Relating Jeffreys' Prior and Reference Prior in Nonregular Problems

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SUMMARY

In nonregular models characterized by a scalar boundary parameter, Jeffreys' prior and the reference prior are identical when the sample minimum or maximum is sufficient for this parameter. Outside this scenario, they are different by *definition*. However the marginal reference prior for the "regular" parameter is the Jeffreys' prior provided it is based on the *concentrated* likelihood function; this is illustrated through an example of a low-price, sealed-bid auction.

Keywords: JEFFREYS' PRIOR; LOW-PRICE SEALED-BID AUCTION; REFERENCE PRIOR; NONREGULAR..

1. INTRODUCTION

There are several examples of nonregular models in economics; these are models where the support of the data depends on the parameters. For example, in stationary job-search models (Lancaster, 1997) the optimal wage of an an unemployed worker is greater than his reservation wage; the latter is a function of model parameters like the workers' preferences, unemployment benefit etc. In random production models of cost-minimizing firms (Brown and Walker, 1994), optimum behavior of a firm requires nonnegativity and concavity of its cost function; both these conditions imply that the support of the data, whether it is cost shares or input demands, will depend on parameters. In structural auction models, submitted bids are greater than the expected value of bidders' second-highest valuation.

A well recognized technical hurdle in estimation and testing of nonregular models is that standard asymptotic theory breaks down; "nonregularity" introducing an asymmetry between the "regular" and "nonregular" parameters. The latter are parameters in the support of the distribution function of the data; the rest of the parameters are "regular". While the MLE of the "regular" parameter conditional on the "nonregular" parameter continues to have standard asymptotics, the MLE of the "nonregular" parameter does not.

Since Bayesian estimation proceeds by examining the posterior distribution of the parameters, it is unaffected by the dependence of the support of the data on parameters. However prior elicitation in nonregular models is complicated by the fact that a prior on the parameters implies a prior on the support. This implicit prior on the support of the data may be inconsistent with the the observed data. For example, the implicit prior on the support of the data may put zero mass on that part of the parameter space within which some of the observed data falls. Prior elicitation has to be done in a manner that ensures that the part of the parameter space where the likelihood and the implicit prior on the support put most of their mass is similar.

In some nonregular models parameters have a natural interpretation as the limiting form of some observables.[1] In these models subjective prior elicitation that is not subject to the kind of inconsistency discussed above is relatively straightforward. It is less obvious how one should proceed outside this scenario. An alternative in this instance is a "rule" for constructing a noninformative prior that takes into account the "nonregularity" in the sampling density of the data. The prior that emerges from this "rule" could also be used to conduct a prior sensitivity analysis. What should this noninformative prior be?

Jeffreys' prior is used as a standard noninformative prior in many instances. Beyond location-scale cases from iid data, it is less certain whether Jeffreys' prior is a candidate for the position of "noninformative" prior.[2] In a pioneering paper, Bernardo (1979) has introduced the so-called reference prior to represent the idea of a noninformative prior. The basic idea is to obtain a prior that maximizes the expected information from data. Formally, the reference prior emerges from maximizing an asymptotic expansion of Lindley's measure of information.

When there are no nuisance parameters and certain regularity conditions are satisfied, Bernardo's reference prior is the Jeffreys' prior. In nonregular models one of these regularity conditions fails; specifically, the support of the data depends on the parameters. If the likelihood of the data is characterized by a scalar boundary parameter, I show that Jeffreys' prior and reference prior are *identical,* if and only if the sample minimum or maximum is a sufficient statistic for this parameter.

The likelihood function in nonregular models rarely satisfy the necessary and sufficient conditions under which the sample maximum or minimum is sufficient for a scalar boundary parameter. Jeffreys' prior and the reference prior are now different by *definition*. However, I observe through an example of a low-price, sealed-bid auction that the marginal reference prior for the "regular" parameter is the Jeffreys' prior for the "regular" parameter based on the *concentrated* likelihood function. In this instance, Jeffreys' prior based on the *concentrated* likelihood function is the prior that maximizes Lindley's measure of expected information from data. An outline of the paper is as follows.

Section 2 describes the statistical framework used in the paper; reference priors are also defined. Section 3 establishes that the reference prior and Jeffreys' prior for a scalar boundary parameter are identical when the sample minimum is sufficient for this parameter. Section 4 considers a low-price, sealed-bid auction studied by Sareen (1999) and Paarsch (1992) to illustrate the results in Section 3. Section 5 concludes.

2. NOTATION AND DEFINITIONS

In the discussion that follows, random variables will be indicated by capital letters, and the realization of a random variable by a lower case letter; in addition, matrices will be indicated by bold faced letters. The assumptions made in this paper are now stated.

When standard regularity conditions (see Poirier, 1995, p. 259) are satisfied so that

[1] Bajari (1999), Bajari and Hortacsu (2000) and Sareen's (1999) work provide some examples in the context of sealed-bid auctions.

[2] Kleinberger (1994), Zellner (1971, pp. 216-220) and Phillips (1991) provide several examples of this.

the support of the data does not depend on θ, I will refer to this as the ***regular*** case. If the ***support of the data depends on the parameter*** θ, but other regularity conditions are satisfied, I will refer to this as the ***nonregular*** case. The definition of reference priors when there are no "nuisance" parameters is based on the following assumption.

Assumption 1. *Given data B_j^i where $i = 1, ..., n$ for each j and $j = 1, ...T$, and the $k \times 1$ parameter vector $\theta \in \mathbf{\Theta}$, (a) $B_j^i \,|\, \theta$ is independently and identically distributed; (b) θ can be partitioned into $\theta = (\eta, \varphi)$, where η is a scalar and φ is a $k-1$ dimensional vector; and (c) $f_b\left(b_j^i \,|\, \theta\right)$, the sampling density of B_j^i, is positive only on an interval $[l(\eta), u(\eta)]$ that depends on η and it is permitted that one of the end points may be plus or minus infinity; (d) $[l(\eta), u(\eta)]$ is monotone in η.*

In case θ is a scalar, then I assume that $\theta = \eta$. For the vector parameter case given η, the density function $f_b\left(b_j^i \,|\, \theta\right)$, is regular with respect to φ. Hence I will refer to η as the "nonregular" parameter and φ as the "regular" parameter.

Ghosal and Samanta (1995, 97) and Ghosal (1997), extending Bernardo's (1979) work, have obtained the reference prior when the support of the data depends on the parameters. The definitions that follow are based on their work.

Definition 1. (Reference prior for scalar parameter nonregular case).

$$\pi^b_{ref}(\theta) \propto |c(\theta)| \,. \tag{1}$$

$\pi^b_{ref}(\theta)$ is the reference prior for θ; $|\quad|$ is the absolute value of $c(\theta)$ which is defined below,

$$c(\theta) = E_{\mathbf{B}\,|\,\theta}\left[\frac{\partial}{\partial\theta}\log f_b(B_j^i|\theta)\right] = l'(\theta) f_b\left(l(\theta)\,|\,\theta\right) - u'(\theta) f_b\left(u(\theta)\,|\,\theta\right). \tag{2}$$

$[l(\theta), u(\theta)]$ defines the support of the sampling density of the data, $f_b(B_j^i \,|\, \theta)$.

Definition 2. (Reference prior for the multi-parameter nonregular case).

$$\pi^b_{ref}(\eta, \varphi) \propto |c(\eta, \varphi)| \sqrt{\det \mathbf{J}_B^{\varphi\varphi}\,(\eta, \varphi)}. \tag{3}$$

$\pi^b_{ref}(\eta, \varphi)$ is the reference prior for [η,φ]. $c(\eta, \varphi)$ corresponds to the quantity given in equation (2) for the multi-parameter case,

$$c(\eta, \varphi) = E_{B\,|\,\eta,\varphi}\left[\frac{\partial}{\partial\eta}\log f_b(B_j^i \,|\, \eta, \varphi)\right] = l'(\eta) f_b\left(l(\eta)\,|\,\eta, \varphi\right) - u'(\eta) f_b\left(u(\eta)\,|\,\eta, \varphi\right). \tag{4}$$

$\mathbf{J}_B^{\varphi\varphi}\,(\eta, \varphi)$is the lower right hand block of $\mathbf{J}_B\,(\eta, \varphi)$, the Fisher information matrix about η, φ from the nT variables B_j^i ,

$$\mathbf{J}_B^{\varphi\varphi}\,(\eta, \varphi) = E_{B\,|\,\eta,\varphi}\left[\frac{\partial}{\partial\varphi}\log f_b(B_j^i \,|\, \eta, \varphi)\right]\left[\frac{\partial}{\partial\varphi}\log f_b(B_j^i \,|\, \eta, \varphi)\right]'. \tag{5}$$

An important feature of the sampling density of the data in nonregular models is that the MLE of the "regular" and the "nonregular" parameters asymptotically converge to *different* distributions at *different* rates.[3]

The difference in the distributions to which the estimator of the "regular" and the "nonregular" parameters converge is taken into account in constructing reference priors through the difference in the contribution made by the "regular" parameter and the "nonregular" parameter to the reference prior for θ. While the "regular" parameter φ contributes $\sqrt{\det \mathbf{J}_B^{\varphi\varphi}(\eta,\varphi)}$, the "nonregular" parameter η contributes $|c(\eta,\varphi)|$ to the reference prior for $\theta=(\eta,\varphi)$. It is in this sense that the reference prior for θ treats the "regular" and the "nonregular" parameters asymmetrically. Note that both the "regular" and the "nonregular" parameters contribute the standard deviation of the asymptotic distribution of the relevant maximum likelihood estimator.

Jeffreys' prior for $\theta=(\eta,\varphi)$, on the other hand, treats η and φ symmetrically since it is proportional to $\sqrt{\det \mathbf{J}_B(\eta,\varphi)}$. Hence Jeffreys' prior *ignores* an important aspect of the sampling distribution in nonregular models.

3. GENERAL RESULTS

In nonregular models, Jeffreys' prior and the reference prior are *identical* if the likelihood of the data is characterized by a scalar boundary parameter and the ***sample minimum is sufficient*** for this parameter. I prove this in Proposition 1. Huzurbazar (1976, pp. 158, 174-186) establishes that the sample minimum or maximum is a sufficient statistic for θ *if and only if* θ is a *scalar* and if the density function for the data and the support of this density function satisfy certain conditions. These *necessary and sufficient* conditions are given in Assumption 2 which follows. I will be using the Huzurbazar results in context of a sample minimum W_*. Thus $W_*=\min_{\substack{i=1,\ldots,n\\ j=1,\ldots,T}}\{B_j^i\}$.

Assumption 2. *Given θ is a scalar and $B_j^i\,|\,\theta$ is independently and identically distributed, (a) the support of the data depends on θ, $B_j^i\geq l(\theta)$; (b) $l(\theta)$ is a strictly monotonic, continuous and differentiable function of θ; (c) $m(b_j^i)$, $q(\theta)$ are strictly positive functions of b_j^i and θ, respectively; and (d) the form of the sampling density of the data, $f_b(b_j^i\,|\,\theta)$, is*

$$f_b(b_j^i\,|\,\theta)=m(b_j^i)q(\theta).$$

Using Assumption 2, it is now proved that Jeffreys' prior and the reference prior are identical.

Proposition 1. *Under Assumption 2 and using Definition 1,*

$$\pi_{jeff}^b(\theta)=\pi_{ref}^b(\theta).$$

[3]The MLE of the "regular" parameter after "concentrating" out the "nonregular" parameter from the likelihood function converges in distribution at rate $\sqrt{T}$ to a Normal distribution; $\sqrt{T}\left(\widehat{\varphi}_{ml}-\varphi_o\right)\sim N\left(0,\mathbf{J}_{\mathbf{B}}^{\varphi\varphi}\left(\widehat{\eta},\varphi\right)\right)$, where $\widehat{\eta}=w_*$ and φ_o is the true value of φ. The MLE of the "nonregular" parameter is the sample minimum, W_*. It converges in distribution at rate T to an Exponential distribution; $T\left(W_*-\eta_o\right)\sim EXP\left(c(\eta,\varphi)\right)$, where η_o is the true value of η and $c(\eta,\varphi)$ is the sampling expectation of the score function given in equation (4).

Proof. The Fisher information for θ is given by $J_B(\theta) = E_{\mathbf{B}\,|\,\theta}\left[\left(\frac{\partial}{\partial\theta}\log f_b(B_j^i\,|\,\theta)\right)^2\right]$. From Definition 1, $[c(\theta)]^2 = \left(E_{\mathbf{B}\,|\,\theta}\left[\frac{\partial}{\partial\theta}\log f_b(B_j^i\,|\,\theta)\right]\right)^2$. Then from Jensen's inequality,

$$E_{\mathbf{B}\,|\,\theta}\left[\left(\frac{\partial}{\partial\theta}\log f_b(B_j^i\,|\,\theta)\right)^2\right] \geq \left(E_{\mathbf{B}\,|\,\theta}\left[\frac{\partial}{\partial\theta}\log f_b(B_j^i\,|\,\theta)\right]\right)^2.$$

Taking the positive square-root of the above expression, gives Jeffreys' prior on the LHS and the reference prior defined in Definition 1 on the RHS. From Jensen's inequality, the above equation holds with equality if $\frac{\partial}{\partial\theta}\left[\log f_b(B_j^i\,|\,\theta)\right]$ is not a function of B_j^i, so that $E_{\mathbf{B}\,|\,\theta}\left[\frac{\partial}{\partial\theta}\log f_b(B_j^i\,|\,\theta)\right]$ equals $\frac{\partial}{\partial\theta}\log f_b(B_j^i\,|\,\theta)$. Under Assumption 2, this is the case. Hence Proposition 1 follows. $\triangleleft$

Intuitively, Proposition 1 follows from the score function not depending on B_j^i. Since both the reference prior and Jeffreys' prior involves the sampling expectation of the score, whether the sampling density $B_j^i\,|\,\theta$ is regular or nonregular, is irrelevant.

Few nonregular models are characterized by a scalar boundary parameter. However *given* the "regular" parameters φ, the sample minimum could be sufficient for η; that is, the likelihood of the data as a function of η satisfies Assumption 2. This is the case in the example discussed in the next Section.

Once Assumption 2 is relaxed, a sufficient statistic does not exist for the scalar boundary parameter θ. Job-search models, random production models and multiparameter structural auction models belong to this category. Jeffreys' prior and the reference prior for θ are now *different* by ***definition***.[4] However the marginal reference prior for the "regular" parameter is the Jeffreys' prior based on the concentrated likelihood function. I illustrate this through an example of a low-price, sealed-bid auction.

4. EXAMPLE: LOW-PRICE, SEALED-BID AUCTION

The sealed-bid auction used as an example is described. The reference prior and the exact reference posterior are obtained for this auction. Finally, a discussion of the results follows.

4.1. *Description of Auction*

In a previous paper Sareen (1999) studies auctions for procurement of crude-oil on the international market by the Indian Oil Corporation, a public sector undertaking in India. Bidders are invited to submit sealed-bids with the lowest bid being declared the winning bid. Bidders are assumed to know the cost at which they can procure crude-oil and submit bids proportional to this cost. The cost of each bidder is drawn from the same distribution which is known to them. The parametric assumption about the cost distribution follows the work of Paarsch (1992). The assumptions ensure that a closed form solution is available for the bidding rule which is a differential equation that emerges from the profit maximization exercise performed by each bidder.

[4]The exception is when both the Jeffreys' prior and the reference prior are lebesgue measures on $[0, \infty]$.

Typically the number of potential bidders in an auction is fixed. Asymptotic arguments in the auction literature are made with respect to the number of auctions. This could be problematic since the auctioned object may not be the same across auctions. For example, if the environment under which subsequent auctions are held is different, the auctioned object is heterogenous. This problem is resolved by conditioning on the heterogenous characteristics of the auctioned object; conditional on these characteristics, subsequent auctions are homogenous. In the current context, Sareen (1999) observes 37 auctions for crude-oil; all these auctions were held in the year 1993-94. The auctioned object is a metric tonne of crude-oil. A bid is the price in US dollars of a metric tonne of crude-oil. The key factor driving these auctions is the domestic demand for crude-oil. Since instability is not noted in any of the economic indicators used by the Government of India in the period 1992-95, it is assumed that the environment under which the auctions are held is unchanged.

An auction is indicated by the subscript j and bidders in an auction by the superscript i. When bids are proportional to cost, then Paarsch (1992, p. 203) shows that the likelihood of the bids is a Pareto distribution with density function

$$f_b(b_j^i \,|\, \eta, \varphi) = \frac{\varphi \eta^{\varphi}}{(b_j^i)^{\varphi+1}}, \qquad b_j^i > \eta. \tag{6}$$

$\eta(\delta, \varphi, n) = \frac{\delta\varphi(n-1)}{\varphi(n-1)-1}$ is the lower bound of the support of the density function of the bids. The dependence of $\eta(\delta, \varphi, n)$ on δ, φ and n will be suppressed for notational convenience. δ is some minimum common cost of procuring crude-oil that is known to all the bidders. φ is the shape parameter of the distributions of B_j^i. n is the number of potential bidders in an auction. This is identical to the number of participants in the auction since there is no reserve price and no costs to preparing and submitting bids.

4.2. *Reference Prior and Posterior*

Definition 2 is used to construct the reference prior for $\theta = [\eta, \varphi]$. The lower bound of the density function for a bid in equation (6) is $l(\eta) = \eta$. The sampling expectation of the score function, $c(\eta, \varphi) = l'(\eta) f_b(\eta \,|\, \eta, \varphi) = \frac{\varphi}{\eta}$. I obtain $\mathbf{J}_{\mathbf{B}}^{\varphi\varphi}(\eta, \varphi)$ as the element in the second row and column of the Fisher information matrix $\mathbf{J}_{\mathbf{B}}(\eta, \varphi)$; this is given in Appendix 1. $\sqrt{\det \mathbf{J}_{\mathbf{B}}^{\varphi\varphi}(\eta, \varphi)} = \frac{1}{\varphi}\sqrt{nT + \frac{(nT)^2\varphi^2}{(\varphi m-1)^2}}$, where $m = n - 1$. Then from Definition 2 the reference $prior$ for $[\eta, \varphi]$ is

$$\begin{aligned} \pi_{ref}^b(\eta, \varphi) &\propto \frac{\varphi}{\eta}\left[\frac{1}{\varphi}\sqrt{1 + \frac{Tn\varphi^2}{(\varphi m - 1)^2}}\right], \qquad \eta, \varphi > 0, \\ &\propto \left\{\frac{1}{\eta}\right\}\left\{\sqrt{1 + \frac{Tn\varphi^2}{(\varphi m - 1)^2}}\right\}. \end{aligned} \tag{7}$$

The exact reference $posterior$ for $[\eta, \varphi]$ is

$$\begin{aligned} \pi_{ref}^b(\eta, \varphi \,|\, \mathbf{b}) &= \pi_{ref}^b(\varphi|\mathbf{b})\, \pi_{ref}^b(\eta|\varphi, \mathbf{b}), \quad 0 < \eta < w_*; \\ &\propto \left[\sqrt{1 + \frac{nT\varphi^2}{(\varphi m - 1)^2}}\varphi^{nT} e^{-\varphi(\sum\sum \log b_j^i)}\right]\left[\eta^{nT\varphi - 1}\right], \end{aligned}$$

$$\propto \left[\sqrt{1+\frac{nT\varphi^2}{(\varphi m_j-1)^2}}GA\left(nT+1,\left(\sum_{j=1}^{T}\sum_{i=1}^{n}\log b_j^i\right)^{-1}\right)\right]$$
$$[PF\,(nT\varphi, \quad w_*)]\,. \tag{8}$$

The first term in the square brackets of (8) is the kernel of the reference posterior for $\varphi, \pi_{ref}^{b}(\varphi\,|\,\mathbf{b})$. The term in the second square brackets is the posterior kernel of $\eta|\varphi$; it belongs to the power-function distribution.

4.3. *Discussion*

There are several points worth noting in this example.

First, the minimum winning bid is sufficient for η *given* φ. Hence the reference prior and Jeffreys' prior for $\eta|\varphi$ are identical and proportional to $\frac{1}{\eta}$. The reference prior for η given φ is proportional to $c(\eta,\varphi)$ from Definition 1. Jeffreys' prior for η given φ is proportional to $\sqrt{\mathbf{J}_{\mathbf{B}}^{\eta\eta}(\eta,\varphi)}$; $\mathbf{J}_{\mathbf{B}}^{\eta\eta}(\eta,\varphi)$ is the element in the first row and column of the Fisher information matrix $\mathbf{J}_{\mathbf{B}}(\eta,\varphi)$ given in Appendix 1.

Second, the Jeffreys' prior for (η,φ) is proportional to $\frac{1}{\eta}$ again. The reference prior for (η,φ) has been derived in equation (7); it is *different* from Jeffreys' prior.

Third, integrating out η from the posterior density function of (η,φ) given in equation (8), I obtain the marginal reference posterior for the "regular" parameter φ,

$$\pi_{ref}^{b}(\varphi\,|\,\mathbf{b}) \propto \left[\frac{1}{\varphi}\sqrt{1+\frac{Tn\varphi^2}{(\varphi m-1)^2}}\right]\prod_{j=1}^{T}\prod_{i=1}^{n}\left[\frac{\varphi w_*^{\varphi}}{(b_j^i)^{\varphi+1}}\right]; \tag{9}$$

this is derived in Appendix 2. It is proportional to the product of two quantities. The term in the second square brackets is the Pareto density function of the bids with a difference. It is as if the bids are truncated at w_* instead of η. This is called the *concentrated* likelihood function in the classical literature. The first term in the square brackets is the square-root of the Fisher information for φ based on this *concentrated* likelihood, $\sqrt{\det \mathbf{J}_{\mathbf{B}}^{\varphi\varphi}(w_*,\varphi)}$.

This result is intuitive for two reasons.

First, W_*, the MLE of the "nonregular" parameter η converges to its asymptotic distribution at rate T; the MLE of the "regular" parameter φ after "concentrating" or substituting the "nonregular" parameter with w_*, converges to its asymptotic distribution at the slower rate of $\sqrt{T}$. Christensen and Kiefer (1991) argue that because W_* converges to its asymptotic distribution at a superconsistent rate, it is almost as if $w_* = \eta$. Hence, it is appropriate to substitute or "concentrate" w_* for η in the likelihood function.

Second, once the "nonregular" parameter has been "concentrated out", the likelihood function becomes regular with respect to φ. Since the "concentrated" likelihood satisfies standard regularity conditions, the marginal reference prior for φ is Jeffreys' prior based on this concentrated likelihood function.

Lancaster (1997) makes a similar point in context of the likelihood function in job-search models with a *difference*.[5] He is unable to obtain the above integral analytically.

[5] In Lancaster (1997), the reservation wage is the "nonregular" parameter and the parameters of the distribution of wage offers are the "regular" parameters.

As a result, he takes a Laplace approximation of the integral as the total number of search periods goes to infinity. The $message$ is that the superconsistency of the sample minimum for the "nonregular" parameter showing up in the marginal posterior distribution for the "regular" parameters could be an $exact$ result. I observe it in my sample of nT bids for Pareto likelihoods.

5. CONCLUSION

An interesting question at this point is how reference analysis for (η, φ) differs when parameters are partitioned into "nuisance" parameters and "parameters of interest". The paper has so far assumed that there are no "nuisance" parameters. While the $\boldsymbol{likelihood}$ function is $\boldsymbol{identical}$ irrespective of whether "nuisance parameters" are present or not, the reference $\boldsymbol{prior}$ is different. Details of this are given in Sareen (2000, pp. 85-87).

The results in the paper extend to the case where parameters are partitioned into "nuisance" parameters and "parameters of interest", but with a $difference$. The marginal reference posterior for φ in the low-price, sealed-bid auction discussed in Section 4 is now the product of the same prior and the $concentrated$ likelihood of $nT-1$ bids $instead$ of nT bids. Hence if a large number of auctions were to be observed, marginal inference about φ would be the same whether "nuisance" parameters were present or not. Typically only a small number of auctions are observed. In that scenario, it is $almost\ as\ if\ nT-1$ bids, when there is a partitioning of parameters into "nuisance parameters" and "parameters of interest", provide the $same$ information about φ as nT bids when there is no partitioning.

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APPENDIX 1. FISHER INFORMATION

When bids are proportional to cost so that the likelihood of the bids is Pareto, the Fisher information matrix for (η, φ) is

$$\mathbf{J_B}(\eta,\varphi) = \begin{bmatrix} (nT)^2\frac{\varphi^2}{\eta^2} & -\frac{(nT)^2\varphi}{\eta(\varphi m-1)} \\ -\frac{(nT)^2\varphi}{\eta(\varphi m-1)} & \frac{nT(\varphi m-1)^2+(nT)^2\varphi^2}{\varphi^2(\varphi m-1)^2} \end{bmatrix}.$$

APPENDIX 2. SUPERCONSISTENCY OF MINIMUM WINNING BID

I have obtained the reference prior for (η, φ) from all the bids in equation (7). The reference posterior is proportional to the product of the likelihood function of the nT bids which I obtain from equation (6) and the reference prior for (η, φ);

$$\pi^b_{ref}\left(\eta,\varphi \,|\, \mathbf{b}\right) \propto \left[\frac{1}{\eta}\sqrt{1+\frac{Tn\varphi^2}{(\varphi m-1)^2}}\right]\prod_{j=1}^{T}\prod_{i=1}^{n}\left[\frac{\varphi\eta^{\varphi}}{(b^i_j)^{\varphi+1}}\right].$$

Integrating out η I obtain

$$\begin{aligned}
\pi_{ref}^{b}\left(\varphi\,|\,\mathbf{b}\right) &\propto \left[\sqrt{1+\frac{Tn\varphi^2}{(\varphi m-1)^2}}\right]\left[\prod_{j=1}^{T}\prod_{i=1}^{n}\frac{\varphi}{(b_j^i)^{\varphi+1}}\right]\int_0^{w_*}\eta^{nT\varphi-1}d\eta, \\
&\propto \left[\sqrt{1+\frac{Tn\varphi^2}{(\varphi m-1)^2}}\right]\left[\prod_{j=1}^{T}\prod_{i=1}^{n}\frac{\varphi}{(b_j^i)^{\varphi+1}}\right]\left[\frac{w_*^{nT\varphi}}{nT\varphi}\right], \\
&\propto \left[\frac{1}{\varphi}\sqrt{1+\frac{Tn\varphi^2}{(\varphi m-1)^2}}\right]\left[\prod_{j=1}^{T}\prod_{i=1}^{n}\frac{\varphi w_*^{\varphi}}{(b_j^i)^{\varphi+1}}\right],
\end{aligned}$$

which is the result in equation (9).