# VI - Fourier, Wavelet and Fractal Analyses in the study of Normal and Turbulent Fluid Flows.

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## Abstract:

**The article is devoted to a brief discussion of Fourier transform analysis and the Wavelet transform analysis of turbulence based on the celebrated Navier-Stokes equations. A classical method for theoretical description of turbulence is the decomposition of the flow into Fourier components, known as Spectral method. Spatial Fourier analysis allows one to give precise meaning to the important notion of different scales of turbulence. The new method of Wavelet transform analysis may enable representation of quantities that depend on scale and position and hence it has the ability to give local information about the turbulent flows. Farge's and Meneveau's wavelet transform analysis of turbulence is discussed in some detail. Special attention is given to the adaptive wavelet method. Finally fractals, multifractals and singularities in turbulence are included.**

## Key words:

**Amplitude spectrum, phase spectrum, local energy spectrum, local and global wavelet spectra, fractals, multifractals, singularities in turbulence, Fractal Housdorff dimension etc.**

# 1. Notations and Terminology:

1) $L^2(\square)=\left\{ f(.)\left| \int_{-\infty}^{\infty} |f(x)|^2\, dx<\infty \right. \right\}$ - - (Hilbert space of L-integrable functions with finite energy)

2) $\langle f,g\rangle=\int_{-\infty}^{\infty} f(x)\,g^*(x)\,dx$, the Inner product of *f* and $g\in L^2(\square)$, $*$ indicates complex conjugation.

3) The Fourier transform of $f\in L^2(\square)$ is defined by $f\overset{\wedge}{(w)}=\frac{1}{\sqrt{2p}}\int_{-\infty}^{\infty} f(t)e^{-iwt}dt$. This is often referred to as "Fourier spectrum" of *f*.

4) The Wavelet transform of $f\in L^2(\square)$ is defined by;

$$W_y\,[F](a,b)=\frac{1}{\sqrt{a}}\int_{-\infty}^{\infty} f(t)y^*\left(\frac{t-b}{a}\right)dt,$$

where $y_{a,b}(t)=y\left(\frac{t-b}{a}\right)$ plays the same role as the kernel $e^{iwt}$ in the Fourier transform.

5) The Navier – Stokes equations for an unsteady motion of an incompressible viscous fluid of constant density $r$ and kinematics viscosity $J$ and the continuity equations are:

$$\frac{\partial U_i}{\partial t}+\underbrace{U_J\frac{\partial U_i}{\partial x_j}}_{\text{Convection}}=-\underbrace{\frac{1}{r}\frac{\partial P}{\partial x_i}}_{\text{Pressuregradient}}+\underbrace{J\nabla^2 U_i}_{\text{Viscosity}};$$

Where $J$ are the kinematical viscosity and the incompressibility condition $\frac{\partial U_i}{\partial x_i}=0$.

6) The fractal Housdorff dimension D of a fractal object of N parts scaled by a ratio r, from the whole is defined by $D=\frac{\log N}{\log\left(\frac{1}{r}\right)}$.

# 2. Introduction:

During the last two decades, the techniques of Fourier and Wavelet analyses have been applied with tremendous success, to various branches, which include Signal processing, Image compression, Pattern recognition, Medical imagery, Fluid mechanics and Geo-physics. As a result of application of Fourier and Wavelet transforms to differential equations, which model a complex phenomenon like turbulence, considerable progress has been made in understanding its unpredictable and chaotic behavior. Most of the progress has been based on the Navier – Stokes equations. However, there are certain major difficulties associated with the *NS* equations. Firstly, in three dimensions there are no results on important issues like existence, uniqueness, regularity and sensitive dependence on initial conditions. Thankfully, such results do exist for two dimensional *N-S* equations.Secondly, there are indications that solutions to 3-D *NS* equations can be singular at certain places and during some time intervals, in the flow.Thirdly, another difficulty arises from the non-linear convective term in the equation. This is, perhaps, the major difficulty of the turbulent theory. Fourthly and most importantly, in the limit as $J \to 0 \left( R \to \infty \right)$, the second order equation reduces to the first order Euler equations. Mathematically, the equations lead to a singular *perturbation problem*.

The purpose of this paper is to introduce Wavelet analysis, which is an active field of research, in the study of turbulence, a complex phenomenon, characterized by disorder, randomness, and instability. This has been discovered physically, but still largely unexplored mathematically. The perspective on the topic presented here is one that comes primarily on the work done in the field of Fluid flow and turbulence theory. In the process it is hoped to address certain relevant issues that are open and challenging. An extensive bibliography is provided for the reader who wants to go into more detail on this particular subject.

## 3. Historical Background:

Fourier transform has been very useful in many areas, including Signal processing, Quantum mechanics, Wave motion and Turbulence. In these areas, the Fourier transform $\overset{\wedge}{f}\left(k\right)$, of a function (signal) $f\left(x\right)$ is defined in the time (space) and frequency (wave number) domains. One of the important features of FT is that the trigonometrical kernel exp (ikx) oscillates indefinitely and hence the localized information contained in the signal in the time-domain is spread over the frequency domain. Although, $\overset{\wedge}{f}\left(k\right)$ does not loose any information about $f\left(x\right)$, due to its spreading over the k-space, it becomes virtually impossible to study its properties from those of $\overset{\wedge}{f}\left(k\right)$. This is probably a major weakness of *FT*, which to a large extent, limits its applicability. According to *Heisenberg uncertainty principle*, the energy spread of a signal and its frequency content, cannot be simultaneously and arbitrarily small. Motivated and guided by this principle, in 1946, Dennis Gabor introduced the Windowed Fourier Transform, rightly called as local version of *FT*, to achieve time and frequency localization. In 1982, Jean Morlet discovered the idea of Wavelet transform, which provides an alternative to *WFT* overcoming its limitations. The basic difference is, in a *WT* the notion of scale is introduced as an alternative to frequency leading to the so-called time-scale resolution, which is tailor-made for analyzing scale-invariant phenomena.

## 4. Basic Theory:

Turbulence is thought to arise via the instability of normal (laminar) flow as the Reynolds number is increased indefinitely, random in time and space, characterized by intermittency, unpredictability and irregularity. Turbulent flow forms such a complicated mathematical problem that its solution must lean heavily on experimental data. One disadvantage is that, experimental measurements commonly concern with quantities that are easy to measure rather than those that have physical significance and the sheer volume of the data may be a bar to the understanding of the nature, structure and dynamics of the turbulent flow. All this is true, but, if the development of the theory is to be the primary goal, use must be made of practicable methods and interpretable results. It might appear that a thorough mathematical analysis of flows would be too difficult or almost impossible. Because, we have no reason to believe that, the basic dynamical equations governing normal flows somehow do not apply to turbulent ones. Our experience suggests that, turbulence is obeyed by the fundamental equations as normal flows. Turbulence is time and space dependent with a very large number of spatial degrees of freedom. Moreover, it is rotational, that is it contains vorticity. Normal flows, of course, possess vorticity, but a characteristic of turbulence is that the vorticity has intense random small-scale variations in both space and time. The transition from normal to turbulence occurs when the instability sets in i.e. due to instability of normal flow at large *R*. Thus a large *R* is a prerequisite for the production of turbulence.

One of the most common approaches to the study of turbulence is to use N-S equations together with the continuity equation in the Fourier space. In tensor notation, the equations are

$$\left.\begin{aligned}&\frac{\partial u_i}{\partial t}+u_m\frac{\partial u_i}{\partial x_m}=-\frac{\partial u_i}{\partial x_i}+J\nabla^2 u_i+F_i\\&\frac{\partial u_i}{\partial x_i}=0\end{aligned}\right\}------(1)$$

Where $u_i = u_i(x,t)$ is the velocity field.

A turbulent flow is an ensemble of different solutions of the above equations.

It is important to point out that the use of *N-S* equations, is perhaps, justified

For the study of turbulence because Mach number of incompressible turbulent flows is relatively small.

Using the continuity equation and the *N-S* equation in the absence of external forces, we have the following equation:

$$\frac{\partial u_i}{\partial t}+u_m\frac{\partial}{\partial x_m}(u_i u_m)=-\frac{\partial p}{\partial x_i}+J\nabla^2 u_i ------(1)$$

In the basic equation, both velocity and pressure terms occur, and one can choose to focus on one and quantify the other, since both are interrelated.

Taking the divergence of (1) combined with the continuity equation gives the Poisson's equation for the pressure in terms of velocity. i.e.,

$$\nabla^2 p=-\frac{\partial^2(u_i u_m)}{\partial x_i\partial x_m}-----(2)$$

Eliminating the pressure from the N-S equation, we obtain

$$\frac{\partial u_i}{\partial t}-J\nabla^2 u_i=-\frac{1}{2}p_{ijm}(\nabla)(u_i u_m)------(3)$$

$$\text{where; } p_{ijm}(\nabla)=\frac{\partial}{\partial x_m}p_{ij}(\nabla)+\frac{\partial}{\partial x_j}p_{im}(\nabla)$$

$$\& \ \nabla=\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_3}\right),\ p_{ij}(\nabla)=d_{ij}-\frac{1}{\nabla^2}\frac{\partial^2}{\partial x_i \partial x_j},$$

Vorticity is defined as a curl of the velocity field i.e. $\Omega=\nabla\times U$; which means it has zero divergence i.e., $\frac{\partial\Omega_i}{\partial x_i}=0$

Taking curl of (1) to eliminate the pressure, we obtain the Helmholtz equation for vorticity:

$$\frac{\partial\Omega_i}{\partial t}+u_m\frac{\partial\Omega_i}{\partial x_m}=\Omega_m\frac{\partial u_i}{\partial x_m}+J\frac{\partial^2\Omega_i}{\partial^2 x_m}------(4)$$

## 5. Fourier Transform in Turbulence and the N-S Equations:

Fourier analysis is one of the most powerful mathematical methods and is used in a wide variety of scientific problems. It allows the decomposition of general functions into sinusoidal components, from which they may be reconstructed using the Fourier inverse transform. Fourier analysis is often employed to reduce linear differential or integral equations to algebraic equations, a technique that allows the solution of many difficult and interesting scientific problems. However, its use in the theory of turbulence is limited, and is intended to turn differential operators into algebraic factors, for the simple reason that turbulent quantities are random functions.

Taking the Fourier transform of the N-S equations gives:

$$\left(\frac{\partial}{\partial t}+Jk^2\right)\hat{u}_i(x,t)=-ikmp_{ij}\int u_j(q)u_m(k-q)d^3q------(5)$$

$$\text{where; } p_{ij}(k)=d_{ij}-\frac{k_ik_j}{k^2}$$

Equation (5) describes the time evolution of the transformed velocity field $\hat{u}_i$ in k-space. The velocity $u_i(x,t)$ is represented as a linear combination of plane waves, in some direction *i*. However, the information related to the position in physical space is completely hidden, which is a serious handicap in dealing with a turbulent flow. It has been recognized that turbulence has a set of localized structures, often called co-herent structures even at a very high Reynolds number *R* or a very low viscosity. There are many examples of such structures which play a central role in the time and space intermittency of turbulence. Although, the *N-S* equations in physical space provide no explicit information about scales of motion, which is essential for gaining physical insight into turbulent flows, their role in analyzing it cannot be underestimated.

In analysis of the spectral properties of turbulence, one often finds that direct use is made use of Fourier transform of flow quantities such as $u_i(x,t)$. In general, the *FT* of a signal $f(x)$ can be expressed as

$$\hat{f}(k)=\left|\hat{f}(k)\right|\exp\left(iq\left(\hat{k}\right)\right)------(6)$$

where; $\left|\hat{f}(k)\right|$ is called the amlitude spectrum and $\hat{q}(k)$ the phase spectrum.

The energy (power) spectrum of $f(x)$ is defined by

$$\hat{E}(K)=\left|\hat{f}(k)\right|^{2}------(6)$$

So that the total energy of the signal is given by;

$$E=\int_{-\infty}^{\infty}\left|\hat{f}(k)\right|^{2}dk------(7)$$

Clearly, it follows from (6) and (7) that the energy spectrum and the total energy depend only on the amplitude and are completely independent of the phase. In other words, the *FT* does not provide any local or structured information of the signal. In spite of this major shortcoming, the *FT* has been useful to analyze stationary stochastic signals. In particular, the *FT* is fairly successful in the theory of a homogeneous turbulent velocity field, confined to a box of volume $a^3$, and then considering what happens as that volume grows very large, thus approaching the infinite case. This round-about procedure was necessary to resolve the technical difficulty, namely lack of convergence of the integral! So, in the final analysis, for the real description of turbulence, there is a need for a transform that decomposes the flow field into components of different length-scales, different positions, and different directions. The *FT* analysis, as we have seen, does not have this ability. The Wavelet analysis representing quantities that depend on scale, position and direction succeeds in this venture and thereby giving a more revealing picture of turbulence.

## 6. Farge's Wavelet Transform Analysis of Turbulence:

It has already been pointed out, that the dynamics of turbulent flows depends not only on different length scales, but on different positions and directions. Consequently, physical quantities like energy, vorticity, entropy and pressure become highly intermittent. The Wavelet transform has the ability to provide a wide variety of local information of the physical quantities associated with turbulence.

Therefore, the *WT* is adopted to define the space-scale energy density by;

$$\overset{\square}{E}(l,x)=\frac{1}{l}\left|\overset{\square}{f}(l,x)\right|^{2}------(8);$$

Where $\overset{\square}{f}(l,x)$ is the wavelet transform of the given function $f(x)$.

It is helpful to introduce a local energy spectrum $\overset{\square}{E}(l,x_0)$ in the *nhd* of $x_0$ by

$$\tilde{E}_c(l,x_0) = \frac{1}{l}\int_{-\infty}^{\infty} \tilde{E}(l,x)\, c\left(\frac{x-x_0}{l}\right) dx ------(9)$$

Where $c$ is considered as a filter around $x_0$.
In particular, if $c$ is the Dirac Delta function, then the local wavelet energy function becomes;

$$\tilde{E}(l,x_0) = \frac{1}{l}\left|\tilde{f}(l,x_0)\right|^2 -----(10)$$

The local energy density can be defined by

$$\tilde{E}(x) = \frac{1}{C_y}\int_{0}^{\infty} \tilde{E}(l,x)\frac{dl}{l} ------(11)$$

On the other hand, the global wavelet spectrum is given by;

$$\tilde{\mathrm{E}}(l) = \int_{-\infty}^{\infty} \tilde{E}(l,x)\, dx ------(12)$$

Another significant feature of turbulence is the so-called intermittency phenomenon. Farge used *WT* to define the local intermittency $I(l,x_0)$. According to Farge et.al.(1990) if $I(l,x_0)=10$, the point $x_0$ contributes 10 times more than the average to the Fourier energy spectrum at scale *l.* This shows a distinct advantage of Wavelet analysis over *FT* analysis. Nevertheless, the *WT* analysis of turbulence does suffer from certain limitations, allowing further innovations and improvements in the tools as well as techniques.

## 7. Adaptive Wavelet Method for analysis of Turbulent flows:

Unpredictability is a key feature of turbulent flows, due to which, it is virtually impossible to analyze a fully developed turbulence in an incompressible viscous flow at a very large Reynolds number. This necessitated the use of a new technique, called the Adaptive Wavelet Method introduced by several researchers including Farge (1997). The new approach seems to be useful for simulating turbulence, because, the inherent structures involved in turbulence are localized co-herent vortices evolving a multiscale non-linear dynamics. Frohlich and Schneider used wavelet basis functions that are localized in both physical and spectral spaces and hence the approach is a reasonable compromise between grid-points and spectral methods. Thus the space-adaptivity of wavelets was fully exploited in this method. This method, they say, gives highly accurate results at very high Reynolds number.

## 8. Meneveau's Wavelet Analysis of Turbulence:

Meneveau (1993) initiated WT analysis for the study of time dependent 3-D computations of the velocity field in a turbulent flow. He also provided the first direct evidence, of the existence of the energy cascade from large to small turbulent scales. In developed turbulence, according to Meneveau, a cascade process is continuously going on in which, small scales are produced inside large ones, yet smaller ones from inside the small scales, and so on. At each stage, some of the energy of the parent eddy goes into creating its small-scale children, so there is an associated flux of energy from large to small eddies, that cascades down to the smallest scales where viscosity is important and energy flux is dissipated. Although, the fascinating description, without doubt contains the essential physics of the phenomenon, it is evidently qualitative in nature. Meneveau's analysis of turbulence is based on the wavelet transformed N-S equations in the orthonormal representation. A 3-D wavelet transform of a function $f(x)$ is defined by

$$w(r,x) = W^{[f]}_{(r,x)} = \frac{r^{-\frac{3}{2}}}{\sqrt{C_y}} \int_{-\infty}^{\infty} y\left(\frac{x-x}{r}\right) f(x)\, d^3x ------(13)$$

Where $y(x) = y(|x|)$ is the isotropic wavelet satisfying the admissiblility condition;

$$\mathrm{C}_y = \int_{-\infty}^{\infty} |k|^{-1} \left|\hat{y}(k)\right|^2 d^3k$$

As in the one dimensional case, the wavelet transform $w(r,x)$ can also be obtained from the $FT\ \hat{f}(k)$, so that

$$w(r,x) = \frac{1}{(2p)^3} \frac{r^{\frac{3}{2}}}{\sqrt{C_y}} \int_{-\infty}^{\infty} \hat{y}^*(rk)\, \hat{f}(k)\, e^{i(k\cdot x)} d^3k ------(14)$$

By applying (14) to the Fourier transformed *N-S* equations, one could find an expression for the total flux of kinetic energy, through scale *r* and at position $x$. Argoul et.al. who has done considerable research, on the turbulent flows using wavelet analysis showed that further progress cannot be achieved without making appropriate assumptions. One of the predictions rather than assumptions was: energy cascade of turbulent eddies reveal fractal structure. This was a turning point in the development of theoretical models, in describing, understanding and analyzing turbulence.

## 9. Fractal Analysis of Turbulence:

Mandelbrot (1982) first introduced the idea of a fractal as a self-similar geometrical figure that consists of an identical motif repeating itself in an ever decreasing scale. Fractal shapes are said to be scale-invariant in the sense that, the magnified portions, no matter how small, look like the whole and to each other. To put it in another way, a fractal object is one, which can be divided into any number of similar parts. Fractals are not smooth lines, surfaces or volumes in the usual sense, because such classical geometrical objects have integer dimension. Instead a fractal object has a "furry" structure in which successive magnifications reveal more and more details at finer and finer scales, which closely resembles the concept of "multiresolution" in Wavelet analysis. Fractal objects are distinguishable and perhaps qualified, be the so-called "fractal dimension" a number that agrees with our intuitive notion of dimension, but need not be an integer. The concept of fractal dimension is strongly connected with self-similarity. Given a self-similar object of N parts scaled by a ratio r from the whole, its fractal dimension (called Housdorff dimension) is given by

$$D = \frac{\log N}{\log\left(\frac{1}{r}\right)} ------(15)$$

This definition agrees with the usual one for simple objects such as the line, surface and volume and in all these cases D is an integer. Mandelbrot also gave a more formal definition of a fractal as a set with Housdorff dimension strictly greater than its topological dimension. Euclidian line fills no space at all. But, the outline of the Kotch curve, which can be constructed geometrically by successive iterations with infinite length, does fill a two-dimensional area, showing that a one-dimensional curve can fill up a two-dimensional space! It is more than a line and less than a plane. Mandelbrot could characterize the fractal dimension of the Kotch curve, as precisely as 1.2628. By the same token, the fractal dimension of a fractal surface is more than two and could be as large as three. This leads to a general result that, the Housdorff dimension of a fractal object is a measure of its space filling ability. Mandelbrot also conjectured that the concept of a fractal and its dimension could be effectively utilized to model many phenomena having self-similarity property and governed by entropy, in the real world.

Although, the above discussion has no direct relevance to the study of turbulence, it was justifiable, for illustrative purposes and an obvious analogy between fractal behavior and the energy cascade with its succession of smaller and smaller scales.

The celebrated "Kolmogorov statistical equilibrium theory" which is one of the foundation stones of the theory of turbulence, is essentially based on some basic assumptions including the hierarchy of self-similar eddies of different scales and the energy cascade from larger to smaller eddies. This lead Mandelbrot to believe that, the structure of turbulence may either be locally or globally self-similar fractals. Then he proposed, for the first time, fractal analysis of turbulent flows and predicted that multiscale cascade models, which continued indefinitely, led to dissipation of energy and they could be quantified accurately by fractal dimension. His fractal approach to turbulence received tremendous boost after the introduction of a simple by Frisch et.al. by means of which they explained the geometrical and physical significance of the fractal model of turbulent flows. The problem of intermittency also stimulated considerable interest in the study of kinematics of turbulence using fractals and fractal dimensions. All these developments, tragically received a setback when experts and specialists Anselmat et.al. proved convincingly that there was no satisfactory fractal model that could fully describe the complexity of the turbulent flows. They introduced the concept of "multifractals" and in their multifractal model of turbulence, they used the scale-invariance property, whish is one of the remarkable symmetries of the Euler's equations. In the mean time, the fractal facet of turbulence attracted the attention of Sreenivasan and

Meneveau (1986) and Vassilicos (1993). Their analysis revealed some complicated geometrical features of turbulent flows.

As a fitting finale to the above discussion, some comments on the possible developments of singularities in turbulence are not, in any way, out of the context. In his landmark observation, Mandelbrot said that "the turbulent solutions of the basic equations involve singularities or near singularities of an entirely new kind". He also stated that the singularities of the solutions of the N-S equations can only be fractals. Sreenivasan (1991), in his review article, while accepting the major influence of fractal formulations in understanding certain aspects of turbulence, says that "the possibility of some inherent problems present in these formulations cannot be ruled out." However, he concludes with an optimistic note, by saying that some magical inspiration or a major breakthrough in the analysis tools, may herald a new era in the progress of the turbulent theory, which remains far from being understood.

## 10. Conclusion:

Indeed, several theoretical works and experimental observations have revealed that the turbulence possesses some singularities in the velocity and vorticity fields. It also follows from solutions of the *N-S* equations that very large deviations exist in isolated eddies with complicated internal structure. Kolmogorov who formulated statistical theory of turbulence (which is now known as the "Universal Equilibrium Theory") way back in 1962, made some useful observations, one of which says that "there must be singularities in the derivations of the velocity field on a scale, where the rate of energy dissipation is locally very large". It remains an open question, whether the nature of these singularities is local or global.

It has been proposed very recently, that there may be some connection between turbulence and Mathematical "chaos," whose onset has triggered new approaches in the solution of differential equations in which it appears. Furthermore, nonlinearity is crucial to the appearance of chaos and is also important in the generation and maintenance of turbulence. The mathematical theory of chaos may, in the future contribute to our understanding of turbulent flows. Chaos a new science in the making is tantalizing, but yet it is unclear whether it will useful in the study of turbulence. An important and relatively recent method for the study of turbulence is by direct numerical solution of the N-S equations. Such solutions have become possible at moderately high Reynolds number in the last decade owing to the rapid development of both the speed and memory of computers. Turbulence has therefore become a very lively area of scientific research and applications, in recent years.

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