and McMillan, 1987); an increase in the number of potential bidders leads to bidders bidding their true valuation so that all the gains from trade go to the seller. This adresses a concern voiced by many researchers about the inability to obtain a measure of potential competition (Paarsch, 1997, pp. 334, 343).

I illustrate the method by estimating a descending-price auction model with publically announced reserve prices within the symmetric independent private-values paradigm previously studied by Laffont, Ossard and Vuong (1995). This model is described in Section 2. The proposed method is discussed in Section 3. In Section 5 I generate an artificial data set from known values of the parameters of the distribution of private values, for a symmetric descending-price auction within the independent private-values paradigm. This Section demonstrates that the proposed method can recover the "truth" about the location and the scale parameters of the distribution of private values. Section 4 applies the proposed method to the data set on descending-price auction of eggplants in the Marmande district of France used by Laffont, Ossard and Vuong. In Sections 2-5, I have assumed that the reserve price of the seller is fixed. Section 6 extends the proposed method to allow for a parametric model for the seller's reserve price. Section 7 concludes.

Section 2: The Descending-Price Auction Framework

In the discussion that follows, auctions are indexed by $j = 1, ..., T$, and individual bids/bidders are indexed by i with $i = 1, ..., N$. As an example, b_j^i refers to the i-th bid in the j-th auction. All random variables are indicated by capital letters, and the realization of a random variable by a lower case letter.

Following the framework developed by Laffont, Ossard and Vuong (1995, pp. 954-956), a single and indivisible object is sold by one seller to N potential risk-neutral bidders in all auctions j. The single and indivisible object that is being auctioned is a case of greenhouse eggplants weighing 15-350 kilos. The sellers in the market are farmers. The buyers are resale-trade firms. The buyers or the resale-trade firms are agents of retail sellers of eggplants; these retail sellers are the N potential risk-neutral bidders. Laffont, Ossard and Vuong assume the buyers or the resale-trade firms are intermediating between the potential bidders and the sellers of the eggplants, with each resale-trade firm representing atleast one potenatial bidder. The resale-trade firms come to the auction with the private values that the potential bidders communicate to them. They bid according to these private values and undertake no startegic bidding. Each potential bidder i has a private value for the auctioned object, v_j^i, which is known only to her. This private value is known to her *ex ante* (before she submits the bid). She does not know the private value of other bidders. However, each bidder knows that all private values including her own are drawn independently from the same probability distribution (assumption of symmetry). The probability density function of the private value of bidder i in auction j, V_j^i, is given by $g(v_j^i|\theta)$ and cumulative distribution by $G(v_j^i|\theta)$, where $g(\cdot)$ and $G(\cdot)$ are defined on the support $[0, +\infty)$; θ is a $k \times 1$ parameter vector that characterizes the distribution of private values. Each potential bidder also knows N, the number of potential bidders in an auction.

When working with data on more than one auction, it is likely that there is heterogeneity across auctions either because the environment in which the auctions are held is different or because the auctioned object is different. This heterogeneity across auctions is taken into account by assuming that the probability density function and the distribution function of the private values of the bidders differ across auctions. That is, V_j^i, the private value of bidder i in auction j comes from a distribution that is characterized by a $k \times 1$ parameter vector θ, a $k_z \times 1$ vector of covariates, $\mathbf{Z_j}$, observed by both the researcher and the potential bidders. The probability density function and distribution function of V_j^i, taking into account heterogeneity across auctions, are given by $g(v_j^i|\theta, \mathbf{Z_j} = \mathbf{z_j})$ and $G(v_j^i|\theta, \mathbf{Z_j} = \mathbf{z_j})$, respectively. Henceforth I will indicate $\mathbf{Z_j} = \mathbf{z_j}$ by $\mathbf{z_j}$ in $g(\cdot)$ and $G(\cdot)$.

An auction begins with a case of eggplants being displayed for inspection by the bidders. The seller of this case of eggplants announces a (minimum) reservation price, p_j^o, in auction j for a kilogram of eggplants. The reservation price is assumed to be fixed in an auction. [1] The auctioner begins the auction by quoting a very high price for a kilogram of eggplants. The auctioner keeps lowering this price till a bidder makes a bid. The auctioner will not lower the price below the reserve price announced by the seller.

[1] Note that only the those potential bidders whose private values are above the reservation price, p_j^o, will participate in the auction. In a descending-price auction the number of participants are not observed.

Once a bidder makes a bid the auction stops. This bidder is the winner of the auction with the winning bid being at least as high as the reservation price of the seller, p_j^o.

The winner of an auction j is the bidder with the highest private value $v_j(1:N)$ for the auctioned object; $V_j(1:N)$ is the highest order statistic for a sample of size N from the distribution of V_j^i. The winning bid in auction j, w_j, is (Laffont, Ossard and Vuong, 1995, p. 955):

$$w_j = e\Big(v_j(1:N), N, p_j^o, G(v_j(1:N)|\theta, \mathbf{z_j})\Big) \tag{1}$$

$$= v_j(1:N) - \frac{\int_{p_j^o}^{v_j(1:N)} \Big[G(\xi|\theta, \mathbf{z_j})\Big]^{N-1} d\xi}{\Big[G(v_j(1:N)|\theta, \mathbf{z_j})\Big]^{N-1}} \quad if \quad v_j(1:N) \geq p_j^o. \tag{2}$$

Once the density of the private values V_j^i, $g(v_j^i|\theta, \mathbf{z_j})$, is specified, the density of the winning bid, $h(\cdot)$, can be obtained from equations (1) and (2) (Paarsch, 1992, pp. 196):

$$h(w_j|N, \theta, \mathbf{z_j}, p_j^o) = \frac{\tilde{g}(e^{-1}(w_j)|N, \theta, \mathbf{z_j})}{e'(e_p^{-1}(w_j))}$$

$$\equiv \frac{\tilde{g}(v_j(1:N))|N, \theta, \mathbf{z_j})}{e'(v_j(1:N))}. \tag{3}$$

In equation (3), $e^{-1}(w_j) \equiv e^{-1}\Big(w_j; N, p_j^o, G(v_j(1:N)|\theta, \mathbf{z_j})\Big)$, is the inverse of the winning bid defined in equation (1) with respect to its first argument, $v_j(1:N)$. Or equivalently, $e^{-1}(w_j) \equiv v_j(1:N)\Big(w_j, p_j^o, \mathbf{z_j}, N, \theta\Big)$, is the solution to equation (2) in terms of $w_j, N, \theta, \mathbf{z_j}, p_j^o$,

$$e^{-1}(w_j) \equiv v_j(1:N)\Big(w_j, p_j^o, \mathbf{z_j}, N, \theta\Big). \tag{4}$$

$e'(\cdot)$ is the Jacobian of the transformation of $v_j(1:N)$ to w_j in equation (1). $\tilde{g}(\cdot)$ is the density function of the highest order statistic for a sample of size N from the distribution of private values, V_j^i, with density function given by $g(\cdot)$. It is given by (Donald and Paarsch, 1996, pp. 525):

$$\tilde{g}(v_j(1:N)|N, \theta, \mathbf{z_j}) = Ng(v_j(1:N)|\theta, \mathbf{z_j})\Big[G(v_j(1:N)|\theta, \mathbf{z_j})\Big]^{N-1}. \tag{5}$$

The density function of the winning bid follows from Donald and Paarsch (1996),

$$h(w_j|N, \theta, \mathbf{z_j}, p_j^o) = \frac{N}{N-1}\frac{[G(e^{-1}(w_j)|\theta, \mathbf{z_j})]^N}{[e^{-1}(w_j) - w_j]}, \tag{6}$$

where $h(\cdot)$ is defined on the support (Laffont, Ossard and Vuong, 1995, p. 958):

$$p_j^o \leq w_j \leq \left[\int_0^{\infty} (N-1)\xi g(\xi|\theta, \mathbf{z_j})[G(\xi|\theta, \mathbf{z_j})]^{N-2} d\xi\right]. \tag{7}$$

The interpretation of (6) and (7) is that if there is at least one participant in auction j then the density of the winning bid, W_j, is given by $\frac{N}{N-1}\frac{[G(\cdot)]^N}{[e^{-1}(w_j)-w_j]}$. There will be at least one participant in auction j if the winning bid is at least as high as the reservation price announced by the seller in auction j, p_j^o. The upper bound on the winning bid is the unconditional expected value of the second highest private value; this is given by $\int_0^{\infty}(N-1)\xi g(\xi|\cdot)[G(\xi|\cdot)]^{N-2} d\xi$ in equation (7).

Equation (7) indicates that the support of the density function and the distribution function of the winning bid is a complicated function of the parameters of interest, θ, N, which varies across auctions. This violates the regularity conditions under which ML estimation is done, making it a difficult exercise. Further the likelihood function in equation (6) requires inverting the winning bid, given by equation (1), with respect to it's first argument, $v_j(1:N)$, to obtain $e^{-1}(w_j)$. In other words a solution for $v_j(1:N)$ in terms of $w_j, N, p_j^o, \theta, \mathbf{z_j}$ is required from equation (1). An analytical expression is available for $e^{-1}(w_j)$ only if the distribution of private values, $G(\cdot)$, is assumed to have some simple specification (Paarsch, 1992). In all other cases the solution has to be obtained numerically, whereby $e^{-1}(w_j)$ has to be obtained for each value of θ for each auction $j = 1, ..., T$.

The next section discusses the method proposed in this paper and how it addresses the problems mentioned above.

Section 3: Proposed Method

The basic relation underlying the proposed method is Bayes formula for revising beliefs about unobservables after observing the data,

$$P(unobservables|data) = \frac{P(unobservables)P(data|unobservables)}{P(data)}. \tag{8}$$

$P(data|unobservables)$ when viewed as a function of the unobservables given the data is called the likelihood function. $P(data)$ is the marginal likelihood of the data.

In the discussion below, $\mathbf{w} = (w_1, ..., w_T)'$ is the column vector of winning bids in the T auctions; $\mathbf{p^o} = (p_1^o, ..., p_T^o)'$ is the T dimensional column vector of reservation prices announced by the seller in the T auctions; $\mathbf{z} = (\mathbf{z_1}, ..., \mathbf{z_T})'$ is a $T \times k_z$ dimensional matrix of k_z covariates observed by both the researcher and the potential bidder in the T auctions. $\mathbf{v} = \Big(v_1(1 : N), ..., v_T(1 : N)\Big)'$ is a T dimensional column vector of latent data or the highest private value in each of the T auctions.

Inference about the unobservable parameters, N, θ, is performed using Bayes formula in (8). Since relation (8) is unaffected by the support of the data depending on the parameters of the model, inference about the parameters is unaffacted.

I need a sample from the posterior distribution $N, \theta | \mathbf{w}, \mathbf{p^o}, \mathbf{z}$ to conduct inference on N, θ. The posterior kernel is

$$f(N, \theta | \mathbf{w}, \mathbf{p^o}, \mathbf{z}) \propto f(N, \theta) f(\mathbf{p^o}, \mathbf{z} | N, \theta) f(\mathbf{w} | N, \theta, \mathbf{p^o}, \mathbf{z}). \tag{9}$$

$f(N, \theta)$ is the prior density function of the parameters. Assuming p_j^o to be fixed in all j auctions, I show in Appendix A, that the distinction between the observed covariates, $\mathbf{Z}$, being fixed or stochastic is irrelevant in Bayesian analysis. Hence,

$$f(\mathbf{p^o}, \mathbf{z} | N, \theta) = 1.$$

$f(\mathbf{w} | N, \theta, \mathbf{p^o}, \mathbf{z})$ is the likelihood function, $L(N, \theta; \mathbf{w}, \mathbf{p^o}, \mathbf{z})$, which is obtained from equation (6);

$$\begin{aligned} L(N, \theta; \mathbf{w}, \mathbf{p^o}, \mathbf{z}) &= \prod_{j=1}^{T} h\Big(w_j | N, \theta, p_j^o, \mathbf{z_j},\Big) \\ &= \prod_{j=1}^{T} \frac{N}{N-1} \frac{[G(e^{-1}(w_j) | \theta, \mathbf{z_j})]^N}{[e^{-1}(w_j) - w_j]}, \end{aligned} \tag{10}$$

where $e^{-1}(w_j) \equiv e^{-1}\Big(w_j; N, p_j^o, G(v_j(1:N) | \theta, \mathbf{z_j})\Big)$. To obtain samples from the posterior distribution $N, \theta | \mathbf{w}, \mathbf{p^o}, \mathbf{z}$, I need to invert $e(\cdot)$ to obtain $v_j(1:N)\Big(w_j, p_j^o, \mathbf{z_j}, N, \theta\Big)$ in (10). In short, it brings me back to the problem mentioned at the end of the last section when a likelihood-based method is used.

In the method I propose, the data augmentation idea of Tanner and Wong (1987) is used, but with a difference. Treating the unobserved highest private values in the T

auctions, $\mathbf{V} = \Big(V_1(1:N), ..., V_T(1:N)\Big)'$, as latent data, I move to a higher dimensional problem and sample from the distribution $N, \theta, \mathbf{V}|\mathbf{w}, \mathbf{p^o}, \mathbf{z}$ instead of the posterior distribution, $N, \theta|\mathbf{w}, \mathbf{p^o}, \mathbf{z}$. I then discard $\mathbf{v}$ to obtain sample from $N, \theta|\mathbf{w}, \mathbf{p^o}, \mathbf{z}$. The move to sampling from a higher dimension helps since it is **easy** to sample the higher dimensional distribution. It is **easy** for me to sample from the higher dimensional distribution, $N, \theta, \mathbf{V}|\mathbf{w}, \mathbf{p^o}, \mathbf{z}$, as I do not need to invert $e(\cdot)$ to obtain $v_j(1:N)$. Draws are obtained from $N, \theta, \mathbf{V}|\mathbf{w}, \mathbf{p^o}, \mathbf{z}$ using the Metropolis-Hastings algorithm. The data augmentation idea of Tanner and Wong also involves a move to sampling from a higher dimensional distribution by augmenting the observed data with latent data. In their case it is **easy** for them to sample from the higher dimensional distribution since they can set up a Gibbs sampler in the higher dimensional problem. [2]

I now explain why I do not need to invert $e(\cdot)$ to obtain a solution for $v_j(1:N)$ in terms of $w_j, p_j^o, \mathbf{z_j}, N, \theta$ when I am drawing from the higher dimensional distribution $N, \theta, \mathbf{v}|\mathbf{w}, \mathbf{p^o}, \mathbf{z}$. The kernel of this distribution is derived in Appendix A. It is given by

$$\begin{aligned} f(N, \theta, \mathbf{v}|\mathbf{w}, \mathbf{p^o}, \mathbf{z}) \propto & f(N, \theta) f(\mathbf{p^o}, \mathbf{z}|N, \theta) f(\mathbf{v}|N, \theta, \mathbf{p^o}, \mathbf{z}) \\ & f(\mathbf{w}|N, \theta, \mathbf{p^o}, \mathbf{z}). \end{aligned} \tag{11}$$

As explained before, $f(\mathbf{p^o}, \mathbf{z}|N, \theta) = 1$. $f(\mathbf{v}|N, \theta, \mathbf{p^o}, \mathbf{z})$ in (11) is the joint density function of the highest private value in each of the j auctions. Since the auctions are assumed to be independent,

$$\begin{aligned} f(\mathbf{v}|N, \theta, \mathbf{p^o}, \mathbf{z}) &= \prod_{j=1}^{T} f(v_j(1:N)|N, \theta, \mathbf{z_j}) \\ &= \prod_{j=1}^{T} \tilde{g}(v_j(1:N)|N, \theta, \mathbf{z_j}) \\ &= \prod_{j=1}^{T} N g(v_j(1:N)|\theta, \mathbf{z_j}) \Big[G(v_j(1:N)|\theta, \mathbf{z_j})\Big]^{N-1}, \end{aligned} \tag{12}$$

[2] The Gibbs sampler involves successive sampling from the distribution of the parameters conditional on the observed data and the latent data, and the distribution of the latent data conditional on the observed data and the parameters. This successive sampling obtains draws from the distribution of parameters and latent data conditional on the observed data. The latent data is then discarded to obtain draws from the posterior distribution of the parameters.

$\tilde{g}(\cdot)$ being the density function of the highest order statistic, given in equation (5). Note that the T highest private values, $v_j(1:N)$, $j = 1, ..., T$, drawn from this distribution will be functions of $N, \theta, \mathbf{p^o}, \mathbf{z}$, though **not necessarily** the solutions to equation (1), $e^{-1}(w_j) \equiv e^{-1}\Big(w_j; N, p_j^o, G(v_j(1:N)|\theta, \mathbf{z_j})\Big)$.

Since $v_j(1:N)$ is a function of $N, \theta, \mathbf{p^o}, \mathbf{z}$, I do not need to condition on $\mathbf{V}$ in the last term in equation (11), $f(\mathbf{w}|N, \theta, \mathbf{p^o}, \mathbf{z})$. This is the augmented likelihood function, $L(N, \theta, \mathbf{v}; \mathbf{w}, \mathbf{p^o}, \mathbf{z})$. It is obtained by replacing $e^{-1}(w_j)$, the solution to equation (1), by $v_j(1:N)$, a draw from the distribution of the highest private value given by equation (12), in the density function for the winning bid, $h(w_j|N, \theta, \mathbf{z_j}, p_j^o)$, given by equation (6). Thus,

$$L(N, \theta, \mathbf{v}; \mathbf{w}, \mathbf{p^o}, \mathbf{z}) = \prod_{j=1}^{T} \frac{N}{N-1} \frac{[G(v_j(1:N)|\theta, \mathbf{z_j})]^N}{[v_j(1:N) - w_j]} \mathbf{1}_{[v_j(1:N) = e^{-1}(w_j)]} + 0\mathbf{1}_{[v_j(1:N) \neq e^{-1}(w_j)]}. \tag{13}$$

$\mathbf{1}_{[\cdot]}$ is an indicator function of the event $[\cdot]$; it indicates that the likelihood function is zero if $v_j(1:N)$ is not the solution to equation (1). Since the posterior kernel given by equation (11) is non-standard, the Metropolis-Hastings algorithm is used to obtain draws from this posterior distribution. Details of the algorithm are given in Appendix B.

The reason for the proposed method working is as follows. Indicate a draw from the posterior distribution by the superscript l, the Metropolis-Hastings algorithm obtains draws, $N^{(l)}, \theta^{(l)}, v_j^{(l)}(1:N)|\mathbf{w}, \mathbf{p^o}, \mathbf{z}$. Many of these draws **do not** solve equation (1); the Metropolis-Hastings assigns a probability of zero to these draws. A large number of the draws solve the equilibrium winning bid defined by equation (1),

$$\begin{aligned} w_j &= e\Big(v_j(1:N), N, p_j^o, G(v_j(1:N)|\theta, \mathbf{z_j})\Big), \\ &= v_j(1:N) - \frac{\int_{p_j^o}^{v_j(1:N)} \Big[G(\xi|\theta, \mathbf{z_j})\Big]^{N-1} d\xi}{\Big[G(v_j(1:N)|\theta, \mathbf{z_j})\Big]^{N-1}}; \end{aligned}$$

the Metropolis-Hastings algorithm assigns a non-zero probability to these draws. A large number of draws from the Metropolis-Hastings algorithm solve the equilibrium winning bid equation (1) that I have reproduced above because the vehicle through which learning about $e^{-1}(w_j) \equiv v_j(1:N)\Big(\cdot\Big)$ occurs is the augmented likelihood function,

$L(N, \theta, \mathbf{v}; \mathbf{w}, \mathbf{p^o}, \mathbf{z})$; and this augmented likelihood function is based on the equilibrium winning bid given by equation (1) and reproduced above.

Section 4: Application

This section applies the method proposed in section 2 to the descending auction of greenhouse eggplants in the Marmande district of France studied by Laffont, Ossard and Vuong (1995). In Section 2, I have described the institutional framework of this auction. The assumptions of a symmetric independent private-values model are justified by Laffont, Ossard and Vuong (1995, pp. 967-969). I will therefore not repeat the justification. The data set comprises of 81 auctions. For each auction data on the winning bid and the covariates observed by the bidder and the researcher that result in heterogeneity across auctions is available. 11 resale-trade firms or buyers are observed to attend all the 81 auctions regularly.

Section 4.1: Likelihood and Prior Specification

To specify the likelihood function given by equation (13), the distribution of private values, v_j^i, has to be specified. Following Laffont, Ossard and Vuong (1995, pp. 968) I assume that the private value of the potential bidder i in auction j follows a log-normal distribution, for all N potential bidders in auction j, and for all T auctions. The private values will not be identically distributed across auctions due to the heterogeneity across auctions represented by the variables $\mathbf{Z_j}$. Specifically,

$$g(v_j^i|\theta, \mathbf{z_j}) = g_{LN}(v_j^i|\mu_j, \sigma) = \frac{1}{v_j^i}\phi(ln v_j^i|\mu_j, \sigma^2), \tag{14}$$

with μ_j given by (Laffont and Vuong, 1995, pp. 968):

$$\begin{aligned}\mu_j = \mathbf{z_j'}\lambda = \lambda_1 + \lambda_2 seller_j + \lambda_3 size1_j + \lambda_4 size2_j \\ + \lambda_5 period_j + \lambda_6 date_j + \lambda_7 supply_j \qquad \forall j = 1, ..., n;\end{aligned} \tag{15}$$

$\phi(\cdot)$ and $\Phi(\cdot)$ are the density function and the distribution function of a normal distribution. $seller_j$ is a dummy variable that takes value 1 or 0; there are two sellers, one of who has reputation for better quality, and is hence preferred by the bidders. $seller_j$ takes value 1 if the seller in auction j is the preferred seller. $size1_j$ and $size2_j$ are dummy variables that take account of heterogeneity in the weight of the eggplants across auctions. There are three official size categories, less-than-three hundred grams, three-to-four hundred grams, and over-four-hundred grams. $size1_j$ takes value 1 if the eggplants in auction j fall in the middle category and 0 otherwise; $size2_j$ takes the value 1 if the eggplants fall in the largest category in auction j, and 0 otherwise. $period_j$ is a dummy variable that takes value 0 if the jth auction is conducted during the months of August or first week of September, and value 1 otherwise. The dummy variable $period_j$ takes acccount of the drop in the prices of eggplants during the period mentioned due to the availability of other types of summer eggplants. The $date_j$ variable is a trend variable that takes account of the increase in prices of eggplants over time. $supply_j$ is the supply of eggplants in Marmande on the day auction j takes place.

In equation (15) the $(k_z \times 1)$ vector of observed covariates, $\mathbf{z_j}$, is a 7×1 dimensional vector,

$$\mathbf{z_j} = [1, seller_j, size1_j, size2_j, period_j, date_j, supply_j]'. \tag{16}$$

The $k \times 1$ parameter vector θ, where $k = 8$, is,

$$\theta = [\lambda', \sigma^2]' = [\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \sigma^2]', \tag{17}$$

where $\lambda = [\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7]'$.

I now specify each term in the kernel of the posterior distribution, $P(N, \theta, \mathbf{v}|\mathbf{w}, \mathbf{p^o}, \mathbf{z})$, given by equation (11). Prior beliefs about N and θ are assumed to be independent.

$P(N)$ is the prior density for the number of potential bidders. There are eleven buyers who attend all 81 auctions being studied. Laffont, Ossard and Vuong (1995, pp. 970) conjecture that of the eleven buyers, one buyer was an agent for a number of potential bidders or retail sellers. [3] As a result, they consider values of N between 11 and 23, and optimize their criterion function with respect to these values to obtain an estimate of

[3] Note that the number of participants and potential bidders will differ in these auctions because of the existence of a reservation price. Since the auctions being studied are Dutch auctions, the number of participants will not be observed. Thus, the eleven buyers who participate in the auction are not the participants; they are the agents of the retail traders who are the potential bidders.

N. This implies that they have prior beliefs about N taking some integer value between 11 and 23, but are uninformative within that range. Hence, I specify a discrete-uniform distribution for N over the interval $[11, 23]$

$$P(N) = P_u(N|11, 23) = \frac{1}{13}. \tag{18}$$

For the parameter vector $(\lambda', \sigma^{-2})'$, a normal-gamma distribution is specified, $\lambda, \sigma^{-2} \sim NG(\underline{\mathbf{a}}, \underline{\mathbf{B}}, \underline{\mathrm{c}}, \underline{\mathrm{d}})$ (Poirier, 1995, pp. 128), where $\underline{\mathbf{a}}$ is a 7×1 dimensional vector, $\underline{\mathbf{B}}$ is a 7×7 matrix, and $\underline{\mathrm{c}}, \underline{\mathrm{d}}$ are scalars. The density function of $\theta = (\lambda', \sigma^2)'$ is:

$$P(\theta) = \phi_7(\lambda|\underline{\mathbf{a}}, \sigma^{-2}\underline{\mathbf{B}}) IG(\sigma^2|\underline{\mathrm{c}}, \underline{\mathrm{d}}), \tag{19}$$

where $\phi(\cdot)$ is the multivariate normal density function and $IG(\cdot)$ is the inverted gamma density function. The reason for specifying a normal-gamma density function for $(\lambda', \sigma^{-2})'$ is that the log of private values of a bidder i in auction j, lnV_j^i, has been assumed to follow a normal distribution with parameters $\mathbf{z_j'}\lambda, \sigma^2$ (equation (14)-(17)). By assuming that prior beliefs about $(\lambda', \sigma^{-2})'$ follow a normal-gamma distribution, I am assuming that the prior is a "fictitious" sample (Poirier, 1995, pp. 298) from the same population that generated the **unobserved** data on the private values, lnV_j^i. $\underline{\mathbf{a}}, \underline{\mathbf{B}}, \underline{\mathrm{c}}, \underline{\mathrm{d}}$ are hyper-parameters that have to be specified. I assume that $\underline{\mathbf{B}}$ is a diagonal matrix with the diagonal elements being $\underline{\mathrm{b}}$. This implies that prior beliefs about the parameters $\lambda_1, ...\lambda_7$ are independent; and that apriori, I consider $\lambda_1, ..., \lambda_7$ to be equally "significant". With these assumptions the prior density function in equation (19) can be written as:

$$P(\theta) = IG(\sigma^2|\underline{\mathrm{c}}, \underline{\mathrm{d}}) \prod_{m=1}^{7} \phi(\lambda_m|\underline{\mathrm{a}}_m, \sigma^{-2}\underline{\mathrm{b}}), \tag{20}$$

where $\underline{\mathbf{a}} = (\underline{\mathrm{a}}_1, \underline{\mathrm{a}}_2, \underline{\mathrm{a}}_3, \underline{\mathrm{a}}_4, \underline{\mathrm{a}}_5, \underline{\mathrm{a}}_6, \underline{\mathrm{a}}_7)'$.

The next term in the posterior kernel given by equation (11) is $P(\mathbf{v}|\mathbf{z}, \theta, N)$. As explained in equation (12), this is the joint density function of the highest private value, $V_j(1 : N)$, in each auction j. The density function of V_j^i, $g(\cdot)$, is specified in equation (14); then the density function of the highest private value is

$$P(\mathbf{v}|\mathbf{z}, \theta, N) = \prod_{j=1}^{81} \frac{N}{v_j^i} \phi(logv_j(1:N)|\mathbf{z_j'}\lambda, \sigma^2) \Big[\Phi(lnv_j(1:N)|\mathbf{z_j'}\lambda, \sigma^2)\Big]^{N-1}. \tag{21}$$

The last term in the posterior kernel given by equation (11) is the augmented likelihood function given by equation (13),

$$L(N,\theta,\mathbf{v};\mathbf{w},\mathbf{p^o},\mathbf{z}) = \prod_{j=1}^{81} \frac{N}{N-1} \frac{\left[\Phi(lnv_j(1:N)|\mathbf{z_j'}\lambda,\sigma^2)\right]^N}{[v_j(1:N)-w_j]}, \quad (22)$$

where $p_j^o \leq w_j \leq \int_0^\infty (N-1)\phi(ln\xi|\mathbf{z_j'}\lambda,\sigma^2)[\Phi(ln\xi|\mathbf{z_j'}\lambda,\sigma^2)]^{N-2}d\xi$.

The Metropolis-Hastings algorithm is used to obtain a sample from the posterior distribution $N,\theta,\mathbf{v}|\mathbf{w},\mathbf{p^o},\mathbf{z}$. The kernel of this density function is given in appendix C.

Section 4.2: Results

As mentioned in section 4.1, Laffont, Ossard and Vuong (1995, pp. 968) assume that the logarithm of the private value of bidder i in auction j follows a normal distribution with mean μ_j and variance σ^2, where $\mu_j = \mathbf{z_j'}\lambda$, is given by equation (15). Observing that the ratio of the standard deviation of retail prices to the mean is 0.05, Laffont, Ossard and Vuong (1995) fix $\sigma^2 = 0.0025$. Thereafter, estimates for λ and N are obtained. [4] These are reported in column 1 of Table 1.

Assuming a normal-gamma prior for $(\lambda',\sigma^{-2})'$ given by equation (21), I obtained draws from $N,\theta,\mathbf{V}|\mathbf{w},\mathbf{p^o},\mathbf{z}$ by specifying two sets of values for the hyperparameters. I specified the mean of λ to be the estimates obtained by Laffont, Ossard and Vuong, $E(\lambda)=\hat{\lambda}$; these are reported in column 1 of Table 1. The mean of σ^2, $E(\sigma^2)=0.0025$, the value at which σ^2 was fixed by Laffont, Ossard and Vuong. The variance of λ and σ^2 were specified to take two different values; $V(\lambda_m)=16$, $\forall m=1...,7$, $V(\sigma^2)=0.25$, and $V(\lambda_m)=0.067$, $\forall m=1...,7$, $V(\sigma^2)=0.0125$. The former values for the variance of λ and σ^2 make the prior for the parameters relatively noninformative and amounts to "letting the data speak for itself". Figure 1 plots the posterior histograms for $\lambda_m, m=1,...,7$, σ^2, and N for these relatively noninformative values of the hyperparameters. Column 2 of Table 1 gives the posterior means with the posterior standard deviation in brackets. The posterior distributions for λ_m,σ^2 are approximately symmetric, though the posterior

[4] An estimate for N is obtained by maximizing their objective function for values of N between 11 and 23.

means are not similar to the estimates reported by Laffont, Ossard and Vuong (1995) in column 1 of Table 1. The posterior distribution for N has a spike at $N = 11$, which is again different from the estimate of $N = 18$ obtained by Laffont, Ossard and Vuong (1995). Further, the posterior variance for N is 1.4. This is high compared to the other parameters especially when account is taken of the fact that the prior variance for N is 12. Thus the conjecture made by Laffont, Ossard and Vuong (1995) that of the eleven buyers who attend all the auctions, one buyer is representing seven retail sellers is not replicated by my results. Note that, I obtained a similar spike at $N = 11$, and similar results for the posterior distribution of $\lambda_m, m = 1, ..., 7$, when I assumed $V(\lambda_m) = 0.067$ $\forall m = 1..., 7$, and $V(\sigma^2) = 0.0125$.

This Section concludes with the result that neither the estimate of the scale parameter, σ^2, or the location parameter λ, of the distribution of private values of bidder i in auction j, lnV_j^i, nor the estimate of potential competition reported by Laffont, Ossard and Vuong is replicated with the method proposed in this paper. To ascertain the reason for these differences, I simulate a Dutch auction in the next section. This will also serve as a check on the proposed method if I am able to recover the "true" parameter values.

Section 5: A Simulated Dutch Auction

In this section, I generate an artificial data set for a Dutch auction with parametric assumptions similar to the Dutch auction for eggplants studied in Section 4. I assume that the log of private values, lnV_j^i, follows a Normal distribution with mean, $\mu = 2$, and variance, $\sigma^2 = 0.75^2$,

$$g(v_j^i|\theta, \mathbf{z_j}) = g_{LN}(v_j^i|2, 0.75^2) = \frac{1}{v_j^i}\phi(lnv_j^i|2, 0.75^2) \quad \forall i, j; \tag{23}$$

the number of potential bidders, N, is 15, with 11 buyers participating regularly in all auctions. I generate data for 81 auctions; that is, the sample size of the simulated data, like the real data set is 81. Note that I have assumed all 81 auctions to be identical by assuming that the mean of lnV_j^i is the same. Further the reserve price in auction j, p_j^o, is assumed to be zero for all of the 81 auctions. The assumptions of identical auctions and absence of reserve price are made to simplify the process of generating the data, and will not affect the analysis being done in this Section.

Prior specification for the parameters of the simulated data, μ, σ^2, N, is similar to the prior specification for the parameters of the real data. A normal-gamma prior is assumed for $(\mu, \sigma^{-2})'$; the density function of $\theta = (\mu, \sigma^2)'$ is

$$P(\theta) = \phi(\mu|\underline{a}, \sigma^{-2}\underline{b})IG(\sigma^2|\underline{c}, \underline{d}); \tag{24}$$

N is assumed to follow a discrete uniform distribution on the interval $[11, 23]$.

Section 5.1: Posterior Results with Artificial Data

In this section, I ascertain how well the proposed method fares in recovering the "truth" about the parameters of the distribution of private values, V_j^i.

For the artificial data, the observable is the vector of winning bids, $\mathbf{w}$; the unobservables are $\theta, N, \mathbf{V}$. $\theta = (\mu, \sigma^2)'$ are the parameters of the distribution of the private value of bidder i in auction j, V_j^i; $\mathbf{V}$ is the 81 dimensional column vector of latent data on the highest private value, $V_j(1:N)$, in each of the 81 simulated auctions. The M-H algorithm samples the kernel of the posterior density function, $P(N, \mu, \sigma^2, \mathbf{v}|\mathbf{w})$; this kernel is given in appendix C, equation (C.2). The hyperparameters for the prior on $\theta = (\mu, \sigma^2)'$ are specified such that $E(\mu) = 2$ and $E(\sigma^2) = 0.75^2$. I have conducted the posterior analysis with two sets of values for the variances, $V(\mu) = 1$, $V(\sigma^2) = 3$, and $V(\mu) = 100$, $V(\sigma^2) = 100$; the latter set of values makes the prior on the parameters relatively noninformative. Figure 2 plots the posterior histograms of μ, σ^2, N with the latter set of values for the variances of the parameters; the posterior mean and standard deviation of these parameters is given in column 2 of Table 2. The posterior means for μ, σ^2, and N are within $3 - 5\%$ of the "true" parameter values given in column 1 of Table 2. Again note the high posterior variance of 2.38 for N. Similar results were obtained with $V(\mu) = 1$ and $V(\sigma^2) = 3$.

Hence the proposed method does a good job of recovering the "truth" about the location and the scale parameters of the distribution of the logarithm of private values in the simulated Dutch auction. In Section 4 I noted at the end that I was unable to replicate the parameter estimates reported by Laffont, Ossard and Vuong. In view of the proposed method recovering the "truth" in the simulated auctions, a possible explanation for the simulated nonlinear leats squares estimates of λ and σ^2 differing from results obtained by the proposed method in Section 4 is the bias in the simulated nonlinear least squares estimates in a sample of 81 auctions.

Section 5.2: Learning About Potential Competition

I have pointed out several times, in Sections 4 and 5, to the high posterior variance of the measure of potential competition, N. Further, the posterior histogram for N in both the real and the simulated Dutch auctions has a spike at the boundary of the parameter space for N, specifically at $N = 11$, and assigns very low "probability" to all other points in the parameter space for N. These features of the posterior for N are repeated when I plot the kernels of the bivariate posterior of μ and N with $\sigma^2 = 0.75^2$ in figure 3, and the bivariate posterior of σ^2 and N with $\mu = 2$ in figure 4. [5] An explanation for these features follows.

Consider the extreme case when the number of potential bidders, N, approaches infinity; the winning bid then tends to the **highest possible** valuation (McAfee and McMillan, 1987, p. 711). Learning about the number of potential bidders from data on winning bids then amounts to learning about the number of potential bidders from data on the highest possible valuation. From properties of order statistics, [6] the posterior distribution of N is proportional to the probability that the private value of the winner is the highest amongst the N potential bidders, and that the remaining $N - 1$ bidders have private values below the highest private value, $v_j(1 : N)$. Since, the highest private

[5] A Normal prior is specified for μ centred at 2 and with variance 100 in figure 3. In figure 4, an Inverted Gamma prior has been assumed for σ^2 with the mean of σ^2 being 0.75^2 and variance 100.

[6] To simplify the discussion, consider a single auction j, and assume that the parameter vector θ that characterizes the distribution of private values, $g(v_j^i|\theta)$, is known. When N the number of potential bidders is large, the winning bid approaches the highest possible private valuation; that is, the highest private value, $v_j(1 : N)$, will equal the highest

value, $v_j(1:N)$, is the highest possible valuation when the number of potential bidders approaches infinity, the probability that all the potential bidders except the winner bid below the highest private value, is one.

At the other extreme is the case where there is just one potential bidder. The winning bid will now be the lowest possible valuation. In this case, from footnote 7, the posterior density function for the number of potential bidders will not be flat. Learning about N will occur because the probability that any bidder will bid belwo this single potential bidder is zero.

The simulated Dutch auction and the auction studied by Laffont, Ossard and Vuong correspond to cases which lie between the above mentioned extreme cases. For these auctions, from properties of order statistics (see footnote 7), the probability that all bidders, except the winner, have private values below the highest private value is small and decreases with an increase in the number of bidders. This explains the spike at the boundary of the parameter space for N, and the near-zero "probability" for all points on the parameter space of N.

Section 6: A Parametric model for the seller's reservation price

Following Riley and Samuelson (1981, pp. 381) there is a single [7] risk neutral seller

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private valuation. Then from the properties of order statistics,

$$\begin{aligned} P(N|w_j) &\cong P\Big(N|v_j(1:N)\Big) \\ &\propto P(N)P(v_j(1:N)|N) \\ &\propto Ng\Big(v_j(1:N)|\theta\Big)\Big[G(v_j(1:N)|\theta)\Big]^{N-1}, \end{aligned}$$

where N is assumed to have a discrete uniform prior distribution with a fixed support. $g(v_j(1:N)|\theta)$ is the probability that the winner has the highest valuation, $v_j(1:N)$; $\Big[G(v_j(1:N)|\theta)\Big]^{N-1}$ is the probability that the remaning $N-1$ bidders have private values below the highest private value, $v_j(1:N)$. When N approaches infinity, the latter is one since $v_j(1:N)$ is the highest possible valuation.

[7] Note that the seller in each auction could be different or the same. In the auction that I have studied in the previous section, there are two sellers, one of whom puts out a case of eggplants in each auction.

in auction j who has a valuation v_j^o for the auctioned object. V_j^o is drawn from a distribution with distribution function $F(v_j^o|\delta, \mathbf{x_j})$ and density function $f(v_j^o|\delta, \mathbf{x_j})$, where δ is a parameter vector that characterizes the distribution of V_j^o, the valuation of the seller in auction j. $\mathbf{x_j}$ is a $k_\delta \times 1$ vector of covariates observed by the econometrician that make the T auctions heterogeneous from the seller's viewpoint. [8] The seller in auction j determines the reserve price, p_j^o, by maximizing her expected revenue (Laffont, Ossard and Vuong, 1995, pp. 956):

$$Max_{p_j^o} E(W_j) + v_j^o \Big(G(p_j^o|\theta, \mathbf{z_j}) \Big)^N. \tag{25}$$

The first term is the seller's expected revenue if the object is sold; and the second term if the object is unsold or all the N potential bidders bid below p_j^o. The optimization exercise in equation (25) yields the following solution for the seller's reserve price in auction j, p_j^o:

$$\begin{aligned} p_j^o &= v_j^o + \frac{1 - G(p_j^o|\theta, \mathbf{z_j})}{g(p_j^o|\theta, \mathbf{z_j})} \\ &\equiv d\Big(v_j^o, G(p_j^o|\theta, \mathbf{z_j}) \Big). \end{aligned} \tag{26}$$

The density function of the seller's reserve price in auction j can be obtained from equation (26) once the distribution of the seller's valuation in auction j, $F(v_j^o|\delta, \mathbf{x_j})$, is specified. This is derived in appendix D. It is given by:

$$\begin{aligned} h(p_j^o|\delta, \theta, \mathbf{x_j}, \mathbf{z_j}) = &\Big[2 + \frac{[1 - G(p_j^o|\theta, \mathbf{z_j})] g'(p_j^o|\theta, \mathbf{z_j})}{[g(p_j^o|\theta, \mathbf{z_j})]^2} \Big] \\ &f(d^{-1}(p_j^o)|\delta, \mathbf{x_j}), \end{aligned} \tag{27}$$

where $d^{-1}(p_j^o) \equiv d^{-1}(p_j^o, G(p_j^o|\cdot))$ is the inverse of the function defined in (26) with respect to its first argument. Equivalently, $d^{-1}(p_j^o) \equiv v_j^o$, where v_j^o is the solution to equation (26) in terms of $p_j^o, \theta, \mathbf{z_j}$.

Inverting the function defined by equation (26) analytically is possible only if the distribution of private values, $G(p_j^o|\cdot)$, is assumed to have some simple specification; in other cases it will have to be obtained numerically. Or alternatively, as outlined in Section 3, the observed data can be augmented by the latent data on the seller's private value for

[8] Some of the k_x covariates, $\mathbf{x_j}$, may be the same as the observed covariates, $\mathbf{z_j}$, that make the auctions heterogeneous for the bidders.

the auctioned object, v_j^o, in each auction j. The augmented likelihood function previously specified by equation (10) will now be given by :

$$L(N,\theta,\delta,\mathbf{v},\mathbf{v^o};\mathbf{w},\mathbf{p^o},\mathbf{z},\mathbf{x}) = \prod_{j=1}^{T} h\Big(w_j|N,\theta,\delta,\mathbf{z_j},\mathbf{x_j}\Big), \tag{28}$$

where $\mathbf{x}$ is the $k_x \times T$ dimensional vector of covariates observed by the researcher that lead to heterogeneity across auctions; and $\mathbf{v^o} = (v_1^o,...,v_T^o)'$ is the vector of the seller's private value in the T auctions that are **unobserved** by the researcher. Note that from equation (26) once the seller announces the reservation price, p_j^o, all N potential bidders will know the seller's private value v_j^o. Thus in each auction v_j^o is common knowledge of the N potential bidders, but is **unobserved** by the researcher. $h(w_j|\cdot)$ is the augmented density function obtained from equation (27) by replacing $d^{-1}(p_j^o)$ by v_j^o since the two quantities are equivalent.

The distribution of interest is now the posterior distribution:

$$N,\theta,\delta,\mathbf{v},\mathbf{v^o}|\mathbf{w},\mathbf{p^o},\mathbf{z},\mathbf{x}. \tag{29}$$

The M-H algorithm is set up in a manner analogous to the one described in Section 3.

Section 7: Conclusions

I have proposed a likelihood based method for estimating a symmetric parametric structural auction model within the independent private-values paradigm. By simulating a Ducth auction, it has been demonstrated that the proposed method recovers the "truth" about the mean and variance of the distribution of private values. The difference in the parameter estimates for the location and scale parameters obtained by Laffont, Ossard and Vuong and the proposed method are attributed to the bias in a sample of 81 auctions of the simulated nonlinear least squares estimators.

The sensitivity of the results to the specification of the distribution of the private values is demonstrated in this paper; this concern has been expressed by other researchers (Laffont, 1997) too. A Dirichlet process [9] (Ferguson, 1973) or a scale-mixture of normal

[9] A Dirichlet process is a probability measure on the space of all measures.

distributions for the log of private values would be a step towards entertaining flexible distributions for the private values. A posterior odds analysis, using the data augmentation idea described above, could be used to ascertain the advantage of moving to nonparametric functional forms for the distribution of private values.

Finally, while the data augmentation idea has been used to estimate a symmetric structural descending-price auction model within the IPVP, it can be extended to other auction models since it provides an alternative to solving computationally burdensome differential equations to obtain solutions for the **unobserved** private values.

Appendix A

Let $\mathbf{Z_j}$, the vector of observed covariates be drawn from a distribution that is characterized by a parameter vector, γ. Then,

$$P(N, \theta, \gamma, \mathbf{v}|\mathbf{w}, \mathbf{p^o}, \mathbf{z}) = \frac{P(N, \theta, \gamma, \mathbf{v}, \mathbf{w}, \mathbf{p^o}, \mathbf{z})}{P(\mathbf{w}, \mathbf{p^o}, \mathbf{z})}. \tag{A.1}$$

The numerator is:

$$\begin{aligned} &P(N, \theta, \gamma, \mathbf{v}, \mathbf{w}, \mathbf{p^o}, \mathbf{z}) = P(\gamma)P(N, \theta|\gamma) \\ &P(\mathbf{z}|N, \theta, \gamma)P(\mathbf{v}|N, \theta, \mathbf{z}) \\ &P(\mathbf{p^o}|N, \theta, \mathbf{z}, \mathbf{v})P(\mathbf{w}|N, \theta, \mathbf{v}, \mathbf{p^o}, \mathbf{z}). \end{aligned} \tag{A.2}$$

I now make four assumptions. First, the parameters are assumed to be variation-free (Poirier, 1995, pp. 461). [10] Second, prior beliefs about γ are statistically independent from N, θ, $P(N, \theta|\gamma)$=$P(N, \theta)$. Third, $\mathbf{Z}$ is weakly exogenous (Poirier, 1995, p. 461) of N, θ, so that $P(\mathbf{z}|N, \theta, \gamma)$=$P(\mathbf{z}|\gamma)$. Fourth, $\mathbf{p^o}$ is fixed. The numerator in (B.2) is:

$$P(N, \theta, \gamma, \mathbf{v}, \mathbf{w}, \mathbf{p^o}, \mathbf{z}) = P(\gamma)P(\mathbf{z}|\gamma)P(N, \theta, \mathbf{v}, \mathbf{w}|\mathbf{p^o}, \mathbf{z}). \tag{A.3}$$

The denominator in (A.1) is

$$P(\mathbf{w}, \mathbf{p^o}, \mathbf{z}) = P(\mathbf{z})P(\mathbf{w}, |\mathbf{p^o}, \mathbf{z}). \tag{A.4}$$

[10] The parameters N, θ, γ are variation-free or the parameter space forms a product space, if

$$(N, \theta, \gamma) \in \Im \times \Theta \times \Gamma,$$

where $N \in \Im$, $\theta \in \Theta$, and $\gamma \in \Gamma$.

Substitute the denominator from (A.4) and numerator from (A.3) in (A.1),

$$\begin{aligned} P(N,\theta,\gamma,\mathbf{v}|\mathbf{w},\mathbf{p^o},\mathbf{z}) &= \frac{P(\gamma)P(\mathbf{z}|\gamma)}{P(\mathbf{z})}\frac{P(N,\theta,\mathbf{v},\mathbf{w}|\mathbf{p^o},\mathbf{z})}{P(\mathbf{w}|\mathbf{p^o},\mathbf{z})} \\ &= P(\gamma|\mathbf{z})P(N,\theta,\mathbf{v}|\mathbf{w}). \end{aligned} \tag{A.5}$$

Equation (11) will follow from (A.5) by integrating out γ.

Appendix B

The basic concept under any Markov Chain Monte Carlo method is that the probability of the next state conditional on the present state and the past states equals the probability of the next state conditional on the present states. The target density from which I want to sample is $f(N,\theta,\mathbf{v}|\mathbf{w},\mathbf{p^o},\mathbf{z})$. I will write this as $f(\mathbf{y}|\mathbf{w},\mathbf{p^o},\mathbf{z})$, where $\mathbf{y}=(N,\theta,\mathbf{v})'$. The draws from this target density will be iindicate by $\mathbf{y}^{(l)}$. The problem with sampling the target density is that it is of a nonstandard from and only it's kernel is available. Instead of sampling from the nonstandard target density, the Metropolis-Hastings algorithm samples form the transition density, $q(\mathbf{y})$, which is of some standard form. I have chosen the transition density to be the multivariate uniform defined on the support $[\mathbf{y}^{(l)}-\mathbf{d},\mathbf{y}^{(l)}+d]$, where d is an increment that is adjustedvso that forty percent of the draws are accepted in the manner shown below. Suppose the chain has been initialized by some start values, $\mathbf{y}^{(0)}$. The chain is at $\mathbf{y}^{(l)}$; the move to $\mathbf{y}^{(l+1)}$ will be accomplished as follows.

- Generate $\tilde{\mathbf{y}}$ from $q(\mathbf{y}^{(l)})$, u from $U[0,1]$.
- If $u\le\alpha(\mathbf{y}^{(l)},\tilde{\mathbf{y}})$, set $\mathbf{y}^{(l+1)}=\tilde{\mathbf{y}}$

$$\alpha\Big(\mathbf{y}^{(l)},\tilde{\mathbf{y}}) = min\left[\frac{f\Big(\tilde{\mathbf{y}}|\cdot\Big)}{f\Big(\mathbf{y}^{(l)}|\cdot\Big)},1\right]$$

- else set $\mathbf{y}^{(l+1)}=\mathbf{y}^{(l)}$;
- repeat for l = 0,1,...,L;
- return values $\{\mathbf{y}^{(1)},\mathbf{y}^{(2)},...,\mathbf{y}^{(L)}\}$.
- Discard initial draws to eliminate sensitivity to start value, $\mathbf{y}^{(0)}$;

•rest constitute sample from target density, $f(\mathbf{y}|\mathbf{w}, \mathbf{p^o}, \mathbf{z})$.

The details of the transition density, $q(\mathbf{y})$, are now given.

$$N^{(l+1)} \sim U\Big[N^{(l)} - d_N, N^{(l)} + d_N\Big];$$

$$\theta^{(l+1)} \sim U\Big[\theta^{(l)} - d_\theta, \theta^{(l)} + d_\theta\Big];$$

$$v_j^{(l+1)}(1:N) \sim U\Big[v_j^{(l)}(1:N) - d_{v_j}, v_j^{(l)}(1:N) + d_{v_j}.$$

The restrictions that I impose on the sampler are that $\sigma^2 > 0$, $N \geq 11$, and that $(N^{(l)}, \theta^{(l)}, v_j^{(l)})$ solve the Bayesian-Nash Equilibrium winning bid function given by equation (1).

Appendix C

The kernel of the posterior density function, $P(N, \theta, \mathbf{v}|\mathbf{w}, \mathbf{p^o}, \mathbf{z})$, from equations (18)-(22) is

$$P(N, \theta, \mathbf{v}|\mathbf{w}, \mathbf{p^o}, \mathbf{z}) \propto \phi_7(\lambda|\underline{\mathbf{a}}, \sigma^{-2}\underline{\mathbf{B}})IG(\sigma^2|\underline{\mathrm{c}}, \underline{\mathrm{d}})$$
$$\prod_{j=1}^{T} \frac{N^2}{N-1} \frac{1}{v_j(1:N)} \frac{1}{v_j(1:N) - w_j} \phi(lnv_j(1:N)|\mathbf{z_j'}\lambda, \sigma^2)\Big[\Phi(lnv_j(1:N)|\mathbf{z_j'}\lambda, \sigma^2)\Big]^{2N-1} \quad (C.1)$$

For the fake descending-price auction, the posterior density function with the likelihood based method is $P(N, \mu, \sigma^2, \mathbf{v}|\mathbf{w})$. The kernel of this posterior density function is

$$P(N, \mu, \sigma^2, \mathbf{v}|\mathbf{w}) \propto \phi(\mu|\underline{\mathrm{a}}, \sigma^{-2}\underline{\mathrm{b}})IG(\sigma^2|\underline{\mathrm{c}}, \underline{\mathrm{d}})$$
$$\prod_{j=1}^{T} \frac{N^2}{N-1} \frac{1}{v_j(1:N)} \frac{1}{v_j(1:N) - w_j} \phi(lnv_j(1:N)|\mu, \sigma^2)\Big[\Phi(lnv_j(1:N)|\mu, \sigma^2)\Big]^{2N-1} \quad (C.2)$$

The kernel of the posterior density function of $V_j(1:N)$ is

$$P(v_j(1:N)|N, \mu, \sigma^2, w_j) \propto \frac{1}{v_j(1:N)} \frac{1}{v_j(1:N) - w_j}$$
$$\phi(lnv_j(1:N)|\mu, \sigma^2)\Big[\Phi(lnv_j(1:N)|\mu, \sigma^2)\Big]^{2N-1}. \quad (C.3)$$

The posterior distribution for the fake descending-price auction when the **exact** solution for $v_j(1:N)$ is available is $P(N,\mu,\sigma^2|\mathbf{v},\mathbf{w})$. The kernel of this density function will be the same as the kernel of the posterior density function $P(N,\mu,\sigma^2,\mathbf{v}|\mathbf{w})$; the latter is given in equation (C.2).

Appendix D

The density function of the seller's private value, V_j^o, in auction j is:

$$\begin{aligned} h(p_j^o|\delta,\theta,\mathbf{x_j},\mathbf{z_j}) &= \frac{f(d^{-1}(p_j^o)|\delta,\mathbf{x_j})}{d'(d^{-1}(p_j^o))} \\ &\equiv \frac{f(v_j^o)|\delta,\mathbf{x_j})}{d'(v_j^o)}. \end{aligned} \tag{D.1}$$

In (D.1), $d^{-1}(p_j^o) \equiv v_j^o$ is the inverse of the function defined in equation (26) with respect to it's first argument; alternatively v_j^o is the solution to equation (26) in terms of $p_j^o, \theta, \mathbf{z_j}$. $d'(\cdot)$ is the Jacobian of the transformation of v_j^o to p_j^o in equation (26). It is given by

$$d'(d^{-1}(p_j^o)) = 2 + \frac{[1-G(p_j^o|\theta,\mathbf{z_j})]g'(p_j^o|\theta,\mathbf{z_j})}{[g(p_j^o|\theta,\mathbf{z_j})]^2}\Big]. \tag{D.2}$$

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Table 1

POSTERIOR RESULTS WITH REAL DATA

	Econometrica	Exercise 1
λ_1	0.0286	0.4320
		(0.102)
λ_2	-0.0240	-0.1191
		(0.039)
λ_3	0.2402	0.1804
		(0.054)
λ_4	0.1213	0.0928
		(0.034)
λ_5	1.1998	0.4659
		(0.059)
λ_6	0.3202	0.3541
		(0.034)
λ_7	-0.0357	-0.0603
		(0.026)
σ^2	0.0025	0.0478
		(0.010)
N	18	13
		(1.297)

Table 2

POSTERIOR RESULTS WITH FAKE DATA

Parameter	True Value	Proposed Method
μ	2.00	1.8617
		(0.139)
σ^2	0.5625	0.5815
		(0.1014)
N	15.00	13.447
		(2.389)

LIKELIHOOD BASED ESTIMATION OF SYMMETRIC PARAMETRIC STRUCTURAL AUCTION MODELS WITH INDEPENDENT PRIVATE VALUES †

by

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