V – Wavelets in Numerical Analysis of Differential and Integral Equations.

(Theory and Practice)

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Abstract:

The aim of this article is to provide a brief introduction to the applications and implications of wavelets in Numerical solutions of Differential and integral Equations. We describe a method, called Wavelet-Galerkin method for solving the two point boundary value problem, for ordinary differential equations. The hierarchical nature of wavelets makes them more suitable for solving the partial differential equations. A brief review of classical methods like Finite difference, Finite element, spectral and collocation methods is given just to highlight their strengths and weaknesses with regard to numerical efficiency, consistency and stability. Finally, we give Wavelet - Galerkin method based on Finite difference and elements methods for the resolution of elliptic problems in a bounded domain, using the so-called connection coefficients. A glimpse into the integral equations and their solution using wavelets is included.

Key words and terms:

Condition number of a matrix, sparsity of a matrix, Ill (well) conditioning and preconditioning of linear systems of equations, Finite difference, Finite element, Spectral and Collocation methods. Wavelet-Galerkin method, Connection coefficients etc.

5.1 Introduction:

Wavelet analysis assumed significance due to the spectacular success in signal and image processing. Currently several investigations are being carried out which aim at harnessing the hierarchical structure of wavelet decompositions of functions and their derivatives, in widely different areas. In areas such as time-series analysis, approximation theory and numerical solutions of differential equations, wavelets are recognized as powerful weapons not just tools. It is not in general, possible to obtain exact (actual) solution to an arbitrary differential equation. This not simply because man's ingenuity or/and machine's intelligence fails, but, because the repertory of standard functions like polynomials, trigonometric, exponential, logarithmic and so on is too limited to accommodate the amazing diversity of differential equations encountered in practice. This necessitates either to go for, discretization of differential equations leading to numerical (approximate) solutions or their qualitative study which is concerned with deduction of important properties of the solutions without actually solving them. After discretizing a differential equation in a conventional manner like the finite difference approximation, wavelets can be used effectively, for purely algebraic manipulations in the resultant linear system of equations, which may lead to a smaller *condition number*, using a technique called *preconditioning*. The aim of this article is to throw some light on this aspect of

wavelet analysis for numerical and qualitative analysis of differential equations. The wavelet transform is a tool for carving up functions, operators, or data into components of different frequency, allowing one to study each component separately. The term *wavelet* was itself coined in 1982, according to **[Daubechies]**. Wavelet analysis may be thought of as a generalization of analysis by the Hilbert space method, wherein one forms an orthogonal basis of the space of interest. Equations in that space may then be solved in terms of the basis. Hilbert space techniques are especially useful in the solution of linear ordinary differential equations (ODEs), and permit one to reduce certain partial differential equations (PDEs) to two or more ODEs related by *variables of separation*.

Our understanding of the fundamental processes of the natural world is based; to large extent, on differential and integral equations. The motivation for studying differential and integral equations is due to their prominent use, as models of various physical phenomena. The laws of nature, it is often said, are written in the language of Mathematics. How else, one can explain the remarkable effectiveness of Mathematics and its allied branches in describing the physical world? Many applications of Mathematics require numerical approximations of differential equations and a brief introduction to the topic, by giving simple examples, is the main focus of the article. It seems natural to try wavelets where traditionally finite elements are used. Solving, for instance, boundary value problems, there are interesting results, showing that this might be fruitful. Wavelet-Galerkin method bears a testimony to this, which uses wavelet differentiation, giving rise to *connection coefficients*. Our focus is on ideas and principles, so no attempt is made to cover any topic in depth, but rather to present a general outline of the various techniques.

5.2 Preliminaries and Prerequisites:

Preliminaries:

We usually subdivide differential equations into ordinary differential equations (ODE's) and partial differential equations (PDE's).ODE's involve derivatives w.r.t. only one variable and PDE's involve partial derivatives. Equations may have constant or variable coefficients. Some equations are referred to as homogeneous (non-homogeneous). Equations are also labeled with orders and degrees. All these notions can be used to characterize equations by their appearance. We have seen many; rather different looking PDE's and it may seem hopeless to try to develop a unified theory that can treat all these diverse equations. This impression is essentially correct and in order to proceed, we look for criteria for classifying PDE's. The situation is slightly different in ODE's. The theory developed for lower order equations, is more often than not, passed on to higher order equations.

An important distinction is, between linear and non-linear equations. A D.E. is linear if it is a polynomial equation of 1^{st} degree in the unknown function and its derivatives. In order to clarify the concept of linearity,(from modern point of view) it is useful to write the equation in the form;

$$L(u) = 0 \text{ ----------} (1).$$

Using this notation, we say that equation. (1) is linear if it satisfies: $L(au + bv) = aL(u) + bL(v)$; for any constants a, b and relevant functions u, v. The *principal part* of the PDE is the part of the equation that contains the

derivatives of the highest order. Non-linear equations with principal part linear are called *semi-linear*. Semi-linear PDE's of 2nd order are classified further (motivated from conic-sections) as under.

The general semi- linear PDE of 2nd order is of the form:

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G(x, y)$$

This can be put in the form $L(u) = G$,

where; $L = A\frac{\partial^2}{\partial x^2} + B\frac{\partial^2}{\partial x \partial y} + C\frac{\partial^2}{\partial y^2} + D\frac{\partial}{\partial x} + E\frac{\partial}{\partial y} + F$

$L(u) = G$, is called Hyperbolic, Parabolic, Elliptic according as $B^2 - 4AC \begin{cases} > 0 \\ = 0 \\ < 0 \end{cases}$

The classical equations of Mathematical Physics: *Heat* equation. *Wave* equation and *Laplace's* equation are parabolic, hyperbolic and elliptic respectively. From computational point of view, the classification into these three canonical types is not very meaningful or at least, not as important as some other essential distinctions. Most of the differential equations, which are consequences of physical phenomena, are subject to either *initial* or *boundary* or both conditions. The region in which the boundary conditions are specified is called the *region of interest*. By solving the differential equation in this region, one can interpret the physics of the problem.

A type of condition $u(x, 0) = f(x)$ is called a *Cauchy-type* initial condition. If the solution is given specific values at the boundaries,

say $u(0, t) = a$, $u(1, t) = b$ ----------(1); where *a* and *b* are given, we have a *Dirichlet-type* boundary conditions. In many applications, other types of boundary conditions appear. If the derivatives rather than the functional values are specified, we have *Neumann-type* boundary conditions. Generally, Neumann conditions are of the form: $u_x(0, t) = a$, $u_y(1, t) = b$ ----------(2), where *a* and *b* are given. By combining equations (1) and (2), We get a *Robin-type* boundary conditions, which can be written as: $au(0, t) + bu_x(0, t) = c, a\,u(1, t) + b\,u_x(1, t) = g$, for given a, b, c, a, b and g.

Finally, we have *periodic* boundary conditions: $u(0, t) = u(1, t)$, $u_x(0, t) = u_x(1, t)$.

Faced with a problem posed by a differential equation and some initial conditions (IC's) and/or boundary conditions (BC's), we can check whether both the differential equation and the extra conditions are satisfied. This allows us to classify equations from "solvability" point of view. Is the problem at hand, an initial value (time evolution) problem? Or is it a boundary value (static solution) problem? The sub classification of IVP into parabolic or hyperbolic is much less important, because many actual problems are of the mixed type.

Basic Mathematical Questions:

Existence, uniqueness and *stability* are the issues of considerable importance in the theory of differential equations. Questions of existence occur naturally throughout mathematics. The question of whether a solution exists should pop up into mathematician's head whenever he/she attempts to solve a problem. Appropriately, the problem of existence of solution of pde's is one of the focal points in theory as well as practice. Once we have asked the question, whether a solution to a given problem exists, it is natural to ask, how many solutions there are? While uniqueness is often a desirable property for a solution of

a problem (often for physical reasons), there are situations wherein multiple solutions are equally desirable. A common mathematical problem involving multiple solutions is an *eigenvalue problem*. The term stability is one that has a variety of meanings within mathematics. One generally says that a problem is stable, if we change the problem "slightly" the solution changes only slightly. Another notion of stability is that of "asymptotic stability". We say that a problem is asymptotically stable if all its solutions get close to some "nice" function as time goes to infinity.

We say that a BVP, IVP, IBVP is *well-posed* (in the sense of Hadamard) if;

i) It has a solution (existence)
ii) The solution is unique (uniqueness)
iii) The solution depends continuously on the initial and/or boundary data (stability).

Accordingly, an equation may be well or ill posed. Lastly, PDE's where all the variables are spatial are called *Steady-state,* and PDE's which combine differentiation w.r.t. space and time are called *Evolutionary.* Elliptic equations, for example, are steady-state.

Prerequisites:

Eigenvalues and Eigenvectors:

Let A be a $n \times n$ matrix over $\mathbb{C}$. Suppose $\exists\, \lambda \in \mathbb{C}$ and a non-zero vector $x \in \mathbb{C}^n$ *such that* $Ax = \lambda x$

Then λ is called an *eigenvalue* and x is a corresponding *eigenvector*.

Basic facts about eigenvalue and eigenvectors:

- A has at most n eigenvalues.
- If A is symmetric then all the eigenvalues are real and the corresponding eigenvectors orthogonal.
- A is non-singular iff all the eigenvalues are non-zero.

Condition number of a matrix: Let A be a non-singular $n \times n$ matrix.

Then the condition number $C_{\#}(A)$ of A is given by $\|A\|\|A^{-1}\|$. It can be shown [4] that;

$C_{\#}(A) = \dfrac{|\lambda|_{\max}}{|\lambda|_{\min}}$ (Spectral C.N.). This only shows the importance of eigenvalues in the qualitative analysis of solutions of differential equations.

Translation invariant and Elliptic operators:

Let $f : \mathbb{R} \to \mathbb{R}$ and $y \in \mathbb{R}$, then $R_y : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is called *translation* if

$$(R_y) f(x) = f(x - y)$$

Let $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be a L.T. Then T is called translation-invariant. If

$$T(R_y f) = R_y T(f)$$

Note that T and R_y commute with each other. Example: $Lu = -u''$ is translation-invariant.

More generally, any constant linear differential operator say $(Lu)(t) = \sum_{i=0}^{n} a_i \dfrac{d^i}{dt^i}$, where each a_i is constant is *translation-invariant*. It may be noted that all translation-invariant operators are diagonalizable in Fourier basis, and they are the most natural operators used in signal processing and image compression.

$Lu = f$ where; $L = \frac{-d}{dt}\left(a(t)\frac{du}{dt} \right) + b(t)u(t)$ is called *elliptic* if

$\exists\, C_1, C_2, C_3 : 0 < C_1 \le a(t) \le C_2, 0 \le b(t) \le C_3$.

A square matrix *A* is called *tridiagonal* if $a_{ij} = 0 \;\forall\; i,\ j$ such that $|i - j| \ge 2$

A square matrix *A* is called *circulant* if $a_{i+1, j+1} = a_{i,j} \forall i, j$

5.3 Some Theoretical Considerations:

Continuous(analytical) and discrete(numerical) solutions: First of all, let us make sure, we know, what we mean when we talk about these solutions. Indeed, the very question of what it means to solve a given differential equation can be subtle. But, we still have not defined carefully, what we mean by a "solution". Should we ask, for example, that a solution must be real analytic or infinitely differentiable. This might be desirable, but, perhaps we are asking too much. The intuitive notion that a solution is a function that satisfies the equation identically is not precise. A solution of a D.E. $L(u) = f$, of order *m* in a domain $\Omega \subset \mathbb{R}^n$ is, a function $u \in C^m(\Omega)$ that satisfies the equation at all points of Ω. Analytical (continuous, exact or actual) solution is the solution, say of $L(u) = f$, when *u* is given as a mathematical expression in the independent variables and parameters. Numerical (discrete, approximate) solution, on the other hand, is obtained by the methods of Numerical-analysis, often given in the tabular form. It is not in general, possible to obtain analytical solutions to an arbitrary differential equation. This is not simply because, our intellectual capability fails, but because the repertory of standard functions like polynomials, trigonometric, exponential etc. in terms of which solutions may be expressed, is too limited to accommodate the infinite variety of differential equations encountered in practice. Even if an analytic solution can be found, the 'formula' displaying the solution is often too complicated to describe the principle features of the solution. This is particularly true, in case of implicit solutions and of solutions which are in the form of integrals or infinite series.
Would we be rather given the analytical solution to the problem or the numerical solution? Can we expect the solution to behave in the way, as it does? Does it make sense physically? The answers to these and many other questions, mainly depend on, what we want to do with the solution. In practical applications, however, it is becoming increasingly clear that, the numerical solutions are preferable to their analytical counterpart .Even when exact solutions are available, they are often difficult and complex to obtain, let alone interpret. Analytical solutions sometimes, may fail to have any practical value.

Advantages of analytical solution:

1) It contains more information than a table of values. We can compute graph, observe, estimate, manipulate and if necessary modify it using explicitly of the solution.
2 If we wanted to evaluate the solution at any specific point, we could do so, with any degree of accuracy, merely by adding more number of terms in case of solution series.
3) It is a solution at any point, not just at grid points of the domain.

Advantages of numerical solution:

1) Major advantage is that, many problems do not have analytical solutions (e.g. non-linear equations). Many realistic models are, in fact, non-linear. The linear models, for the most part, are the linearized versions (approximations) of the non-linear equations. There are only a limited number of systematic procedures for solving them and these apply to a very restricted class of equations.
2) It allows qualitative as well as quantitative analysis of the behavior of the solution.(Graphical analysis, for example, appeals to our intuition)

Having said, something about their relative advantages, we shall go deeper in to these matters, as we are going to see that almost all essential properties of the exact solution are somehow present in the approximate solution. For this purpose, we find it useful to introduce some notations and terminology. At the end of this section, we will also arrive at the conclusion: *discrete solutions converge towards the continuous solution as the mesh size h tends to zero.*

Some Typical difficulties:

- Non-linear equations are more difficult to solve than linear equations.
- Higher order pde's are more difficult to solve than lower order equations.
- Pde's Containing many independent variables are harder to solve than pde's containing few independent variables.
- For most pde's it is not possible to write explicit formula for the solutions.
- In general, it is more difficult, but far reaching to solve a BVP than IVP.

Case – study

Let us start by considering a two-point BVP,

$(Lu)(x) = f(x), \forall x \in (0,1)$; where $(Lu)(x) = -u''(x)$, and $f \in C(0,1)$ and

$$u(0) = u(1) = 0 \text{-------}(1).$$

We want the solution $u \in C^2(0,1)$, and satisfying the BC's. Thus our u is a candidate for the continuous solution.

Now, let us introduce a similar formalism for the discrete case. First, we let D_h be a collection of discrete functions v defined at the grid points x_j, $j = 0,1,--n+1$.For convenience, we write v_j for $v(x_j)$. Next, we let $D_{h,0}$ be the subset of D_h containing discrete functions, with the special property that they are zero at the boundary. For a function $w \in D_{h,0}$; we define the operator L_h by;

$(L_h w)(x_j) = -\dfrac{w(x_{j+1}) - 2w(x_j) + w(x_{j-1})}{h^2}$, which we recognize as the finite difference approximation of the second derivative. Now, we can formulate the discrete version of equation (1) as follows. Find $v \in D_{h,0}$; such that;

$(L_h v)(x_j) = f(x_j); \forall j = 0,1,--n$ ---- (2). Since some of the properties of L and L_h are connected to the inner product of the functions, we define the inner product of two

continuous functions u and v by $\langle u,v\rangle = \int_0^1 u(x)v(x)dx$. For discrete functions u and v, their inner product can be evaluated by numerical integration, namely the *Trapezoidal rule.*

Having established, suitable notations for the continuous and discrete problems, we are in a position to state some important properties [1]

1. The operators L and L_h are symmetric in the sense that :

$\langle Lu,v\rangle = \langle u,Lv\rangle \ \forall u,v \in C^2(0,1)$ and $\langle L_h u,v\rangle_h = \langle u,L_h v\rangle_h \ \forall u,v \in D_{h,0}$

2. The operators L and L_h are positive definite in the sense that:

$\forall u \in C^2(0,1)\langle Lu,v\rangle > 0$ and $\forall v \in D_{h,0} \ \langle L_h v,v\rangle_h > 0$

The above properties bring out several similarities between the continuous problem and the corresponding numerical approximation. Our final goal is to show that the discrete solution v will indeed converge to the continuous u when the spacing h approaches zero. To this end, we need to introduce the concept of *truncation error* and *consistency*.

Definition: Let $f \in C(0,1)$ and let $u \in C^2(0,1)$, be the solution of (1). Then we define the discrete vector t_h; called the truncation error by;

$t_h(x_j) = (L_h u)(x_j) - f(x_j)$; for $j = 0,1,----n$.

We say that the finite difference scheme (2) is consistent with the differential equation (1) if; $\|t_h\|_{h,\infty} \to 0$ as $h \to 0$.

Thus, roughly speaking, a scheme is consistent if the exact solution almost solves the discrete problem. However, the difference approximation to the D.E. does not guarantee that the solution of the difference equation approximates the analytical solution of the D.E. This is where; notions like convergence and consistency enter the picture. For sufficiently smooth functions, the scheme (2) is consistent.

Lax (1954), proved a remarkable theorem [2] which establishes the relationship between consistency, stability (continuous dependence on the initial data) and convergence:

Lax Equivalence Theorem: Given a well-posed linear IBVP and a FD scheme (which approximates that), satisfying consistent criterion, stability is the necessary and sufficient condition for the convergence.

To put it in a nutshell, as long as we have a consistent scheme, the convergence is synonymous with stability.

Error analysis: In general, FD schemes cannot produce the solution exactly, because the numerical computation is carried, up to a finite number of decimal places. This necessitates the introduction of error called the *round off error* which occurs during the actual processing. This also depends on the type of computer used. In practice, the actual numerical solution is v_{ij}^* and not v_{ij}, so the difference $r_{ij} = v_{ij} - v_{ij}^*$ is the round off error at (i,j). In contrast to truncation error, the round off error cannot be made small by decreasing the mesh size.

Propagation of errors: Floating-point operations (which approximate arithmetic operations, as well as possible) need not satisfy the well-known laws of arithmetic. This results in the propagation of errors. Error propagation plays a decisive role in the ultimate analysis of the solutions.
We call round off errors *harmless* if their contribution to the final solution is considerably small. If all the round off errors are harmless then the algorithm is said to be *well-behaved or numerically stable*. Finding numerically stable algorithms is a primary task of the present day Numerical Analysis.

5.4 Numerical Solutions of Ordinary Differential Equations:

The method we discuss for solving numerically a linear ordinary differential equation comes down to solving a linear system of equations or equivalently a matrix equation $Ax = y$. Theoretically such a system is well understood: for a square matrix A, the existence and uniqueness of a solution are guaranteed iff A is non-singular. However, in applications, there are larger and deeper issues, that need to be addressed, as they are of crucial importance. One of these has to do with a condition number of a matrix. We say that, a system $Ax = y$; is badly (well) conditioned if small relative errors in the input data y, cause large (small) errors in the solutions x. This is undesirable, in applications since it defeats the very purpose of the exercise. In almost all computations, there are bound to be errors, either due to rounding off, or due to imperfect measurements of the data. For a badly conditioned system, the solution can be virtually meaningless physically. The condition number of a matrix measures the *stability* of the system $Ax = y$ under perturbations of y. In practical applications, a small condition number is both necessary and desirable. If the condition number of A is large, we would like to replace the system $Ax = y$ by an equivalent system, whose matrix has a low condition number, which can be accomplished by a technique called *preconditioning.*

Suppose $f : [0,1] \to \mathbb{R}$ is a continuous function. Our goal is to obtain a C^2 function u that is a solution of (P) $(P)\begin{cases} -u''(t) = f(t); 0 \le t \le 1 \\ u(0) = 0 \text{ and } u(1) = 0 \end{cases}$

It is easy to see that, a unique solution exists. However, if f is not in the *closed form*, it may not be possible to express explicitly the formula:

$$u(x) = -\int_0^x \int_0^t f(s)\,ds\,dt + x\int_0^1 \int_0^t f(s)\,ds\,dt$$

Our approach to approximating the solution u is to numerically estimate the integrals. Another method which is more general, because it applies to equations where solutions are not easy to write explicitly, is the *finite difference method.* It is based on approximating the derivatives in equation in our model equation (P) by differences evaluated on a finite set points in [0,1].We discretize and reformulate (P) on the partition $t_j = \frac{j}{N}$, $j = 0,1,--N$ .The smallest step size we can choose is $\frac{1}{N}$, so we let $h = \frac{1}{N}$ .We abbreviate, $x(j) = x_j$ and $y(j) = y_j$ .From Numerical Analysis, we arrive at:

$x_{j+1} + 2x_j - x_{j-1} = y_j$, $j = 1,2,----N$; with boundary conditions (BC's). $x(0) = 0, x(N) = 0------(1)$. Thus we have a linear system of *N-1* equations in *N-1* unknowns, represented by $A_N x = y$ where;

$$A_N = \begin{bmatrix} 2 & -1 & 0 & - & - & - & 0 & 0 \\ -1 & 2 & -1 & 0 & - & - & - & 0 \\ 0 & -1 & 2 & -1 & 0 & - & - & 0 \\ 0 & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}, \; x = \begin{bmatrix} x_1 \\ x_2 \\ - \\ - \\ - \\ - \\ x_{N-1} \end{bmatrix}, \; y = \begin{bmatrix} y_1 \\ y_2 \\ - \\ - \\ - \\ - \\ y_{N-1} \end{bmatrix}$$

For obvious reasons we call A_N a *tridiagonal* matrix. One can check that $\det A_N = N$, hence there is a unique solution x to the equation (1).

As we let $h \to 0$, i.e., $N \to \infty$, we expect our solution to approximate, the true value of u with increasing accuracy. However, it is important numerically, for a linear system to be well conditioned. Note that A_N is real and symmetric, hence Hermition. Therefore A_N is normal. We can therefore find its condition number using eigenvalues. A natural question arises: how to find the eigenvalues? There is a trick, we consider a matrix (which we denote by B_{N-1}) which agrees with A_N, except that the entries of B_{N-1} at the extremities of the principal diagonal are -1 instead of 0. Then B_{N-1} is *circulant*. Hence we can diagonalize B_{N-1}, and determine the eigenvalues.

For the simple equation (P), we were able to explicitly diagonalize the matrix A_N, arising out of F.D. approximation. Partially this was due to the fact that, the operator L defined by Lu=-u″ is *translation invariant*. This and a bit of luck enabled us to obtain the eigenvalues of A_N .More generally an operator L of the form

$$L(u)(t) = \sum_{j=1}^{N} b_j \frac{d^j}{dt_j} u(t) \text{ where } b_j \text{; is constant, is translation invariant. If the}$$

coefficients b_j are allowed to vary with t (time dependent) the operator L will not be translation invariant. For such an operator, the matrix A arising out of FDM, in the solution of $Lu = f$; with BC's $u(0) = u(1) = 0$ on $[0,1]$, will not be, even close to circulant. So we cannot follow the methods applied to A_N above. An alternative approach using wavelets, that includes the variable coefficients, due to Galerkin, comes to our rescue.

Wavelet-Galerkin Method for ODE's:

For certain class of equations, using wavelets in conjunction with the Galerkin method, gives the two primary desired features for the associated linear systems: sparseness and low condition number of the matrix. We consider the class of ODE's (known as *Sturm – Liouville's problems*). A typical SLP

$$(Q)\left\{ Lu(t) \;=\; -\frac{d}{dt}\left(a(t)\frac{du}{dt} \right) + b(t)u(t) = f(t), 0 \le t \le 1 \right.$$

along with Dirichlet's boundary conditions $u(0) = u(1) = 0$ is chosen as a "model equation" for the purpose of illustration. Incidentally, SLP's are named after J.C.F.Sturm and J.Liouville who studied such problems in 1830's. Here a, b, f are continuous and have continuous derivatives on [0, 1]. We assume that the operator L is *uniformly elliptic* on [0, 1]. By a well-known result (Picard's theorem) there is a unique solution u satisfying (Q). The simplest case is $a(t) = b(t) = 1$, which reduces to (P). For the Galerkin method, we suppose that $\{v_j\}$ is a complete orthonormal subset of $L^2[0,1]$ and that every v_j is C^2 on [0, 1] and satisfies the BC's of (Q) i.e.,

$$v_j(0) = v_j(1) = 0 \text{----------} (1)$$

We select some finite set Λ of indices j and consider the subspace $S = span\{v_j, j \in \Lambda\}$.

We look for an approximation to the solution u of (Q) in the form

$$u_s = \sum_{k \in \Lambda} x_k v_k,\ \mathrm{x_k} \in \square \text{ ----------} (2)$$

Our criterion for determining x_k, is that u_s should behave like the true solution on the subspace S,

i.e., $\langle Lu_s, v_j \rangle = \langle f, v_j \rangle \ \ \forall j \in \Lambda \text{----------} (3)$.

Notice that u_s satisfies automatically the BC's: $u_s(0) = u_s(1) = 0$.

It turns out that, u_s is the "best approximation" of u in S, w.r.t. the norm induced by $\langle f, g \rangle = \langle L(f), g \rangle$.

If we substitute, equation (2) in equation (3), we obtain

$\left\langle L\left(\sum x_k v_k\right), v_j \right\rangle = \langle f, v_j \rangle \ \ \forall j \in \Lambda$

Equivalently, $\sum \langle Lv_k, v_j \rangle x_k = \langle f, v_j \rangle \ \ \forall j \in \Lambda \text{----------} (4)$

Let x = $x = [x_k]_{k \in \Lambda}$, $y = [y_k]_{k \in \Lambda}$ and $A = [a_{j,k}]_{j,k \in \Lambda}$; where $a_{j,k} = \langle Lv_k, v_j \rangle$

Then equation (4) is the system of linear equations represented by

$$Ax = y \text{----------} (5)$$

We expect that as we increase our set Λ, our approximation u_s should converge to the true solution u. Now, our main concern is the nature of the linear system, resulting from the choice of wavelet basis as opposed to some other basis, for example, Fourier basis. Numerically, there are two 'requests' rather than demands, we would like A to fulfill, (as

already discussed): A should be sparse and it should have a low condition number. In fact, the best case is, A is diagonal, but the next best case is, A is sparse. To this end , we assume the possibility of modifying the wavelet system for $L^2(\Omega)$, so as to obtain a complete orthonormal system for $L^2[0,1]$. Clearly $\forall j,k \in \Gamma$, y_{jk} is a C^2 function and satisfies the BC's $y_{jk}(0) = y_{jk}(1) = 0$.

Now, we rewrite the equation (2) using the fact that the wavelets are indexed by two integers, in the form

$$u_s = \sum_{j,k\in\Gamma} x_{jk} y_{jk} \quad \text{and equation (4) as}$$

$$\sum_{j,k\in\Gamma} \langle L y_{jk}, y_{lm} \rangle x_{jk} = \langle f, y_{lm} \rangle \text{----------} (6)$$

We can still regard this, as a matrix equation $Ax = y$, where x and y are indexed by the pairs (j,k) and $A = [a_{l,m;j,k}]$ defined by $a_{l,m;j,k} = \langle y_{jk}, y_{lm} \rangle$.

Actually, A itself does not have a low condition number, but we can replace the system $Ax = y$ by an equivalent system $Mz = v$ by a "preconditioning" technique. Thus the matrix in the preconditioned system has a condition number bounded independently of the set Λ. So, as we increase the size of Λ, we achieve more and more accuracy, the condition number remaining bounded, all the while. This is much better than FDM in performance, in which case the condition number grows quadratic ally. This proves the justifiability of the use of Wavelet – Galerkin method, the algorithm of which, literally frees from the worry of step-size and round-off error.

One might be tempted to ask: Are wavelets indispensable, in the sense that, can they not be replaced by other systems, without compromising on numerical efficiency and accuracy? The answer is, indeed yes. There are, complete orthonormal systems for which a similar preconditioning can be done, to yield a bounded condition number. The Fourier basis is one such example, but they don't satisfy the BC's since $e^{2\pi \text{int}} \neq 0$ for any n. But $\{\sqrt{2}\sin 2\pi nt\}$ certainly qualifies for the trial. So although we see the advantage of the Galerkin method over FDM, the advantages over the Fourier basis are mainly due to the compact support of wavelets, which plays a role in establishing the sparseness of the matrix. In conclusion, we can say that the matrices we obtain using FD are sparse, but have large condition number. Using Galerkin method with Fourier basis, we can obtain a bounded condition number, but the matrix is no longer sparse. Using Wavelet-Galerkin method we can achieve both: the sparseness and boundedness of the condition number of a matrix, it is something like having best of both the worlds!

5.5 Numerical Solutions of Partial Differential Equations:

An Overview of Conventional and Classical Methods:

I – Finite Difference Method (FDM):

The basic idea underlying all the different methods is to replace the partial derivatives in a differential equation by suitable difference quotients and to solve the discretised equation so obtained.

We illustrate this with the following Dirichlet's boundary value problem, (P) given by:

$$\begin{cases} u_{xx} - u_{yy} = f(x,y),\ 0 < x,y < 1 \\ u(x,y) = 0,\ \forall (x,y) \in \partial\Omega \end{cases}$$

For the unit-square $\Omega = \{(x,y) \mid 0 < x, y < 1\} \subset \mathbb{R}^2$. We assume that $f(x,y)$ is continuous on $\Omega \cup \partial\Omega$. Since the various methods, for the solution of BVP's are usually compared with this problem, (P) is also called the "model problem".

To solve (P) by means of difference method, one replaces the differential operators by difference operators for BVP, as described in ordinary differential equation, above.

One covers $\Omega \cup \partial\Omega$ with a grid (mesh): $\Omega_h \cup \partial\Omega_h$; where,

$$\Omega_h = \{(x_i, y_j) \mid i, j = 1,2,--N\},\ \partial\Omega_h = \{(x_i,0),(x_i,1),(0,y_j),(1,y_j) \mid i, j = 0,1,--N+1\}$$

We abbreviate, $x_i = ih, y_j = jh,\ u_{ij} = u(x_i, y_j),\ i, j = 0,1,--N+1$ and $h = \dfrac{1}{N+1}$. Then

$\forall (x_i, y_j) \in \Omega_h\ -u_{xx} - u_{yy}$; can be replaced by the difference operator;

$\dfrac{4u_{ij} - u_{i-1,j} - u_{i+1,j} - u_{i,j-1} - u_{i,j+1}}{h^2}$ up to an error t_{ij}

Because of the boundary condition the $u_{ij} = 0$ are known for $(x_i, y_j) \in \partial\Omega_h$

Therefore the unknowns $u_{ij}\ 1 \le i, j \le N$ obey the system of linear equations of *th* form.

$$4u_{ij} - u_{i-1,j} - u_{i+1,j} - u_{i,j-1} - u_{i,j+1} = h^2 f_{ij} + h^2 t_{ij}, \forall (x_i, y_j) \in \Omega_h$$

$$\text{with } f_{ij} = f(x_i, y_j) \text{----------(1)}$$

Here the errors t_{ij} depend on the mesh size *h*. Under appropriate assumptions, for the exact solution *u*, one shows that $t_{ij} = O(h^2)$. For sufficiently small *h*, one can expect that the solution: $z_{ij},\ i, j = 1,2,--N$ of linear system of equations,

$$4z_{ij} - z_{i-1,j} - z_{i+1,j} - z_{i,j-1} - z_{i,j+1} = h^2 f_{ij},\ i, j = 1,2,--N$$

Where $z_{0j} = z_{N+1,j} = z_{i,0} = z_{i,N+1} = 0$, for $i, j = 0,1,--N+1$----------(2)

obtained from (1), omitting the error terms t_{ij}, agrees approximately with u_{ij}.

To every grid point $(x_i, y_j) \in \Omega_h$, there corresponds exactly one component z_{ij} of the solution of (2).
Collecting N^2 unknowns z_{ij} and the right sides of $h^2 f_{ij}$ row-wise into vectors:

$z = (z_{11}, z_{21}, -- z_{NN})^T$, $b = h^2 (f_{11}, f_{21} -- f_{NN})^T$ equation (2) is equivalent to a system of linear equation of the form $Az = b$, with $N^2 \times N^2$ matrix A. It may be noted that A is quite sparse.

II – Finite Element Method (FEM):

Finite difference method can be ill-suited for other than regular rectangular regions. But the world is not regular! The FEM is much better adapted to irregular regions. One advantage of the FEM over FDM is the relative ease with which BC's of the problem are handled. Many physical problems have BC's involving derivatives and irregularly shaped boundaries. BC's of this type are difficult to handle using FD techniques, since each BC involving derivatives must be applied by a difference quotient at the grid points and irregular shape of the boundary makes placing of the grid points difficult, if not impossible. FEM uses a series of complex computations. But computers can take care of that. Such methods are of great importance and growing popularity in solving PDE's. FEM's are especially useful when the domain of BVP's are two or three dimensional and particularly for irregular regions. Using finite elements also facilitates local mesh refinement, in the parts of the region where the variables of interest vary rapidly or where discontinuities occur.
In breaking a 2-D region into sub-regions, there is wide choice of elements that span the region. Truly, rectangles can be used, but they don't fit as well to irregular boundaries as triangles do. Other polygonal shapes are possible and even elements with curved boundaries may be considered. But, these are considerably more complex to use. Triangles are very popular, or a mixture of triangles and quadrilaterals is most commonly used. Although in many respects FEM is quite similar to FDM, FEM is preferable when it comes to complicated geometries in several space dimensions. FEM's are often preferred by practitioners in structural engineering, for the simple reason that, these methods allows considerable freedom in putting elements, wherever one wants them. FEM on the whole is as efficient and as accurate as the best FD scheme with the "bonus" of reduced computer memory requirements.

III – Spectral methods (SM):

Assume one has to find an unknown function u, satisfying a differential equation. SM starts by expanding the function u in a series of Eigen functions of a SLP. Then using orthogonality and various properties of "special functions" one may define approximations to the derivatives of u and employ them to compute u. In practice, the Eigen functions will be usually trigonometric functions or orthogonal polynomials.
As a simple example consider the classic wave equation:

$$\begin{cases} u_t = u_x, t > 0 \\ u(x,0) = f(x) \quad ----------(1) \\ f(x) = f(x+2p) \end{cases}$$

The solution is periodic, suggesting that a Fourier series, rather than a polynomial expression is appropriate.

$$u_N(x) = \sum_{k=0}^{N-1} a_k(t) e^{ikx} ----------(2)$$

$$\frac{\partial u_N}{\partial x} = \sum_{k=0}^{N-1} a_k(t) ike^{ikx}$$

and (1) reduces to $\frac{da_k}{dx} = ika_k,\ 0 \le k \le N-1$

The only input needed for solving the above equation is the set of initial values $a_k(0)$. These are *the Fourier coefficients* of the function f; they are defined by

$a_k(0) = \frac{1}{N}\sum_{l=0}^{N-1} e^{-ik2pl/N} f\left(\frac{2pl}{N}\right)$, which can be computed in $O(N \log N)$ operations using FFT.

An alternative approach, but generalisable to arbitrary orthogonal series is the so-called Galerkin method.

Substitute (2) in (1), multiply by e^{ikx} for $k = 0,1,----N-1$ and integrate over the period. We now impose a BC on the wave equation: $u(1,t) = y(t)$. A Fourier series is no longer adequate, for the solution of the problem. So we expand the function u in terms of *Chebyshev's polynomials* to find

$$u_N(x) = \sum_{k=0}^{N-1} b_{k(t)} T_k(x), \text{ where } T_n(x) = \cos\left(n\cos^{-1}(x)\right) ----------(3)$$

The formula for the derivative is more complicated since

$$\frac{\partial u_N}{\partial x} = \sum_{k=0}^{N-1} b_k(t) T_k(x) ----------(4)$$

where; $c_k b_k = \sum_{\substack{p=k+1 \\ p+k\ odd}}^{N} pa_p$, and $c_k = \begin{cases} 2, k = 0 \\ 1, k > 0 \end{cases}$

But the final equation for the coefficients is,

$\frac{d}{dt}(a_0, a_1 --a_{N-1})^T = A(a_0, a_1 --a_{N-1})^T$ for some known matrix A.

The initial values for the coefficients can again be taken as the Chebyshev coefficients of the initial data and can again be effectively computed using FFT. There is, however, an additional equation to be satisfied,

$$\sum_{k=0}^{N-1} a_k(t) T_k(1) = y(t) ----------(5)$$

Finally, consider a wave equation with variable coefficients,

$$\begin{cases} u_t = \dfrac{\partial}{\partial x}\big(e(x)u\big); |x| < 1, t > 0,\ u(x,0) = f(x),\ u(1,t) = y(t) \end{cases}$$

One should again approximate *u* by an N^{th} degree polynomial, but the coefficients of $e(x)u$ cannot be usually be defined in terms of the coefficients of *u*. This forces us to adopt a different approach.

Take *N+ 1 point* $x_0, x_1 - - x_N$ in [0, 1]. These define a unique polynomial of degree N, which is identical with u at the points x_i and represents the interpolant of u. We now replace $c(x)u$ by the interpolant of $c(x)u$ and compute its derivatives to advance the solution of the problem in time. The BC is satisfied by having

$x_0 = 1$ and by setting $u(x_0, t) = y(t)$.

This procedure, which can obviously be applied to non-linear equations too and is called *Collocation method or Pseudo-spectral method*.

It is more general than the Galerkin method mentioned above. However, this is an efficient numerical needing $O(N^2)$ operation. For a special set of collocation points, the matrix multiplication can be done by FFT in $O(N \log N)$ operations. This is, in fact, one of the reasons why trigonometric functions and Chebyshev polynomials are usually employed in spectral methods. We have singled out collocation methods using trigonometric functions or Chebyshev polynomials as the most useful for two reasons.

1. They may be used variable coefficient and non-linear problems.
2. They allow FFT to be implemented.

We now address, rather precisely, the question: why we use spectral methods?

To answer this, we remark that, the spectral approximation obtained in the above three types of equations, differs from the exact solution of the problem, the quantity:

$\sum_{k=N}^{\infty} a_k e^{ik(x+t)}$; where a_k are the Fourier coefficients of y . Now, if we assume that y is smooth i.e., one possessing continuous derivatives of all orders, it turns out that the coefficients decay faster than any power of *k*. Therefore, a spectral method using *N* modes when applied to a smooth function will admit an error estimate $< \dfrac{C(M)}{N^M}$ for any *M* with some constant $C(M)$. In contrast, a FDM with *N* grid points will usually have an error of the form $\dfrac{C}{N^p}$ where $p = 2$ for 2nd order method.

Computationally, it means that one can obtain, a very accurate solution, using few data points. We should also put a word or two about, non-smooth functions. For instance, consider the Euler's equation of aerodynamics, which produces solutions with shock waves, contact discontinuities. In this case, one cannot expect high accuracy, may be due to Gibb's phenomenon which occurs at discontinuities will produce an oscillating error component which does not vanish as $N \to \infty$.

In fact, the Gibb's phenomenon itself may serve as an error locator, accurately pinpointing sharp transitions in the solution. Although, there exist several satisfactory spectral methods for the treatment of non-smooth solutions, and this area is very much in need of a firm theoretical basis and therefore, requires considerable research.

IV – Collocation Method (CM):

The collocation method, for approximating the solution of an equation of the second kind:

$f - Af = f \text{ ----------} (1)$ Consists of seeking an approximate solution function in a finite dimensional subspace of $C[a,b]$ by requiring that, the equation (1) can be satisfied at only finite number of so-called *collocation* points.

Assume that $A: C[a,b] \to C[a,b]$ is a bounded linear operator and

$X_n = span\{u_0^n, u_1^n, --u_n^n\} \subset C[a,b]$; denote a sequence of subspaces with

$\dim X_n = n+1$, Choose $n+1$ points: $a = x_0^n < x_1^n < -- < x_n^n = b$ in such a manner that the interpolation at these grid points w.r.t the subspace X_n is uniquely possible. Typical examples of elements of X_n are $B-$ splines, polynomials, trigonometric polynomials etc. For convenience, we write x_i instead of x_i^n, u_i instead of u_i^n.

$L_n : C[a,b] \to X_n$, we mean an operator, that map $f \in C[a,b]$ into uniquely determined interpolating polynomial. $L_n(f) \in X_n$, with the property that;

$(L_n f)(x_j) = f(x_j)$, $j = 0,1, -- n$; Representing L_n in terms of Lagrange's basis i.e., $l_0, l_1 -- l_n \in X_n$ such that $l_k(x_j) = \delta_{jk}$, $j,k = 0,1, -- n$, in the form $L_n f = \sum_{k=0}^{n} f(x_k) l_k$.

It can be seen that, L_n is a bounded linear operator, moreover, $\because L_n f = f$, $\forall f \in X_n$ the operator L_n is a projection operator implying $L_n^2 = L_n$.

The collocation method approximates the solution of (1) by an element $f_n \in X_n$ satisfying $f_n(x_j) - (Af_n)(x_j) = f(x_j)$, $j = 0,1, -- n \text{ ---------} (2)$

We express $f_n \in X_n$ as a linear combination $f_n = \sum_{k=0}^{n} g_k u_k$ and immediately see that (2) is equivalent to the linear system

$$\sum_{k=0}^{n} g_k \{u_k(x_j) - A(u_k)(x_j)\} = f(x_j),\ j = 0,1, -- n \text{ ----------} (3),$$

for the coefficients g_k. If we use Lagrange's basis for X_n and write

$f_n = \sum_{k=0}^{n} g_k l_k$ then of course $g_j = f_n(x_j)$, $j = 0,1, -- n$ and (3) becomes

$$g_j - \sum_{k=0}^{n} g_k A(l_k)(x_j) = f(x_j)$$

So the basic idea behind, collocation methods can be summarized as follows.

One tries to approximate the solution by an element $f_n \in X_n$, as a linear combination of its basis elements. To this end one selects a finite number of collocation points and determines f_n such that the differential equation is to be satisfied exactly at these points. There are many possibilities of implementing such methods, by different choices X_n, of bases of X_n, and of collocation points. With a suitable choice, one may obtain very efficient methods.

Among other methods of descretization, *Finite Volume Method* and *Multi(Adaptive) Grid Method* deserve mention. In FVM the assumed solution is integrated over the space-time rectangle and approximated using Trapezoidal rule. In AGM we refine the grid locally where the solution is difficult to approximate and use a coarse grid, where the solution is easy to approximate. One can also think of changing both grid and underlying basis and the method as well. Little has been done in this direction for PDE's. Of the many different approaches to solving PDE's, much of the work that is being done today, involves non-linear equations. Often the method developed to solve a non-linear PDE does not serve the purpose. In addition, the discrete version of the equation may fail to be solvable.

5.6 Wavelet Calculus and Connection Coefficients:

Wavelet – Galerkin Method for PDE's

As we have seen, one can use wavelets to represent a function of one or more variables, which can be either an infinite series or an approximating finite sum. The role of Calculus in computing derivatives and solving D.E's cannot be over-emphasized, in modeling the physical world. Against this background, the question before us is: How can one compute efficiently derivatives of functions in terms of wavelet expansion coefficients? How can they be used in solving D.E's numerically? The answer to these can be given, using the concept of *Connection coefficients.* We will illustrate with a special and useful case, which will ultimately lead to the general case. We are going to show that the knowledge of connection coefficients is an essential ingredient in the Algebra and Calculus of wavelet expansions.

To begin with, consider a wavelet system $W = \{f_k, y_{jk}\}$. Let f be a smooth L^2 function, whose derivative f' is also L^2 (i.e., f is an element of Sobolev space $H^1(\square)$). Represent f in a wavelet series and differentiate:

$$f(x) = \sum f_k f_k(x) + \sum f_{jk} y_{jk}$$

$$f'(x) = \sum f_k f'(x) + \sum f_{jk} y'_{jk}(x)$$

Expand $f'_k(x)$ and $y'_{jk}(x)$; again in wavelet series in the form:

$$f'_k(x) = \sum_l \Gamma^l_k f_l + \sum_{il} \Gamma^{il}_k y_{il}$$

$$y'_{jk}(x) = \sum_l \Gamma^l_{jk} f_l + \sum_{il} \Gamma^{il}_{jk} y_{il}$$

The formulas for $\Gamma's$, are uniquely determined (using orthogonality):

$$\Gamma^l_k = \int f'_k(x) f_l(x)\,dx;\ \ \Gamma^{il}_k = \int f'_k(x) y_{il}(x)\,dx$$

$$\Gamma^l_{jk} = \int y'_{jk}(x) f_l(x)\,dx;\ \Gamma^{il}_{jk} = \int y'_{jk}(x) y_{il}(x)\,dx$$

We call these expressions connection coefficients for *W*.

We state some useful properties and theorems [5] which show that, all these C.C's can easily be computed as linear combinations of the *Fundamental C.C's*

$\{\Gamma^k_0\}; k = 1, --2g-2, g$ being the genus of the wavelet system:

1) $\Gamma^m_l = -\Gamma^l_m$ 2) $\Gamma^{jk}_l = -\Gamma^l_{jk}$ 3) $\Gamma^{mn}_{jk} = -\Gamma^{jk}_{mn}$

Theorem: If $j > 0$, then 1) $\Gamma_0^{0k} = \sum_{k,l} a_k b_l \Gamma_k^{2k+l}$ 2) $\Gamma_0^{jk} = \sqrt{2}\sum a_l \Gamma_l^{j-1,k}$

3) $\Gamma_{00}^{jk} = \sqrt{2}\sum_l b_l \Gamma_l^{j-1,l}$

First, we look at the special cases of Haar and Daubechie's wavelets.

1) $\Gamma = \left(-\frac{1}{2}, 0, \frac{1}{2}\right),\ g = 1$ 2) $\Gamma_0^l = \left(\frac{1}{12}, \frac{2}{3}, 0, \frac{2}{3}, -\frac{1}{12}\right),\ g = 2$

The Wavelet-Galerkin Approximation:

We want to formulate a wavelet-Galerkin approximation to the solution of the problem:

Let Ω be an open set in $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$.

Given $f \in L^2(\Omega)$, and $g \in H^{1/2}(\partial\Omega)$ find a function $u \in H^1(\Omega)$:

$-\Delta u + u = f$ in Ω; $u = g$ on $\partial\Omega$.

Let us consider an approximation solution to the actual solution;

u_ϵ given by;

$$u_\epsilon^j(x,y) = \sum_{(p,q)\in\Lambda_D^j} u_{p,q}^j f_p^j f_q^j$$

where; $\Lambda_D^j = \left\{(p,q) \in \mathbb{Z}\times\mathbb{Z} : \sup p\left(f_p^j f_q^j\right)\right\}$

The goal is to apply Galerkin method to obtain a linear system for the unknowns u_{pq}^j

We use the connection coefficients $\Gamma_m^l = \int_{\mathbb{R}} f_l'(x) f_m(x)\,dx$.

Then using the orthonormality of the wavelet basis and the properties of connection coefficients, the desired linear system for unknown u_{pq}^j's follows. We call the solution of this linear system the wavelet-Galerkin approximation to the actual solution of the Dirichlet problem.

Wavelets can be used as basis functions for Galerkin method, as already mentioned above. If we just do this at one level, they have no particular advantage over other types of functions. Their real power lies in the multiresolution/scale approach.

Multiscale Representation

e.g. $\partial^2 u/\partial x^2 = f$

Expand as

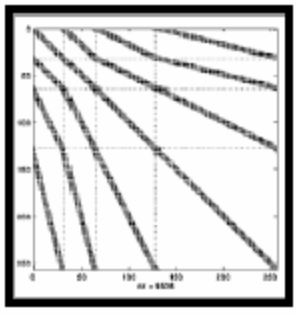
$$u = \sum_k c_k \phi(x-k) + \sum_{j=0}^{J}\sum_k d_{j,k} w(2^j x - k)$$

Galerkin gives a system

$Ku = f$

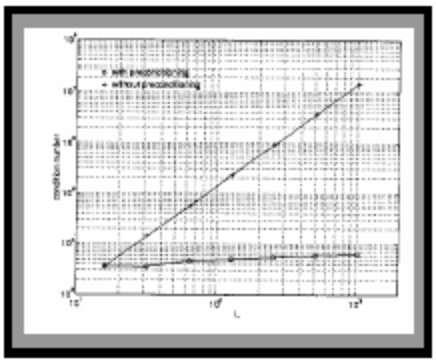
with typical entries

$$K_{m,n} = 2^{2j}\int_{-\infty}^{\infty} \frac{\partial^2}{\partial x^2} w(x-n)\, w(x-m)\,dx$$

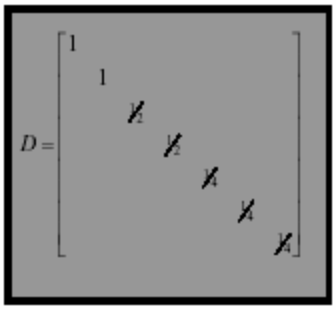


Effect of Preconditioner

- Multiscale equations: $(WKW^T)(Wu) = Wf$
- Preconditioned matrix: $K_{prec} = DWKW^TD$



Simple diagonal preconditioner



5.7 Wavelets in Integral equations:

Recently wavelet based methods have been developed for the numerical solution of linear and non-linear integral equations of both Volterra and Fredholm types.

Let us first make a general observation about the representation of a linear operator T and wavelets. Suppose that f has the representation

$$f(x) = \sum_{j,k} \langle f, \psi_{j,k} \rangle \, \psi_{j,k}(x).$$

Then,

$$Tf(x) = \sum_{j,k} \langle f, \psi_{j,k} \rangle \, T\psi_{j,k}(x),$$

and, using the wavelet representation of the function $T\psi_{j,k}(x)$, this equals

$$\sum_{j,k} \langle f, \psi_{j,k} \rangle \sum_{i,l} \langle T\psi_{j,k}, \psi_{i,l} \rangle \, \psi_{i,l}(x) = \sum_{i,l} \left(\sum_{j,k} \langle T\psi_{j,k}, \psi_{i,l} \rangle \, \langle f, \psi_{j,k} \rangle \right) \psi_{i,l}(x).$$

In other words, the action of the operator T on the function f is directly translated into the action of the infinite matrix $A_T = \{ \langle T\psi_{j,k}, \psi_{i,l} \rangle \}_{i,l,j,k}$ on the sequence $\{ \langle f, \psi_{j,k} \rangle \}_{j,k}$. This representation of T as the matrix A_T is often referred to as the "standard representation" of T [15]. There is also a "nonstandard representation". For virtually all linear operators there is a function (or, more generally, a distribution) K such that

$$Tf(x) = \int K(x,y) f(y) \, dy.$$

The nonstandard representation of T is now simply the decomposition one gets by considering K as an image and calculate the 2D wavelet transform.

5.8 Conclusion:

There are several procedures that come under the banner of numerical methods. The usual procedures employed to find the numerical solutions of DE', as noted above, are FDM, FEM, SM, CM, FVM, AGM and Galerkin methods.Spectral methods are preferred, for very regular geometries and smooth functions. They converge more rapidly than FDM's and FEM's, but they do not work well, for problems with discontinuities. The philosophical view-point that runs through out all these methods, has been to view the PDE as an operator equation $L(u) = f$. The next step is, to identify the solution

subspaces S_1, S_2 -- on which L^{-1} is continuous. From this point of view, it is taken as an 'article of faith' in Numerical Analysis that, continuous operators can be approximately discretised, to yield stable and convergent schemes. The strategy to achieve stability and convergence is either to approximate the solution by replacing DE's by a system of algebraic equations or to approximate the solution spaces involved, by finite dimensional spaces. Solving algebraic equations, in turn amounts to minimizing the condition number of the underlying matrix, using properties of basis elements. Constructing finite dimensional spaces, on the other hand, involves the use of exponential functions, B-splines or wavelets as basis functions. Well, the crucial question is: How accurate are the approximate solutions? The situation with FDM and FEM is not bad. But, the problem of tracking down singularities of the solutions still looms large over numerical-analysts. This is the proverbial "tough nut to crack". Once, this is resolved, refinement of the mesh in and around the localities of singularities can be performed. This results in minimal increase in the dimension, guaranteeing, at the same time, an enhanced accuracy of the approximate solution and reducing CPU time. In recent years, several advanced techniques have been developed, to take advantage of the high-speed computing machinery coupled with enormous storage capacity. Added to this, in view of the widespread accessibility of today's computing techniques, numerical methods are becoming increasingly popular. However, the fact remains that, the availability of powerful computers is gradually shifting the focus away from the analytical solution and towards both numerical analysis and the qualitative theory. Non-linear problems in Fluid-Mechanics are being solved today, that were not even considered, ten years ago. Interestingly, many equations modeling the rich variety of physical phenomena are non-linear. It has been realized that, numerical methods with strong stability are *economic* and *trustworthy* from computational view-point. Infact, to obtain accurate numerical solutions to DE's governing the physical systems, has always been an important problem engaging the minds of scientists and engineers.

Turning our attention, to the solution of PDE's, it must be pointed out that, there is no general theory concerning the solvability of all PDE's. Such a theory is unlikely to exist, given virtually endless variety of physical, geometrical and probabilistic phenomena which can be modeled by PDE's. However, the localization property of wavelets comes to our rescue, in resolving many of the difficulties encountered in the traditional methods. Secondly, the fact that a wavelet system nearly diagonalizes a very broad class of continuous operators is one of the key properties of wavelets. Normally, in FE approaches to a BVP, a finite element grid must be adapted to the boundary. But, this not necessary for the wavelet approach, which gives it great flexibility in applications. Can one, then develop a non-linear Galerkin method based on wavelets? There are numerical evidences and mathematical justifications, for answering this in affirmative. By using, for instance, connection coefficients a non-linear equation can be reduced to a system of non-linear algebraic equations, which can be solved iteratively. Moreover, the inherent wavelet structure permits for a natural *multigrid* solution, of the corresponding algebraic equations. This is, perhaps, one of the major issues confronting the scientific community, in the foreseeable future. All said and done, the use and subsequent effect of wavelets for differential and for integral equations. For differential equations, finite element matrices are already sparse, but they tend to be ill-conditioned. Integral equations, on other hand lead to dense matrices. However, these matrices tend to be well-conditioned. The focus of the current research is, on the discovery of ways and means to solve differential and integral equations of all sorts, within and outside the realm of mathematics.

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