In a comment on Thiel (1988), Levin and Smith (1991), derive the symmetric Bayesian Nash bidding strategies under the common value paradigm. One specific example considered by Levin and Smith (1991) is that the estimator of the unknown value of the object being auctioned follows a normal distribution. This case has been implemented by Paarsch (1992) for a low-price, sealed-bid auction held in the province of British Columbia to plant trees on Crown land. The Forest Service (a government agency) conducts these auctions. Bidders are invited to submit sealed bids, the bidding variable being price per tree planted. The winner is the bidder who has submitted the lowest bid.

One result obtained by Paarsch (1992, pp. 208-209) is that the Nash bidding strategies are linear in the estimate of cost. This conclusion seems to imply that for the auctions that Paarsch (1992) is studying, when faced with the choice between bidding a linear or a nonlinear bidding strategy, bidders choose a linear bidding strategy.

This note claims that the assumption made by Levin and Smith (1991) that the estimator of the unknown value of the auctioned object is Normally distributed, and certain restrictions on the parameters of the common value model constitute sufficient conditions to obtain the result that the Nash bidding strategies are linear. These parametric restrictions are derived in this note. Paarsch's (1992) data set satisfies these parametric restrictions; he also assumes, following Levin and Smith (1991) that the estimator of the unknown value of the object being auctioned is Normally distributed. His conclusion that the Nash bidding strategies are linear follows as a result.

1. Normal Distribution

In the discussion that follows individual bids/bidders are indexed by i and auctions by j, with $i = 1, .., N_j$, and, $j = 1, ..., n$. All random variables will be indicated by capital letters, and the realization of a random variable by small letters.

Consider a low-price, sealed-bid auction, in which N_j bidders submit sealed-bids for the object that is to be auctioned. As in Thiel (1988), Levin and Smith (1991) and Paarsch (1992), the following assumptions are made about this low-price, sealed-bid auction.

The value, c, of this object is unknown to each bidder. The prior density function of the unknown value is given by $g(c)$, and is assumed to be diffuse; that is, C follows a uniform distribution. Each bidder in the jth auction forms an independent and unbiased estimator, X^i_j, of this unknown value, c, and then submits a sealed-bid based on the estimate, x^i_j. That is, $E(X^i_j|c) = c$. The probability density function of the estimator, X^i_j, conditional on the unknown value of the object, c, is assumed to be identical for all i bidders in an auction j. It is normally distributed with mean c and variance σ^2. The probability density function of X^i_j is indicated by $\phi(x^i_j|c,\sigma^2)$ and the distribution function by $\Phi(x^i_j|c,\sigma^2)$.

Bidder i, in auction j, obtains her Nash bidding strategy, b^i_j, by maximizing posterior expected profits. That is, the equilibrium bidding strategy, b^i_j, is obtained as a solution to the following problem (Paarsch 1992, p. 194 or Levin and Smith 1991, p. 372)

$$max_{b^i_j} \int_{-\infty}^{\infty} \Big(b^i_j - c\Big)\Big(1 - \Phi(x^i_j|c,\sigma^2)\Big)^{N_j - 1} \phi(x^i_j|c,\sigma^2) g(c) dc. \quad (1)$$

It is given by

$$b^i_j \equiv b(x^i_j) = x^i_j - \alpha_{N_j}\sigma + \gamma exp\Big[\frac{-x^i_j \xi(1:N_j)}{\sigma}\Big], \quad (2)$$

where $b^i_j > 0$, $b'(\cdot) > 0$, and, $\gamma \geq 0$. $\gamma \geq 0$ ensures that the equilibrium bidding strategy satisfies individual rationality; that is, the expected profit of a bidder is nonnegative. Further, $\gamma = 0$ implies linear bidding strategies. Other quantities in (2) are now described.

$\xi(1:N_j)$ is the expected value of the smallest order statistic for a sample of size N_j from a standard normal distribution. It is given by

$$\xi(1:N_j) = \int_{-\infty}^{\infty} N_j u \Big[1 - \Phi(u|0,1)\Big]^{N_j - 1} \phi(u|0,1) du, \quad (3)$$

where u is a variable that follows a standard Normal distribution. $\xi(1:N_j)$ is strictly less than zero. This result follows from the assumption of Normality of X^i_j. α_{N_j} in equation (2) is given by

$$\alpha_{N_j} = \frac{\int_{-\infty}^{\infty} N_j u^2 \Big[1 - \Phi(u|0,1)\Big]^{N_j - 1} \phi(u|0,1) du}{\xi(1:N_j)}. \quad (4)$$

α_{N_j} is the ratio of the expected value of the square of the smallest order statistic to the expected value of the smallest order statistic, for a sample of size N_j, from a Normal distribution. $\xi(1:N_j) < 0$ implies that $\alpha_{N_j} \leq 0$. If the equilibrium strategy of the bidders is linear ($\gamma = 0$), then from equation (2), α_{N_j} can be interpreted as the increase in the equilibrium bid as a result of an increase in the standard deviation, σ, of the estimator of cost, X_j^i. That is, an increase in the variance of the estimator of cost will lead to less aggressive bidding or higher bids.

The winner of auction j is the bidder with the lowest estimate of the object, $x(1:N_j)$, and hence the lowest bid. The winning bid in auction j, w_j is (Paarsch 1992, pp. 202-203)

$$w_j \equiv b(x(1:N_j)) = x(1:N_j) - \alpha_{N_j}\sigma + \gamma exp\Big[\frac{-x(1:N_j)\xi(1:N_j)}{\sigma}\Big] \tag{5}$$

and the density of the winning bid is

$$f(w_j|c,\sigma,\gamma,N_j) = \frac{N_j\Big[\Phi(\frac{c-x(1:N_j)}{\sigma})\Big]^{N_j-1}\phi(\frac{c-x(1:N_j)}{\sigma})}{\sigma\Big[1-(\frac{\gamma\xi(1:N_j)}{\sigma})e^{\frac{-x(1:N_j)\xi(1:N_j)}{\sigma}}\Big]} \qquad \forall j = 1, ..., n, \tag{6}$$

where $x(1:N_j)$ solves equation (5).

If the auctions are assumed to be independent, then the likelihood function is given by the product of n density functions given by equation (6); it is

$$L(c,\sigma,\gamma;\mathbf{w},N_1,...,N_n) = \prod_{j=1}^{n} f(w_j|c,\sigma,\gamma,N_j). \tag{7}$$

The model given by the likelihood equation in (7), is estimated by Paarsch (1992) under two scenarios. In the first scenario, he assumes $\gamma = 0$ or linear bidding strategy. In the second scenario, γ is not set to zero; Paarsch obtains the maximum of the likelihood function on the boundary of the parameter space for γ; that is, the maximum likelihood estimate of γ, $\hat{\gamma}_{mle} = 0$.

On the basis of these results, Paarsch concludes that bidders choose linear bidding strategies.

2. Sufficient Conditions For Linear Bidding Strategies

The likelihood function specified in equation (7) assumes that the estimator of the unknown value of the auctioned object, X_j^i, is Normally distributed with mean c and variance σ^2. This assumption, and either of the following restrictions (8) or (10) on the parameter c, the unknown value of the auctioned object, constitute sufficient conditions for concluding that the Nash bidding strategies are linear: [1]

$$c \geq l_1, \tag{8}$$

where

$$l_1 = x(1:N_j) + \Big[\sigma(N_j - 1)\frac{\phi(\frac{x(1:N_j)-c}{\sigma})}{\Phi(\frac{c-x(1:N_j)}{\sigma})} + \frac{\gamma\xi^2(1:N_j)e^{\frac{-x(1:N_j)\xi(1:N_j)}{\sigma}}}{\Big(1-(\frac{\gamma\xi(1:N_j)}{\sigma})e^{\frac{-x(1:N_j)\xi(1:N_j)}{\sigma}}\Big)}\Big], \tag{9}$$

with the terms enclosed in $[\cdot]$ being strictly positive; and,

$$c < l_2, \tag{10}$$

where

$$l_2 = x(1:N_j) + \Big[\sigma(N_j - 1)\frac{\phi(\frac{x(1:N_j)-c}{\sigma})}{\Phi(\frac{c-x(1:N_j)}{\sigma})} + \frac{\sigma\xi(1:N_j)}{\Big(1-(\frac{\gamma\xi(1:N_j)}{\sigma})e^{\frac{-x(1:N_j)\xi(1:N_j)}{\sigma}}\Big)}\Big]. \tag{11}$$

I next determine whether Paarsch's (1992) data set satisfies conditions (8) or (10). He assumes, following Levin and Smith (1991) that the estimator of the unknown value of the auctioned object, $X(1:N_j)$, is Normally distributed. To ascertain that his data set satisfied conditions (8) or (10), I first obtained the lowest estimate of the unknown value of the auctioned object, $x(1:N_j)$, by solving (5) numerically. $x(1:N_j)$ was obtained for the minimum, average and the maximum winning bid reported by Paarsch (1992); these estimates are indicated by $x_{min}(1:N_j)$, $x_{avg}(1:N_j)$, and $x_{max}(1:N_j)$ respectively. I varied the number of bidders, N_j, from 3 to 28, the minimum and maximum number of bidders respectively, in the auctions studied by Paarsch. Thus each of $x_{min}(1:N_j)$, $x_{avg}(1:N_j)$, and $x_{max}(1:N_j)$ was obtained for 3 to 28 bidders. I then obtained l_1 and l_2 given by equations (9) and (11) respectively, for $x_{min}(1:N_j)$, $x_{avg}(1:N_j)$ and $x_{max}(1:N_j)$ to ascertain that either condition (8) or (10) were satisfied. The estimates of c and σ used are the ones reported by Paarsch (1992, p. 209, Table 3).

In Tables 1-3, I report $x_{min}(1:N_j)$, $x_{avg}(1:N_j)$, $x_{max}(1:N_j)$, and corresponding to them, l_1 and l_2 for 3, 16 and 28 bidders respectively. [2] For example, table 1 reports

[1] A proof of this is available from the author on request.

[2] As mentioned before, I varied the number of bidders from 3-28 to obtain $x_{min}(1:N_j)$, $x_{avg}(1:N_j)$, $x_{max}(1:N_j)$ and the limits l_1 and l_2 corresponding to the three. Since conditions (8) or (10) were not violalted in any case, I report the results for 3, 16 and 28, the minimum, average and the maximum number of bidders, respectively.

$x_{min}(1:N_j)$, $x_{avg}(1:N_j)$, $x_{max}(1:N_j)$, and corresponding to them, l_1 and l_2 for 3 bidders (that is, $N_j = 3$). In the first row in table 1, $x_{min}(1:N_j)$ is the lowest estimate of the unknown value of the auctioned object corresponding to the minimum winning bid of 0.130 reported by Paarsch (1992, p. 214, table 7) when the number of bidders, N_j, is 3. Then l_1 and l_2 are obtained from equations (9) and (11) respectively, with $x(1:N_j)$ taking the value $x_{min}(1:N_j)$. Paarsch (1992) reports the maximum likelihood estimate of C, $\hat{c}_{mle} = 0.2$; clearly, $\hat{c}_{mle} > l_1$ here, so that condition (8) is satisfied. The rest of the table is interpreted similarly.

Tables 1-3 indicate that $x_{min}(1:N_j)$, $x_{avg}(1:N_j)$, $x_{max}(1:N_j)$ satisfy either condition (8) or (10). Hence Paarsch's (1992) data set satisfies conditions (8) or (10), which along with the assumption of Normality of the unknown value of the object being auctioned, x_j^i, leads him to conclude that the Nash bidding strategies are linear.

Conclusion

This note is a warning to empirical researchers studying auctions under the common value paradigm with diffuse priors as in Thiel (1988), Levin and Smith (1991), and, Paarsch (1992). The empirical researcher could be mislead into concluding that the Nash bidding strategies are linear; an implicit assumption of linearity of the bidding strategies may underlie the specification of the distribution of the estimate of the unknown value of the object being auctioned.

Bibliography

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Proof

The likelihood function specified in equation (7) assumes that the estimator of the unknown value of the auctioned object, X_j^i, is Normally distributed with mean c and variance σ^2. The likelihood function will be maximized at $\gamma = 0$ or linear bidding strategies will be obtained if and only if the likelihood function is a strictly decreasing function of γ; that is,

$$\frac{dL(c,\sigma,\gamma;\mathbf{w},N_1,...,N_n)}{d\gamma} < 0. \tag{A.1}$$

The derivative of the likelihood function with respect to γ, $\frac{dL(c,\sigma,\gamma;\cdot)}{d\gamma}$, is obtained from equation (7). It is

$$\frac{dL(c,\sigma,\gamma;\mathbf{w},N_1,...,N_n)}{d\gamma} = \sum_{j=1}^{n}\Big[\prod_{i\neq j} f(w_j|\cdot)\Big]\frac{df(w_j|\cdot)}{d\gamma}, \tag{A.2}$$

where $f(w_j|\cdot)$ is the density function of the winning bid given by equation (6). Since the terms in $[\cdot]$ are strictly positive, a sufficient condition for $\frac{dL(\cdot)}{d\gamma} < 0$ is that $\frac{df(w_j|c,\sigma,\gamma,N_j)}{d\gamma} < 0$ for all auctions $j = 1,...,n$; that is,

$$\frac{df(w_j|c,\sigma,\gamma,N_j)}{d\gamma} < 0, \quad \forall \quad j=1,...,n \Rightarrow \frac{dL(\cdot)}{d\gamma} < 0. \tag{A.3}$$

The total differential of the density function, $f(w_j|\cdot)$, is

$$\frac{df(w_j|\cdot)}{d\gamma} = \frac{\partial f(w_j|\cdot)}{\partial x(1:N_j)}\frac{dx(1:N_j)}{d\gamma} + \frac{\partial f(w_j|\cdot)}{\partial\gamma} \qquad \forall \quad j=1,...,n. \tag{A.4}$$

Each term of $\frac{df(w_j|c,\sigma,\gamma,N_j)}{d\gamma}$, given by equation (A.4) is now derived.

The first term in equation (A.4) represents the indirect effect of γ on $f(w_j|\cdot)$ through the lowest estimate of the unknown value of the auctioned object, $x(1:N_j)$. Differentiating the density function in equation (6) with respect to γ,

$$\frac{\partial f(w_j|\cdot)}{\partial x(1:N_j)} = -\frac{N_j\phi(\frac{x(1:N_j)-c}{\sigma})\Big[\Phi(\frac{c-x(1:N_j)}{\sigma})\Big]^{N_j-1}}{\sigma^2\Big[1-(\frac{\gamma\xi(1:N_j)}{\sigma})e^{\frac{-x(1:N_j)\xi(1:N_j)}{\sigma}}\Big]^2}\Big[\sigma\Big\{1-(\frac{\gamma\xi(1:N_j)}{\sigma})e^{\frac{-x(1:N_j)\xi(1:N_j)}{\sigma}}\Big\}$$
$$\Big\{\frac{N_j-1}{\sigma}\frac{\phi(\frac{x(1:N_j)-c}{\sigma})}{\Phi(\frac{c-x(1:N_j)}{\sigma})} + \Big(\frac{x(1:N_j)-c}{\sigma^2}\Big)\Big\} + \frac{\gamma\xi^2(1:N_j)}{\sigma})e^{\frac{-x(1:N_j)\xi(1:N_j)}{\sigma}}\Big] \tag{A.5}$$

The influence of γ on $x(1:N_j)$, $\frac{dx(1:N_j)}{d\gamma}$, is obtained from equation (5), the Bayesian-Nash equilibrium bid of the winner of the jth auction; $\frac{dx(1:N_j)}{d\gamma}$ is obtained by the total differentiation of equation (5). It is

$$\frac{dx(1:N_j)}{d\gamma} = \frac{1}{\frac{\gamma\xi(1:N_j)}{\sigma} - e^{\frac{x(1:N_j)\xi(1:N_j)}{\sigma}}}. \tag{A.6}$$

$\frac{dx(1:N_j)}{d\gamma} < 0$ since $\xi(1:N_j) < 0$ and all other quantities in (A.6) are positive.

The second term in equation (A.4) is the direct effect of γ on $f(w_j|\cdot)$. $\frac{\partial f(w_j|\cdot)}{\partial\gamma}$ is obtained by partial differentiation of the density function given by equation (6) with respect to γ. It is

$$\frac{\partial f(w_j|\cdot)}{\partial\gamma} = \frac{N_j\xi(1:N_j)e^{\frac{-x(1:N_j)\xi(1:N_j)}{\sigma}}\left[\Phi(\frac{c-x(1:N_j)}{\sigma})\right]^{N_j-1}\phi(\frac{c-x(1:N_j)}{\sigma})}{\sigma^2\left[1-(\frac{\gamma\xi(1:N_j)}{\sigma})e^{\frac{-x(1:N_j)\xi(1:N_j)}{\sigma}}\right]^2}. \tag{A.7}$$

With $\xi(1:N_j) < 0$ and all other quantities in (A.7) positive, $\frac{\partial f(w_j|\cdot)}{\partial\gamma} < 0$.

From equations (A.6) and (A.7), $\frac{dx(1:N_j)}{d\gamma} < 0$ and $\frac{\partial f(w_j|\cdot)}{\partial\gamma} < 0$ follows from the assumption made by Levin and Smith (1991) that the estimator of the unknown value of the auctioned object, X_j^i, is Normally distributed with mean c and variance σ^2. The sign of $\frac{\partial f(w_j|\cdot)}{\partial x(1:N_j)}$ cannot be ascertained from the assumption of Normality of X_j^i made by Levin and Smith (1991); it depends on the data for the auctions being studied.

If $\frac{\partial f(w_j|\cdot)}{\partial x(1:N_j)} \geq 0$, then from (A.5) the condition given by equation (8) is obtained.

If $\frac{\partial f(w_j|\cdot)}{\partial x(1:N_j)} < 0$, then from equation (A.4), $\frac{df(w_j|\cdot)}{d\gamma} < 0$, if and only if

$$\left|\frac{\partial f(w_j|\cdot)}{\partial x(1:N_j)}\right|\left|\frac{dx(1:N_j)}{d\gamma}\right| < \left|\frac{\partial f(w_j|\cdot)}{\partial\gamma}\right|. \tag{A.8}$$

Given that $\frac{dx(1:N_j)}{d\gamma} < 0$ and $\frac{\partial f(w_j|\cdot)}{\partial\gamma} < 0$ from equations (A.6) and (A.7) respectively, equation (A.8) is

$$\frac{\partial f(w_j|\cdot)}{\partial x(1:N_j)}\frac{dx(1:N_j)}{d\gamma} < -\frac{\partial f(w_j|\cdot)}{\partial\gamma}. \tag{A.9}$$

The condition given by equation (10) follows from (A.9).

Table 1

$N_j = 3$		
$x(1:N_j)$	l_1	l_2
$x_{min}(1:N_j)$	0.1056	-0.0854
$x_{avg}(1:N_j)$	0.4045	0.2105
$x_{max}(1:N_j)$	1.3089	0.9369

Table 2

$N_j = 16$		
$x(1:N_j)$	l_1	l_2
$x_{min}(1:N_j)$	1.0667	0.7351
$x_{avg}(1:N_j)$	2.1949	1.8633
$x_{max}(1:N_j)$	7.6668	7.3351

Table 3

$N_j = 28$		
$x(1:N_j)$	l_1	l_2
$x_{min}(1:N_j)$	1.2978	1.0968
$x_{avg}(1:N_j)$	2.9922	2.7912
$x_{max}(1:N_j)$	12.3235	12.1226

A Comment on
"Deciding Between the Common and Private Value Paradigms in Empirical Models of Auctions" †

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Abstract

This note obtains sufficient conditions under which linear Nash bidding strategies result when it is assumed that the estimator of the unknown value of the auctioned object is Normally distributed in a symmetric parametric structural low-price, sealed-bid auction model under the common value paradigm. These conditions are satisfied by the auctions studied by Paarsch (1992); hence his result that Nash bidding strategies are linear.

Key Words: Low-price, sealed-bid auction; Common value paradigm; Symmetry; Bayesian-Nash equilibrium.
JEL classification: C1; C7; D8

† I have benefitted from comments by Harry J. Paarsch, Mike Peters, and Dale J. Poirier. The usual disclaimer applies.