II-Multiresolution Analysis and construction of wavelets.

(A Tutorial)

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Abstract:

***Multiresolution analysis* (MRA) was formulated based on the study of orthonormal, compactly supported wavelet bases. Wavelets theory and its applications are rapidly developing fields in applied mathematics and signal analysis. Wavelet basis representation of certain signals show advantages over the traditional Fourier basis representation both theoretically and practically. The MRA concept was initiated by Meyer and Mallat which provides a natural framework for the understanding of wavelet bases. Here, we give a brief description of orthonormal, compactly supported wavelet bases.**

2.1 Introduction:

The concept of Multiresolution analysis (in short) MRA is intuitively related to the study of signals at different levels of resolution. In 1986 Mallat and Meyer first formulated the idea of MRA in the context of wavelet analysis. This is a new and remarkable concept, which deals with general formalism for construction of an orthonormal base of wavelets. MRA indeed is central to all construction of wavelet bases. With this theory the duo Mallat and Meyer could link wavelets with the so-called "Filters" in signal processing. One off shoot of their work was FWT, which turned out to be faster than FFT. Another was a mathematical theory on sound footing of orthogonal bases. This theory in turn gives a "recipe" for constructing new orthogonal wavelets. In addition, MRA provides a natural framework for the understanding of wavelet bases and for the construction of new examples. Interestingly, the history of the formulation of MRA is a beautiful example of applications stimulating theoretical development (not the other way around). We present here the essence of MRA by considering the properties of subspaces and their orthogonality and projectability. Our main aim in this article is to construct wavelets, given a multiresolution analysis.Of course we want our wavelet to be closely connected with the given multiresolution analysis.

2.2 Definition, Description and Derivation:

We have already seen that the Haar wavelet "a good friend" of wavelet analysts provides an example of orthonormal basis (1.4). To develop a systematic method for producing wavelets, we introduce another notion axiomatically, which generalizes Haar construction. A formal definition of MRA reads like this. A MRA is an increasing sequence (Chain or ladder) of subspaces $\{V_j\}_{j\in Z}$ of $L^2(R)$ together with a function f, having the following properties.

(a) $\{f(t-k)\}_{k\in Z}$ is an orthonormal basis of V_0.

More generally

$$V_j = \mathrm{Span}\{f(2^j t - k)\} \quad j,k \in Z$$

Evidently, all translates of f by integers are orthogonal to each other.

(b) $V_{j+1} = D_2(V_j)$ Where D_a is the scaling operator, $\forall$ $a \neq 0$. (In practice, a=2).

Referring to the signal f, this can be expressed as $f(x) \in V_j \Leftrightarrow f(2^j x) \in V_0$. In plain English, this simply means that a signal at a given resolution contains all the information of the signal of coarser resolution.

(c) $\bigcap V_j = \{0\}$ Implying that the zero function is the only object common to all subspaces V_j. In other words V_j's are O-centric by inclusion.

(d) $\overline{\cup V_j} = L^2(R)$ Consequently $\forall$ $f \in L^2(R)$ $\exists$ a sequence $\{f_n\}$ such that $\|f_n - f\| \to 0$ as $n \to \infty$ which amounts to the approximation of any $f \in L^2(R)$ by a elements of $\bigcup V_j$ to any degree of accuracy.

Please Note: The function f plays a key role in everything that follows. It is not a wavelet; however it is called the "Scaling function", sometimes the "father wavelet".

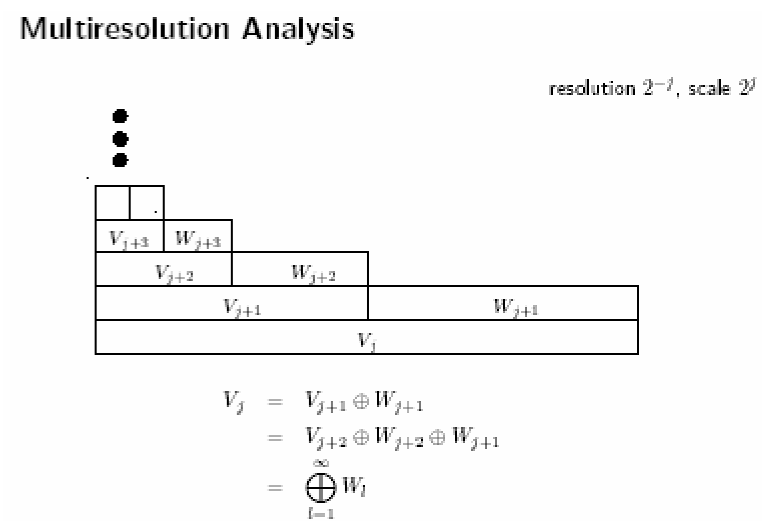
By virtue of (a), the space V_0 can be described as set of time signals f in the following way $V_0 = \left\{f \in L^2(R)/\ f(t) = \sum c_k f(t-k), \sum |c_k|^2 < \infty\right\}$ ------------------- (1)

Using f as the template if we define the functions

$$f_{j,k}(t) = 2^{-\frac{j}{2}} f\left(\frac{t - k2^j}{2^j}\right)_{j,k\in Z} \quad \text{--- (2)}$$

It is easy to see that $\{f_{j,k} / k \in Z\}$ $\forall$ fixed j is an orthonormal basis of V_j.

The beauty of MRA approach is that whenever a chain of subspaces V_j satisfies the 4-conditions listed above, it guarantees the existence of ? the translates and dilates of which form an orthogonal basis of $L^2(R)$. Let us see how we can go about it. Consider W_j an orthogonal complement of V_j. i.e, Suppose for the moment as in the case of V_0, $\exists$ a function ? in W_0 such that $\{?(t-k),\ k \in Z\}$ is an orthonormal basis of W_0. We can reasonably hope (Analogous to the remarks made earlier) that $\{?_{j,k}\}, k \in Z$ is an orthonormal basis of W_j. Further form general principles, Linear Algebra $\oplus W_j = L^2(R)$. The family $\{?_{j,k}\}\ \forall\, j,k \in Z$ would be an orthonormal wavelet basis we are searching for, such a ? would then be our "Mother wavelet". Thus our task having (chosen f) suitably reduces to finding $? \in W_0$ such that the collection $\{?(.-k\}, k \in Z$ is an orthonormal basis of W_0.



2.3 Scaling Equation:

Since $f \in V_0 \subset V_{-1}$ and $f_{-1,k}(t) = \sqrt{2}\, f\,(2t-k)$ we first construct an orthonormal basis of V_{-1}. Accordingly $\exists$ $h_k = \langle f, f_{-1,k} \rangle$ such that

$$f\,(t) = \sqrt{2} \sum h_k f(2t-k) \text{ --------------------------------------- } (3)$$

This identity goes by the name the "Scaling equation" and controls the entire MRA. As a matter of fact, the coefficients h_k when appear in the corresponding algorithm of FWT, they determine more or less everything. We don't need the scaling function f or the mother wavelet. This is in sharp contrast to Fourier analysis, where one has to compute function values e^{ix} time and again.

2.4 Generating Function:

The next point on our agenda is to create a Fourier series by taking the FT on both sides of (3) to obtain

$$\hat{f}(x) = \frac{1}{\sqrt{2}} \sum h_k e^{ik\frac{?}{2}} \hat{f}\left(\frac{?}{2}\right)$$

Looking at this formula we are led to introduce the function

$$H(?) = \frac{1}{\sqrt{2}} \sum h_k e^{-ik?} \text{ -- (4)}$$

This is called the "generating function" or "engine" of the MRA under consideration. Because of $\|h_k\| = 1$, the series (4) is convergent i.e., the equation (3) now takes the form

$$\hat{f}(x) = H\left(\frac{x}{2}\right) \hat{f}\left(x/2\right) \text{ -- (5)}$$

One great thing about H is it is uniformly bounded on R. This of course follows from the Fourier version of the consistency relations.

i.e., $|H(?)|^2 + |H(? + p)|^2 = 1$ --- (6)

Furthermore since $\hat{f}(0) \neq 0$, it follows that H (0) =1 and hence $H(p) = 0$. Our next goal is to describe W_0 as explicitly as possible. Having such a description in hand we shall be able to give a concrete formula for a possible mother –wavelet ? corresponding to the given scaling function f .

Since $V_{-1} = V_0 \oplus W_0$ we return to V_{-1}. Any $f \in V_{-1}$ has a representation of the form $f = \sum f_k f_{-1,k}$ where $f_k = \langle f, f_{-1,k} \rangle$ and taking the FT on both sides we get

$$\hat{f}(?) = \frac{1}{\sqrt{2}} \sum f_k e^{-ik?} \hat{f}\left(\frac{?}{2}\right) \text{ -- (7)}$$

By making use of the generating function we arrive at

$$\hat{f}(?) = e^{i\frac{?}{2}} V(?) \overline{H\left(\frac{?}{2} + p\right)} \hat{f}\left(\frac{?}{2}\right) \text{ -- (8)}$$

Where $V(? + 2p) = V(?)$

Inspired by this identity we are in a position to define ? , by the following formula

$$\hat{f}(x) = e^{i\frac{?}{2}} H\left(\frac{?}{2} + p\right) \hat{f}\left(\frac{?}{2}\right) \text{ -- (9)}$$

Or equivalently;

$$y(t) = \sqrt{2} \sum_k g_k f(2t - k) \text{ -- (10)}$$

According to the properties of FT, we have $g_k = (-1)^k h_{1-k}$

2.5 Illustations:

1) Piecewise constant functions:

MRA can best be illustrated by taking the simplest but historically important example of MRA as follows;

Choose $f = c[0,1]$ and set $V_0 = \{ f \in L^2(R) / \text{f is constant on}[k,k+1] \}$

$$V_0 = D_{2^j}(V_0) \quad j \neq 0.$$

The conditions a), b) are obviously satisfied and c) is guaranteed and 4) is the immediate consequence of the fact that the step functions that jump at the binary rationals $k2^j$ are dense in $L^2(R)$. Because of $?[0,1] = ?[0, 1/2] + ?[1/2, 1]$ the scaling function $f = c[0,1]$

This defines the Box-function.

which satisfies $f(t) = f(2t) + f(2t-1)$

$$\text{i.e.,} \quad f = \frac{1}{\sqrt{2}} f_{-1,0} + \frac{1}{\sqrt{2}} f_{-1,1}$$

Thus we have $h_0 = h_1 = \frac{1}{\sqrt{2}}$

$$h_k = 0 \quad \forall \ k \in Z - \{0,1\}$$

It is to easy to verify (see 1.4) that

$$\hat{f}(x) = \frac{1}{\sqrt{2p}} \frac{\sin\left(x/2\right)}{x/2} e^{ix/2}$$

We now insert the values of h_k and obtain the generating function.

$$H(x) = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} (1 + e^{-ix})$$

$$= \cos\frac{x}{2} \, e^{-ix/2}$$

The "recipe" (9) now gives

$$?(?) = e^{i?/2} \cos\left(\frac{?}{4} + \frac{p}{2}\right) e^{i\left(\frac{?}{4}+\frac{p}{2}\right)} \frac{1}{\sqrt{2p}} \sin\frac{?/4}{?/4} \; e^{-i?/4}$$

$$= \frac{-i}{\sqrt{2p}} \sin^2 \frac{\left(?/4\right)}{?/4} \; e^{-i?/2}$$

In order to obtain ? we have to translate back into the time domain using inverse FT to arrive at the "official" *Haar-wavelet*.

Alternatively, using $y(t) = \sqrt{2}\sum_k g_k f(2t-k)$ and $g_k = (-1)^k h_{1-\bar{k}}$ we have

$$y(x) = \begin{cases} 1 & 0 \le x < \frac{1}{2} \\ -1 & \frac{1}{2} \le x < 1 \\ 0 & \text{otherwise} \end{cases}$$

This multiresolution analysis is called the *Haar multiresolution analysis*.

2) Piecewise Linear Functions:

Let V_0 be the space containing all continuous functions $f(t)$ that are linear between $[k, k+1]$, and let V_j be the space of functions linear on the intervals $[k/2^j, (k+1)/2^j]$. Then $f(2t)$ is a function linear on the intervals $[k/2, (k+1)/2]$ and $f(2t) \in V_1$. Again we can verify that $\{V_j\}$ satisfy the conditions (i)-(iv). The translates of the hat function $\{\beta_1(t-k)\}$ form a basis of V_0, where the hat function $\beta_1(t)$ is defined by

$$\beta_1(t) = \begin{cases} 1-|t| & -1 \le t \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Any function $f(t)$ in V_0 can be written as the superposition of these basis functions,

$$f(t) = \sum_k f(k) b_1(t-k) \qquad (1)$$

Note that $\beta_1(t)$ vanishes for all integer values of its argument except 0. This shows that the expansion (1) agrees with the function at the integers. When t is between two integers $t \in (k, k+1)$, $f(t)$ is a combination of two linear functions $\beta_1(t-k)$ and $\beta_1(t-k-1)$, and this combination of two linear functions gives rise to a linear function which is specified entirely by its two values at k and $k+1$. So (1) is a correct expansion, and $\{\beta_1(t-k)\}$ is a basis for V_0. However, this basis is not orthogonal.

This defines the hat function.

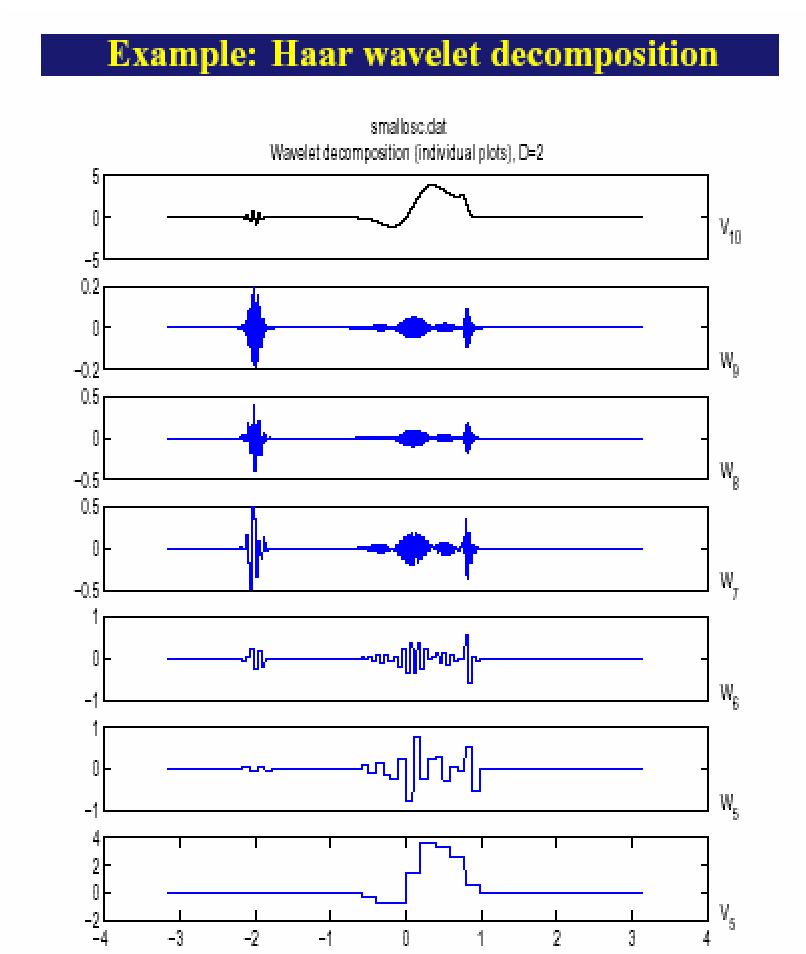
We see that the integral of $\beta_1(t)\beta_1(t-1)$ is nonzero; therefore, the basis $\{\beta_1(t-k)\}$ is not orthogonal. In this case we call $\beta_1(t)$ a nonorthogonal scaling function. We see that from this nonorthogonal scaling function we can obtain an orthogonal scaling function by a well-known orthogonalization procedure. Therefore, the $\{V_j\}$ under consideration still form a multiresolution analysis of L^2 although we have yet to find the orthogonal scaling function ϕ.

Spline Multiresolution Analyses:

In fact, the above two examples of multiresolution analysis can be thought of as the first two of a whole family of multiresolution analyses called the spline multiresolution analyses. Each multiresolution analysis is labeled by an index n, and the space V_0 consists of all functions $f(t)$ that are polynomials of degree at most n between $[k, k+1)$ and have $n-1$ continuous derivatives for $n > 0$. So $n = 0$ corresponds to the Haar multiresolution analysis, while $n=1$ is the multiresolution analysis of the piecewise linear functions. For general n, the space V_0 is the space of n^{th} order polynomial spline functions, and a basis for V_0 is the basis splines or B-splines. Apart from the $n=0$ case, these basis functions are not orthogonal and orthogonalization is required to get the desired scaling function.

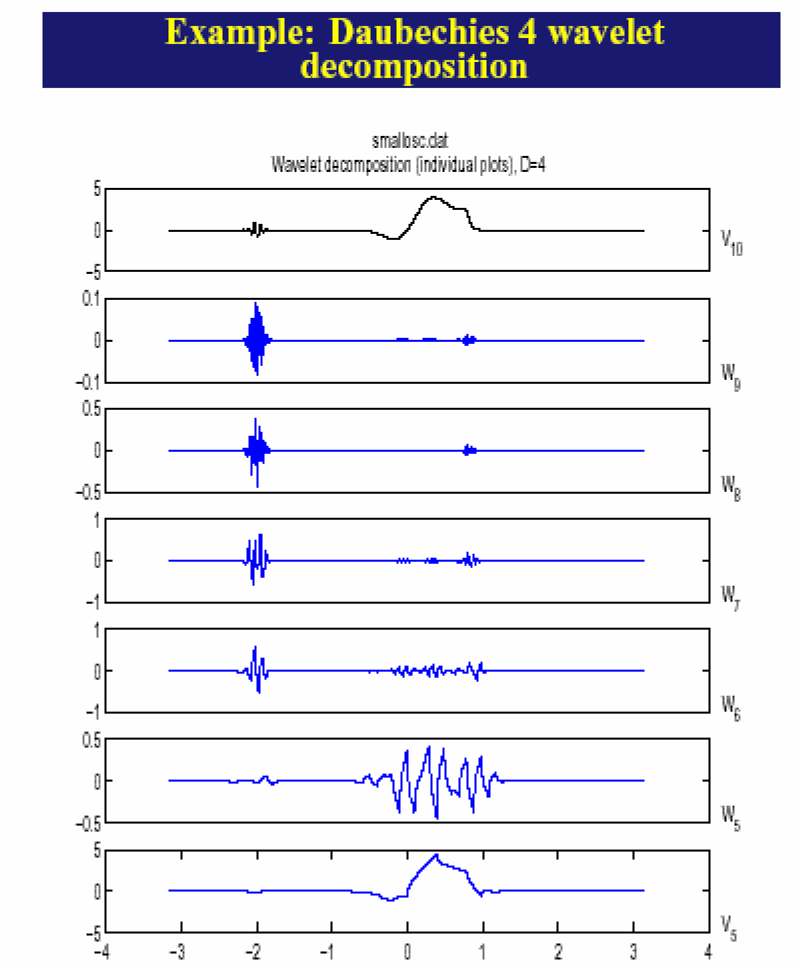
3) Daubechie'swavelets:

Daubechies was the first to give explicit examples of scaling function and associated wavelets that have arbitrary high order of differentiability, have compact support and are orthonormal.



2.6 Conclusion :

With the theory of MRA, wavelets became increasingly understandable and even more natural. Mallat gave a systematic way to construct new orthogonal wavelets.
More than "recipe" he gave an explanation. Meyer's orthogonal wavelets had emerged almost miraculously from his computations. But mathematicians don't like miracles even those supported by proof in good standing. In conclusion it may be recalled that the FT on the one hand, succeeds only in breaking down the signal into frequencies in exactly the same way as the prism breaks light into various colors, on the other, the procedure of MRA outlined above through its FWT allows one to study the signal at a coarser resolution to get an overall picture and at higher resolution to see finer details. This closely resembles the optical procedure of "zooming in and out" that is carried out in a microscope. So FT may be labelled as a *mathematical prism* whereas WT, a *mathematical microscope*. All said and done from the standpoint of practical applications MRA is really an efficient and effective mathematical tool for hierarchical decomposition of an image (signal) into components of different scales (frequencies).



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