III-Wavelets - A mathematical tool for signal processing and Image compression (An Overview)

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Abstract:

With the advent of modern technologies such as digital signal processing, digital satellite imaging and computer simulation programs, the amount of data available to researchers and scientists has grown enormously. To this end *wavelets* have been employed in many fields of study to make large databases more manageable; while compromising little in terms of accuracy and true representation of the original data. In particular, the use of *multiresolution analysis* (MRA), allows the user to approximate the original data, to any desired level of accuracy, and provides a convenient way to analyze and store only as much information as needed.

3.1 A Brief Preview:

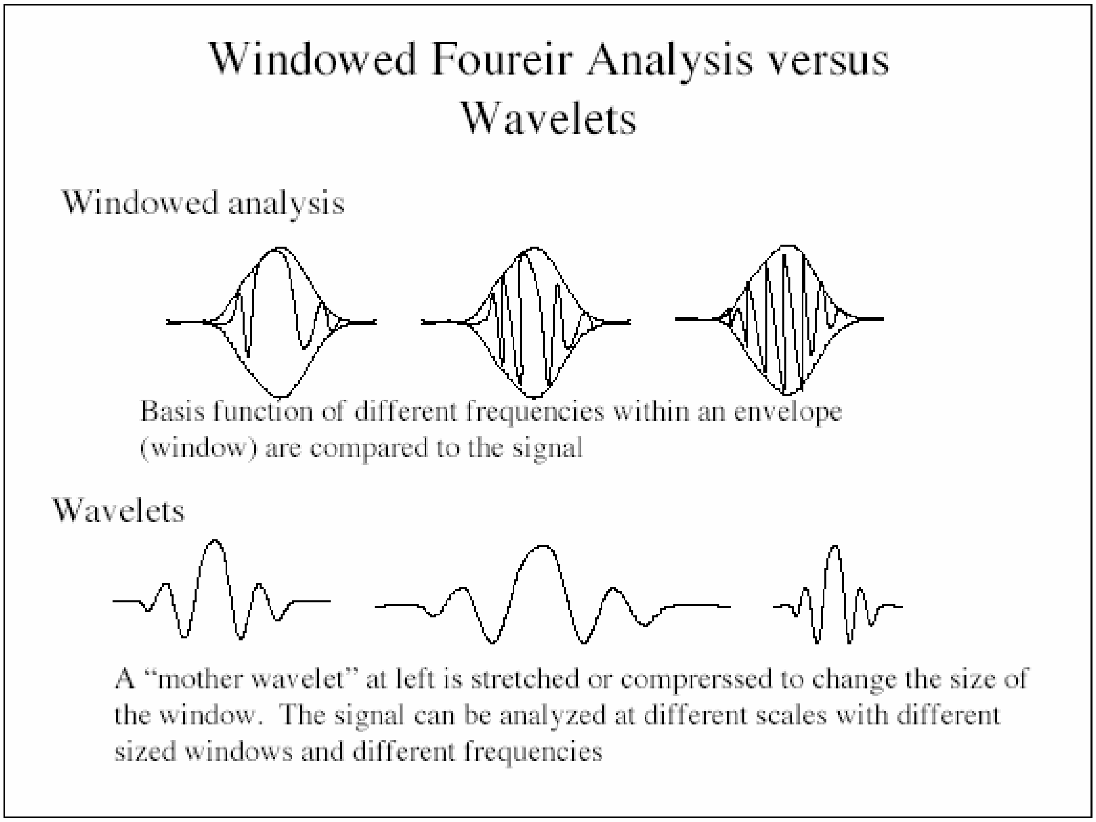
In the process of communication, one needs to represent and analyze a signal or an image. A musical note/speech, an electrical signal, a satellite image are some examples of signals and images. In most real world applications, we come across signals which are analog in nature. One of the classical tools in representing/analyzing analog and digital signals is *Fourier analysis* where a signal is represented as superposition of waveforms. The sinusoidal waves work as basic building blocks in Fourier analysis of signals. Although, the *Fourier transform* is a powerful tool for such signals and has been in use for more than a century, it becomes inadequate and unsatisfactory, when one is interested in analyzing local frequency content of a signal. Keeping in view of the transient nature of a large class of signals, a need was felt to localize the analysis and make it more useful and cost effective. Gabor transform and Short Time Fourier Transform were subsequently introduced for local time-frequency analysis. But, they had their own limitations. Wavelet analysis arose out of a need to represent signals localized in both time/space and frequency/wave-number. Applications of this field include signal and image processing, data compression, numerical methods etc. Therefore, Wavelet Analysis witnessed a tremendous growth during the past decade.

Current research is to a large extent motivated by industrial and commercial applications of Wavelet analysis, driven by globalised economy. These revolve naturally around signal and image processing. But the signal and images arrive from diverse fields: Astronomy, Geophysics, Biology, Meteorology, Medicine, Hydrodynamics and of course Telecommunications, to mention just a few. In all cases one wishes to extract from the signal the pertinent information as discrete numerical values. This set of digital information must be rich enough to characterize the signal, but it should not be larger than the necessary task at hand. If, for example, it is a question of speech over a telephone, then one wants enough numerical information at the receiver end, to reconstruct a recognizable voice. But economy dictates the need to minimize the amount of information that must be stored/transmitted. With the advent of television and transmission of digital pictures over the internet, there is a great deal of interest in compressing audio/video signals. For instance, if the image is broken into grid of 200 pixels by 250 pixels, there will be 50,000 gray-scale values to assign, so the image is represented in a computer as vector of length 50,000. This is a data of enormous size, so we may need to compress it for the purpose of storage, transmission and if necessary retrieval.

3.2 A long and fascinating story:

Fourier analysis is the oldest of various techniques available for signal analysis and synthesis. In 1965, two American scientists Cooley and Tukey developed an algorithm known as Fast Fourier Transform (FFT), one of the major technological breakthroughs of twentieth century. Since the invention of the FFT, it has become an efficient tool and has enjoyed enormous success for analyzing sufficiently smooth periodic signals. In theses cases the Fourier coefficients c_n decrease rapidly as $|n| \to \infty$ and relatively few numerical coefficients are needed to reconstruct the signal for most practical purposes. Unfortunately as soon as the signal becomes irregular, for example, a transient, and the number of coefficients necessary to reconstruct the signal (and hence the amount of data that must either be stored or transmitted) becomes too large & often economically impractical. Before the invention the FFT Fourier analysis was mainly theoretical tool and indeed one of the most important and all pervasive. This quickly changed with the arrival of the FFT, a marvelous discovery in efficient digital computing. In the last quarter of twentieth century this technique has given scope for wide ranging applications. In fact, it has been the back bone of signal and image processing. Nevertheless, even with FFT and modern computing, Fourier analysis does not provide a satisfactory analysis for all kinds of signals. Although the FT $\hat{f}$ contains all the information about f, much of the information is hidden. For example, none of the temporal (spatial) aspects of the f are revealed by $\hat{f}$. If f is a finite signal, the spectrum does not indicate the beginning and end of the signal and if there is a singularity, the time of occurrence is hidden throughout $\hat{f}$. Faced with these kinds of issues one would like to have an analytical tool that provides information both in time and frequency. These technical constraints simply motivated researchers to refine existing tools and develop new ones. One of the first ideas was to truncate the signal and analyze only what happens in a finite interval [-A, A]. One is forced to do this when making numerical computations. Mathematically these amount to multiplying the signal f by characteristic function $c_{[-A,A]}$ and taking Fourier transform of the product. Unavoidably, the computations for this process quickly become voluminous.

To overcome these problems one replaces $c_{[-A,A]}$ with more regular functions called *Windows.*



Gaussian window $w(t) = Ae^{-at^2}\ (a, A > 0)$ among others served as a model. One is led to naturally slide this window along the graph of a function and there by analyze the whole function. One then obtains a family of coefficients depending on two real variables l and b given by;

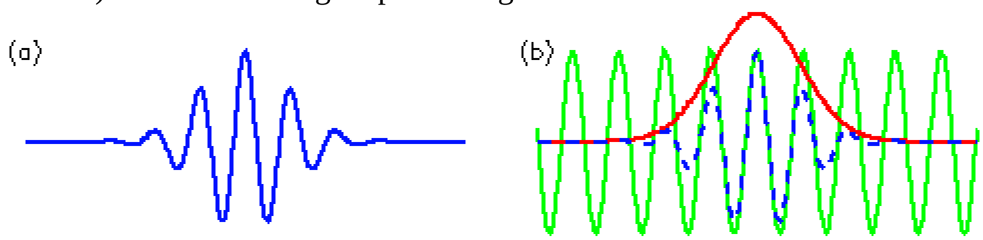
$$W_f(l, b) = \int_{-\infty}^{\infty} f(t)\overline{w}(t-b)e^{-2ip l t}dt \text{ ----------------------------- } (1)$$

Which replaces $\hat{f}(l)$. The mapping $f \to W_f$ is called sliding window Fourier transform or simply the Windowed Fourier transform (WFT). $W_f(l, b)$ thus provides an indication of how the signal behaves at time $t = b$ for the frequency l. We use the function $\overline{w}$ rather than w for reasons of convenience and because we wish to allow the possibility of windows being complex valued. Dennis Gabor's formula changed the course of history in signal processing. Intuitively, one might expect that knowing $W_f(l, b)$ completely determines the signal *f*. But the information contained in $W_f(l, b)$ is redundant. Since we have replaced a one parameter family $\hat{f}$ with a two parameter family. In his 1946 paper Dennis Gabor (Winner of 1971 Nobel prize in Physics) proved that our speculations are well founded. He used essentially the Gaussian window $w(t)$ which has the advantage of approximating the square window while avoiding the disadvantage of introducing abrupt discontinuities. One of the Gabor's important contributions was to show that $W_f(l, b)$ can be inverted to recover *f*, since in signal processing the crucial thing is the ability to recover *f* from its transform. In practice, one generally uses w that is well localized around t =0, for example, a Gaussian. Then w_{lb} is localized around t = b, while $\bar{w}_{lb}$ is localized around $z = l$ where $w_{lb}(t) = w(t-b)e^{2ipl t}$ $l, b \in R$. For numerical computations the coefficients $W_f(l, b)$ are evaluated on a grid (ml_0, nb_0) $m,n \in Z$ and $l_0, b_0 > 0$. We thus obtain a double sequence $W_{m,n}(f) = W_f(ml_0, nb_0)$ which is discretized version of $W_f(l, b)$. Fourier and Gabor analyses have their own relative merits and demerits. The transforms of Fourier and Gabor which we can write formally as;

$$\mathrm{f(t)} = \int_{-\infty}^{\infty} \hat{f}(z)e^{2ip tz}\, dz \quad \text{--------------------------------------(2)}$$

$$f(t) = \iint W_f(l, b) w_{lb}(t)\, dl\, db \quad \text{------------------------------ (3)}$$

over RxR, can be interpreted as decomposing the signal *f* in terms of function that plays the role of basis functions (Building blocks) except that the sums are replaced by integrals. In the FT these functions are sinusoids, in WFT they are modulated Gaussians (attenuated sinusoids). Next, in Fourier method the bases functions are fully concentrated in frequency and totally distributed in time (extending from $-\infty$ to ∞). This is another way of explaining that, taking the FT gives the maximum amount of information about the distribution of the frequencies but completely looses information relative to time. With WFT (or GFT) the time-frequency information remains coupled although there is a compromise. The Heisenberg's uncertainty principle (which won him 1931 Nobel Prize) limits its simultaneous localization in time and frequency. In spite of this which is fact of life, for any time frequency analysis WFT does have advantages over FT for certain applications. A signal *f* of finite duration (compact support) provides one of the best illustrations of the difference between the methods. The reconstruction of f using the IFT (Integral Fourier Transform) necessitates knowing the values of $\hat{f}$ with considerable precision over a very large range of values, for although $\hat{f}(z) \to 0$ it can be frustratingly slow. The situation is quite different for Gabor's analysis. If *f* vanishes on a long enough intervals then the coefficients $W_f(l, b)$ will be negligible for *b* in the neighborhood of b_0. On the other hand if *f* oscillates strongly at $t = b_0$ the value of $W_f(l, b)$ will be large for *b* near b_0 when the values of l match the frequency of *f* near b_0. This is how one can localize the frequency component of *f*. In spite of its advantages for certain applications the Gabor's method has the serous disadvantage that the size of the window is fixed. In terms of the uncertainty principle, this limits its ability to localize events in time. Apart from this, the most important property, the GFT does not possess is $\int_{-\infty}^{\infty} w(t)dt = 0$. This gives us an extra degree of freedom to introduce dilation (scaling) parameter in order to make time-frequency windows flexible. Problems arise when one wishes to analyze signals that contain features on scales that range over several orders of magnitude (i.e. tremor, turbulence) from macroscopic to microscopic. This led Jean Morlet to introduce new method where the window is not only translated but is also dilated and contracted. This was the beginning of the use of wavelets (serving as windows) for the use of signal processing.



(a) Morlet wavelet (b) Windowed wavelet

3.3 From MRA to Wavelets bases:

With wavelets we enter into a dynamic contemporary research environment, what is now known as Wavelet theory (Analysis). Starting with function y called the analyzing wavelet or "Mother wavelet" we construct the family of functions called "Baby-wavelets" given by $y_{ab}(t) = \frac{1}{\sqrt{a}} y\left(\frac{t-b}{a}\right)$, $b \in R$ $a > 0$. The wavelet coefficients of a signal f are the numbers $C_f(a,b) = \int_{-\infty}^{\infty} f(t)\bar{y}_{ab}(t)dt$ ---------------- (4)

A series expansion similar to a Fourier series except that here the series is double and *f* is not required to be periodic, given by Yves Meyer (1985) is

$$f = \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} \left\langle f, y_{j,k} \right\rangle y_{j,k} \text{ -------------------- (5)}$$

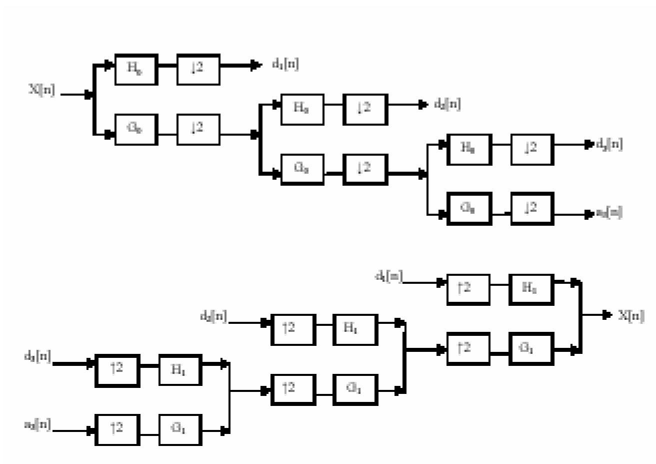
The wavelet first used by Morlet is $y(t) = e^{-\frac{t^2}{2}}\cos 5t$. The simplest example of wavelet is the Haar wavelet. The derivatives of Gaussian are widely used in practice as wavelets. An investigation of wavelets in general and Haar wavelets in particular is already done in some detail in articles I and II. Like Fourier transform wavelet transform (WT) is both a theoretical and practical tool, but unlike FT there is an opportunity to choose different analyzing wavelets depending on the problem at hand. In the light of what we know about wavelets associated with MRA, it allows us to look at different constructions of wavelets such as orthogonal, bi-orthogonal, semi-orthogonal and splines. Examining these constructions in a unified setting, we will be in a position to make comparisons between them.

On the discrete side, the discovery of Daubechies wavelets having compact support stands as a landmark in the theory. Remarkably there exists an orthonormal basis of the form. $2^{j/2}\Psi(2^j t-k),\ j,k\in Z,$ Such that Ψ_r has support

$[0,2_{r+1}]\, and \int t^n y\,(t)dt = 0,\ for 0\le n\le r.$

3.4 Low and High pass filters, Filter banks:

Given a MRA we can employ what is referred to as a "Pyramidal algorithm" for processing signals. This algorithm depends on two sequences called filters. For any wavelet family there will always be equations of the form ? (t) = 2? h $_k$? (2t-k) and ? (t) =2? g $_k$? (2t-k) [Please see article-II] .The sequences {h $_k$ }, {g $_k$ } arising from these equations must satisfy certain conditions. The most useful, for practical purposes is g_k = (-1)^k h$_{1-k}$ which allows us to construct ? knowing the refinement coefficients h_k and the scaling function ? . The sequences {h $_k$ } and {g $_k$ } can be used to process signals and are ostensibly called *"low-pass"* and *"high-pass"* filters. The term filter is used to indicate a convolution operator, because such an operator can cut-out various frequencies. In digital signal processing (DSP) terms a low (high) pass filter suppresses the high (low) frequency components and allows the low (high) frequency components to pass through. The Haar low (high) pass filter, for example, calculates the average of an even and odd element (difference- between them) resulting in a "smoother" signal. In many applications, one never has to deal with the scaling function or wavelets, only the coefficients h_k and g_k need only be considered. A general *Filter-bank* is a sequence of convolutions and other operators that decompose a signal into a collection of sub-signals denoted by LL(low-low),LH(low-high),HL(high-low) and HH(high-high). These sub-signals depending on the type of application help emphasize specific aspects of the original signal or facilitate to work effectively with the original signal. We have linear and non-linear filter banks depending on whether or not the sub-signals depend linearly on the original signal. The study of filter banks is an entire subject in Engineering called "Multirate Signal Analysis" or "Subband coding." A mention must be made about another technique called "Quadrature Mirror Filter" (QMF) in which orthogonal decomposition by low and high pass filters {h $_k$ }, {g $_k$ } takes place at every step in signal processing. In QMF the decomposition step consists of applying a low (high) pass filter followed by down sampling ($\downarrow 2$) retaining only the even index samples.
The reconstruction step consists of up sampling *($\uparrow$ 2)* putting *0* between adjacent samples followed by filtering and addition. In down sampling there is every possibility of loosing information since half of the data is discarded. The effect of this in the frequency domain is a phenomenon called *aliasing* which means mixing up of frequency components resulting in an image not reflecting reality.



Tight frames and (Riesz) Unconditional bases:

For obvious reasons, the orthonormal basis (o.n.b) is most desirable. While the conditions for a set of functions to be an o.n.b are sufficient for the calculations of the signal from the coefficients they are decidedly not necessary. To be a basis it requires that the set be independent, meaning no element can be written as a linear combination of others. If the set is dependent and yet does allow the expression (5) then the set is called a *frame* which is essentially a spanning set. The term frame comes from a definition that puts finite limits (called Riesz bounds) on an inequality

$$A\|f\|^2 \le \sum_k |\langle \varphi_k , f\rangle|^2 \le B\|f\|^2 \quad \text{for some } 0<A\le B<\infty$$

Where $\{\varphi_k\}$ is an expansion set and $f \in L^2(R)$

If A=B, the expansion set is called a *Tight frame*.

If A=B=1, we have the definition of the Hilbert basis. A Riesz (Unconditional) basis is a countable frame whose elements are linearly independent. The use of frames and tight frames rather than bases and orthogonal bases means existence of a certain amount of redundancy. In some cases, redundancy is "blessing in disguise" in giving robustness to the representations, so that the errors become less troublesome. In finite - Dimensional cases the elements can always be removed from a frame to get a basis but the same thing cannot be said about infinite dimensional cases. The usefulness of wavelets in representing functions stems from the fact that, for most of the spaces, the wavelet basis is an unconditional basis which has a "near optimal property" i.e. the wavelet coefficients decay rapidly and this perhaps explains why wavelets are so effective in signal and image compression, de-noising and edge-detection. To put it differently wavelets are the ultimate choice in signal processing and data compression.

3.6 Biorthogonal and Semi- orthogonal wavelets:

Now, an orthonormal wavelet is a function ? such that {?(x-k)}, k∈ Z is an orthonormal basis of *Wo* in MRA. The orthogonolity property (which is at the heart of any MRA) is a more restrictive condition on wavelets and puts a strong limitation on the construction. As we have seen before, Haar wavelet is the only real valued wavelet that is symmetric, compactly supported and orthogonal. Although the advantages of an o.n.b. are many, there are cases where the basis required by the problem is not and cannot be made orthogonal. The generalization to *biorthogonal* wavelets has been considered to gain more flexibility. Here a dual scaling function $\breve{f}$ and dual wavelet function $\breve{y}$ exist, that generate a dual MRA. Moreover, the dual functions have to satisfy $< \breve{f}, f(t-k) >= d_k$ and $< \breve{y}, y\,(t-k) = d_k$ .Because, this type of orthogonality requires two sets of functions the system is called biorthogonal. For these cases one can still have the expression f(x) $= \sum a_k\ \breve{y}_k$ (x) A biorthogonal scaling function and the wavelet are semi-orthogonal if they generate an orthogonal MRA.

3.7 Vanishing Moments and smoothness (Regularity):

We can give several examples of orthonormal wavelet bases. However the only example we have seen so far of compactly supported wavelets has been the Haar wavelet. The compact support enables one to represent signals with small support efficiently. But the disadvantage of these Wavelets lies in the poor decaying of Haar co-efficients of smooth functions. The question is: Is it possible to construct wavelet basis that have the advantages of the Haar system, namely compact support, but that are also smooth? This should result in good time localization, but also better decay of the co-efficients, for smooth functions and higher quality image reconstruction. The answer is given using what are called *vanishing moments*. We have seen that $\Psi(x)$ coming from a MRA must satisfy $\int_R \Psi(x)dx = 0$. The integral is referred to as the 0th moment of $\Psi(x)$. In general if $\int_R x^k \Psi(x) = 0$, for k = 0,1,- - k-1, we say that it has k vanishing moments. The two important properties of the wavelet $\Psi(x)$ related to its vanishing moments are smoothness and approximation. It has been established that if $\{\Psi_{j,k}(x)\}$ is an orthonormal system in $L^2(R)$ then $\Psi(x)$ is smooth and it will have vanishing moments. The smoother $\Psi(x)$ is, the greater the number of vanishing moments. Vanishing moments have far reaching implications for the efficient representation/approximation of functions. Specifically the wavelet series of a smooth function will converge rapidly to the function as long as the wavelet has a lot of vanishing moments, resulting in a good approximation. For image compression, the rule of thumb is; where the image is smooth we need to keep only few co-efficients and where it is rough (where there are edges and corners) we need more co-efficients.

3.8 Wavelet Packets and "Best basis" selection:

A simple but powerful extension of wavelets and MRA are wavelet packets. It is well known that the classical MRA is obtained by splitting V_j into V_{j-1} and W_{j-1}. Then doing the same for V_{j-1} recursively, results in the decomposition $L^2(R)$.The wavelet packets are the basis functions that we obtain if we also use the "splitting trick" on the W_j spaces. For example: If $V_3 = V_0 \oplus W_0 \oplus W_1 \oplus W_2$, we obtain after applying the splitting trick three times, a wavelet packet - basis functions $\{\Psi_0^1(4x-k), \Psi_{1,1}^2(2x-k), \Psi_{0,0,1}^3(x-k), \Psi_{1,0,1}^3(x-k),\ k\in Z\}$. For the dual functions, a similar procedure has to be followed. In the Fourier domain the splitting trick corresponds to dividing the frequency interval essentially represented by the original space into two parts. So the wavelet packets allow more flexibility in adopting the basis to the frequency content of the signal.

It is easy to develop a Fast Wavelet Packet Transform (FWPT). It just evolves applying the same low and high pass filters, also to the co-efficients of the functions of W_j again in a iterative manner. This means that starting from a signal of N samples we construct a full binary tree with $N\log_2 N$ entries. The power of this construction lies in the fact that we have much more freedom in deciding which basis functions we will use to represent a given function. We can select the N co-efficients of the tree to represent the given function that is optimal with respect to a certain criterion. This procedure is called *best basis selection.* One can design fast algorithms that make use of this tree structure. For application in image processing, entropy based criteria are used. The best basis selection in that case has computational complexity O (N).

3.9 Cardinal Splines and Multi-dimensional Wavelets:

The use of cubic splines in image processing is well established. In recent years, application of wavelets and spline wavelets are receiving much attention. Basically any function square integrable or not which is continuous in L^2 (R) whose restrictions to the intervals [k, k+1] are polynomials of degree ≤ 1 is called a cardinal spline of degree 1. By simultaneously increasing the degree of the polynomials and the global regularity, quadratic and cubic splines can be obtained.

Until now we have focused on functions of one variable and the one dimensional situation. However there are also wavelets in higher dimensions. Images, for example, are bi-variate. A simple way to obtain these is to use to tensor products. To fix ideas, let us consider the case of the plane. Let $f(x,y) = f(x).f(y) = f \otimes f(x,y)$ and define $V_0 = \{f; f(x,y) = \sum_{i,j} l_{i,j} f(x-i, y-j),\ l \in l^2(Z^2)\}$. Of course if $\{f(x-k)\}_{K\in Z}$ is an orthonormal set, then $\{f(x-i, y-j)\}$, form an orthonormal basis of V_0. By dyadic scaling we obtain MRA of $L^2 (R^{2})$. The complement W_0 of V_0 in V_1 is generated similarly by the translates of the three functions. $\Psi^{(1)} = f \otimes \Psi, \Psi^{(2)} = \Psi \otimes f, \Psi^{(3)} = \Psi \otimes \Psi$.

There is another perhaps, even more straight forward decomposition. By carrying out one dimensional wavelet decomposition, for each variable separately, we obtain

$$f(x, y) = \sum_{i,j} \sum_{k,l} \left\langle f, \Psi_{i,j} \otimes \Psi_{k,l} \right\rangle \Psi_{i,j} \otimes \Psi_{k,l}.$$

In general, given a multivariate function g, defined on $R^{d,}$ we use the notation $g_{j,k} = g(2^k.- j), \tilde{g}_{j,k} = 2^{k(d/2)} g_{j,k}, j \in Z^d, k \in Z.$ to denote shifted dilates and its L_2 – normalized shifted dilates.

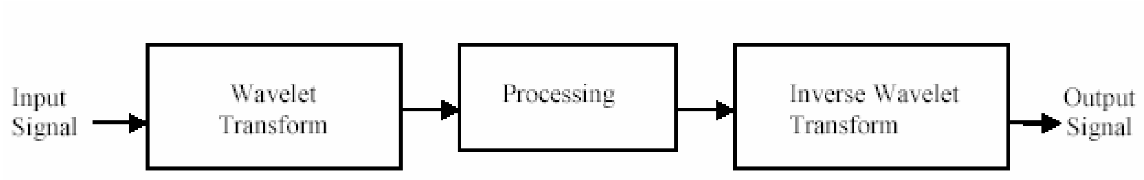
3.10 Understanding Compression:

How does one use wavelets to save space in storing (archiving) and transmitting data? When sending data electronically each bit costs time and space, as we have pointed out in the beginning. Current research is being conducted into how a given amount of information can be stored or transmitted in as few bits as possible through a technique called *data compression*. One of the most common applications of wavelets theory is data compression. There are two kinds of compression schemes. *Lossless* and *Lossy*. In the case of lossless compression one is interested in reconstructing the data exactly without any loss of information. In the lossy compression scheme we compromise a little on quality in the sense that we are ready to accept an error as long as the quality of compression is acceptable. Interestingly, with lossy compression, we can achieve much higher compression ratio than with lossless compression. To be specific let us assume that we are given a digitized image, of 512x512 pixels, 24 bit color image. which takes about 75 megabytes to store it in a computer. This is only for a still image, in case of video, the situation becomes still worse. The interest in compression in general has grown enormously due to the fact that the volume of information obtainable has increased. This is easy to understand when we consider the fact that, to store a moderately large image, one needs to compress it, for the purpose of transmission and if necessary for retrieval. The use of either type of compression can become an economic factor for reducing the capacity of the memory or band width of a channel in the context the explosive growth of the internet and graphical or visual representation of information. In the present scenario, these issues have become more and more important and have engaged the attention of the policy makers all over the world. Now returning to the question asked in the beginning of the section, for certain signals many of the wavelet co-efficients are close to or equal to zero. Through a method called *thresholding* these co-efficients may be modified or nullified, so that the sequence of co-efficients contains long strings of zeros. These long strings can be stored or can be sent electronically in much less space/time, through a type of compression known as *entropy coding*. One way to ignore the small co-efficients is through *hard- thresholding*. In which a tolerance limit (cut-off) l is selected. Any wavelet co-efficient whose absolute value falls below l is set to zero, with the aim to introduce lots of zeros (zero padding) without losing significant detail. There is no hard and fast rule to choose l . The penalty for a larger l is more error in the process. Another way to ignore data is by *soft-thresholding*. Again a tolerance l is set, if any entry is less than l in absolute value it is set to zero. In addition, all other entries 'd' are replaced with $sign(d)\left| \ |d|- l \ \right|$. The use of wavelets and thresholding serves the purpose of only processing the original signal. The actual compression of data takes place using *Huffman entropy coding.*

3.11 Wavelet Image Compression and Quantization:

We are given an image and we are interested in removing some of the information in the original image, without degrading the quality too much. In this way we obtain a "compessed" image which can be stored using less storage and which also can be transimitted, more quickly or using less bandwidth, over a communication channel. The compressed image, in a sense, an approximation of the original image. To fix the ideas, first let us define somewhat mathematically, what we mean by an image. Let us for the sake of simplicity discuss an LxL grey scale image with 256 grey scales. This can be considered to be a piece wise constant function *f* defined on a square $f(x, y) = p_{i,j} \in N$ for $i \le x \le i+1, j \le y \le j+1$; and $0 \le i, j \le L$; where $0 \le p_{i,j} \le 255$. there are several ways to use wavelets transform for compression purposes. Today the successful image compression method is based on two dimensional product basis constructed from the Daubechies wavelets. However the underlying principle of image compression is to consider compression to be an approximation problem, more concretely let us fix an orthogonal wavelet y . Given an integer $M \ge 1$, we try to find the *best approximation* of *f* using representation; $f_M(x) = \sum_{i,j} b_{i,j} y_{i,j}(x)$, with M non zero co-efficients $b_{i,j}$. The basic reason why this might be potentially useful is that each wavelet picks up information about the image *f* essentially at a given location and scale. In other words the wavelet transform allows us to focus on the most relevant parts of *f*. Now to give a mathematical meaning to a best approximation of *f*, we need to agree on an error measure. So we are interested in finding an optimal approximation minimizing the error. $\| f - f_M \|_2$. Because of the orthogonality of the wavelets, this equals; $\left(\sum \left| \langle f, y_{i,j} \rangle - b_{i,j} \right|^2\right)^{1/2}$. A moment's reflection shows that the best way to pick *M* non-zero co-efficients minimizing the error is by simply picking the *M* co-efficients with largest absolute value, and setting $b_{i,j} = \langle f, y_{i,j} \rangle$. This then surely yields the desired optimal approximation of *f*.

Image Compression is also based on a slightly different concept *quantization*. Instead of having 256 levels of grey we might have only 16. Such an image would look blurred or distorted. But we can do just this on the *LH, HL* and *HH* sub-band signals without losing much. All we are going to do is, rounding-off different amounts depending on the sub-band. This is called *dead-zone uniform scalar quantization.*



3.12 Noise Reduction and Edge Detection:

In any given image edges and textures are the two most important characteristics from the point of view of human visual perception. Edges in images are characterized by sharp variation in intensity values. Whereas texture carries contextual information about lighting, surface features, depth and other perceptual clues of objects in an image. The question then is: can we faithfully represent objects and images by means of edge and texture information alone. In other words, how close is the image recovered from its edge-texture representation to the original image. The *Laplacian pyramidal algorithm* (*Mallat's fast wavelet algorithm*)is currently used in edge detection and de-noising techniques.

The distinction between signal and noise depends upon how the measurement of signal models reality, in other words, upon the assumed relations between the signal and the phenomenon represented by the signal. Thus depending on the nature of noise, several methods exist to eliminate or attenuate noise, rightly labeled as de-noising techniques. If the noise has a random nature then removing wavelets with random co-efficients can help in a noiseless signal. If a signal represents a band-limited phenomenon, which may contain frequencies only within a certain range (band); then all frequencies outside the band cannot arise from the phenomenon under consideration, but arise from noise. Removing the wavelets with frequencies outside the band restores the original signal free from noise.

3.13 Concluding Remarks:

We have presented only a few topics which serve as pointers to the literature from what has become a dynamic and productive area of research with rich theory and a wide range of applications. What we call *"Modern Wavelet Theory"* today has been characterized by a healthy interplay between theory and applications. Simply put, mathematicians have worked in close collaboration with researchers from other areas of science and engineering and wavelet theory has been strongly influenced by applied problems. In a rapid developing field overview papers are particularly useful and several good ones concerning wavelets are already available. All said and done, wavelets have generated a tremendous interest in theoretical and applied sciences especially in the past few years. Infact, the advancements in the areas are occurring at such a rapid rate that the very meaning of wavelet analysis keeps changing. The real challenge before us, therefore, is to find a field of science or engineering where wavelet techniques have not been applied or at least tried.

<u>Bibliography:</u>

1. Fourier Analysis and Applications – C.Gasquet, P.Witomski. Springer Publications. (1998).
2. Introduction to Wavelets and Wavelets Transforms (A primer) – C.Sydney Burrus, Ramesh A.Gopinath and Haitao Guo. Prentice Hall International Inc.
3. Discovering Wavelets – Aboufadal and Schlickr.
4. A Wavelet Tour of Signal Processing, Estimations and Approximations – Stéphane Mallat.
5. Finger Prints go Digital – AMS, Vol. 42, November 1995.
6. Adaptive Wavelet Basis and Image Compression – Ronald A.DeVore.,"Wavelets,Images and Surface fitting".
7. An Overview of Wavelet Based MRA – Bearn Jawerth, Wim Sweldens. SIAM Review, Vol. 36, No.3 September 1994.
8. Ten Lectures on Wavelets – Ingrid Daubechies.