King Mongkut's University of Technology Thonburi

FDE 618: Transport Phenomena in Food Processing
ChE610: Fundamentals of Transport Phenomena

Chapter 7
Selected Problem in Transport Phenomena: Melting and Freezing Heat Transfer

7. 1 Introduction

In the late 19th century J. Stefan formulated the problem of finding the temperature distribution and freezing front history of a solidifying slab of water (Alexiades and Solomon, 1993). Over the last century, particularly in the last 30 years, the problem bearing his name has been extended to include such complex phenomena as the solidification of alloy systems and melting due to laser irradiation. Melting and freezing also find themselves involving in a wide range of technologies, encompassing such diverse applications as freeze drying of foods, growth of crystal, cryo-preservation of biological cells as well as latent heat thermal energy storage system.

In this chapter the classical Stefan-type model of melting and freezing is discussed. This problem is non-linear and its principal difficulty lies in the fact that one of its unknown is the region in which it is to be solved. For this reason it is called a "moving boundary problem (or Stefan problem)." For most reasonable phase change processes the heat equation and its differential expression hold within the liquid or solid and a jump condition expressing energy conservation prevails along the curves separating solid from liquid. Fuzzy or fat regions are sometimes seen in lieu of sharp interfaces separating solid from liquid, however.

This chapter starts with an overview of the physical ideas relevant to phase change processes. An introduction to the precise mathematical formulation of the basic physical facts leading to the Stefan problem and the prototype of all phase change models is then given. Some words about the free convection-dominated melting are also mentioned.

7.2 Mathematical Formulation
7.2.1 Overview of the Phenomena Involved in a Phase Change

Several mechanisms are at work when a solid melts or a liquid solidifies. Such a change of phase involves heat (and often also mass) transfer, possible supercooling, absorption or release of latent heat, changes in thermophysical properties, surface effects, etc.

The transition from one phase to the other, i.e., the absorption or release of the latent heat, occurs at some temperature at which the stability of one phase breaks down in favor of the other according to the available energy. This phase change temperature, T_m, depends on pressure. Under a fixed pressure, T_m may be a particular fixed value characteristic of the material or a function of other thermodynamic variables.

Since most solids are crystalline, their particles (atoms, molecules or ions) are arranged in a repetitive lattice structure extending over significant distances in atomic terms. Since formation of a crystal may require the movement of atoms into the solid lattice structure, it may well happen that the temperature of the material is reduced below T_m without formation of solid. Thus, supercooled liquid may appear; such a state is thermodynamically metastable. Note that no such structuring is required in the melting process.

The phase transition region where solid and liquid coexist is called the interface. Its thickness may vary from a few Angstroms to a few centimeters and its microstructures may be very complex. For most pure materials solidifying under ordinary freezing conditions at a fixed T_m the interface appears (locally) planar and of negligible thickness. Thus, it may be thought of as a sharp interface. In other cases, typically resulting from supercooling, the phase transition region may have apparent thickness and is referred to as a mushy zone.

Most thermophysical properties of a material undergo sudden changes at T_m. Such discontinuities complicate the mathematical problems because they induce discontinuities in the coefficients of differential equations. The most fundamental and pronounced effects are due to changes in density; typically in the range of 5 to 10% but can be as high as 30%. The density variation with temperature induces flow by natural convection in the presence of gravity, rapidly equalizing the temperature in the liquid and greatly affecting heat transfer. In microgravity, there is no natural convection but Marangoni convection, due to surface tension effects, may arise and dominate heat transfer.

7.2.2 Formulation of the Stefan problem

For the mathematical description of a melting or freezing problem, the following assumptions are made (Table 7.1):

Table 7.1. Assumptions of the Stefan-type problem

Physical factors involved in phase change processes	Simplifying assumptions for the Stefan problem
Heat and mass transfer by conduction, convection, radiation with possible gravitational, elastic, chemical and electromagnetic effects	Heat transfer isotropically by conduction only
Release or absorption of latent heat	Latent heat is constant; it is released or absorbed at the phase change temperature
Variation of phase change temperature	Phase change temperature is a fixed known temperature, and is a property of material
Nucleation difficulties, supercooling effects	Assume not present
Interface thickness and structure	Assume locally planar and sharp
Surface tension and curvature effects at the interface	Assume insignificant
Variation of thermophysical properties	Assume constant in each phase, for simplicity ($c_l \neq c_s$ and $k_l \neq k_s$)
Density changes	Assume constant ($\rho_l = \rho_s$)

The situation arises, very often in real engineering applications, with the result that the Stefan problem is by far the most frequently applied model of a phase change process.

Based on the aforementioned assumptions the governing conservation equations for a one-dimensional problem are (see Figure 7.1):

Solid region:

$$\rho c_s \frac{\partial T_s}{\partial t} = \frac{\partial}{\partial x}\left(k_s \frac{\partial T_s}{\partial x} \right) \qquad (7.2\text{A})$$

Liquid region:

$$\rho c_l \frac{\partial T_l}{\partial t} = \frac{\partial}{\partial x}\left(k_l \frac{\partial T_l}{\partial x} \right) \tag{7.2B}$$

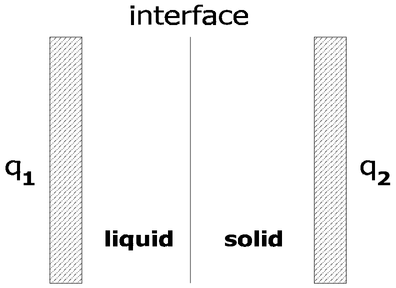

Figure 7.1. Physical model of two-phase Stefan problem

Initial conditions
The initial conditions are:

$$T_i(x,0) = T_0(x) \tag{7.2C}$$

Boundary conditions
The boundary conditions at the interface are:

$$T_s(S(t),t) = T_l(S(t),t) = T_m \tag{7.2D}$$

$$k_s \frac{\partial T_s}{\partial x} - k_l \frac{\partial T_l}{\partial x} = \rho L \frac{dS(t)}{dt} \tag{7.2E}$$

The temperature boundary conditions can be specified in one of the following forms:

$$T(0,t) = T_L \tag{7.2F}$$

$$-k \frac{\partial T}{\partial x} = q \tag{7.2G}$$

$$-k \frac{\partial T}{\partial x} = h(T - T_f) \tag{7.2H}$$

Equations (7.2A) to (7.2H) define a one-dimensional Stefan problem and constitute the foundation on which progressively more complex models can be built by incorporating some of the effects initially left out.

7.3 Solution Procedures

Early analytical works on the solution of phase change problems include those proposed by Lame and Clapeyron and Stefan in the early and late 19th century, respectively. The general phase change problem was also discussed by Neumann in his lecture in 1860's. It was not published until 1912, however. Although a number of phase change problems have appeared in the literature, the exact solutions are limited to only a small number of idealized situations involving semi-infinite or infinite regions subjected to simple boundary conditions.

When exact solutions are not possible approximate, semi-analytical methods can be used to solve the phase change problems. These approximate methods include integral method, perturbation method, variational formulation, etc. These approximate solutions are also confined to limited geometries (generally one-dimensional case) and simple initial and boundary conditions.

7.4 Exact Solution

7.4.1 The Neumann Solution (One-Phase Stefan Problem)

Consider melting of a semi-infinite slab, initially solid at the melting temperature T_m, by imposing a constant temperature $T_L > T_m$ on the face $x = 0$ (Figure 7.2). Based on the assumptions listed in Table 7.1 the governing conservation equation as well as initial and boundary conditions are:

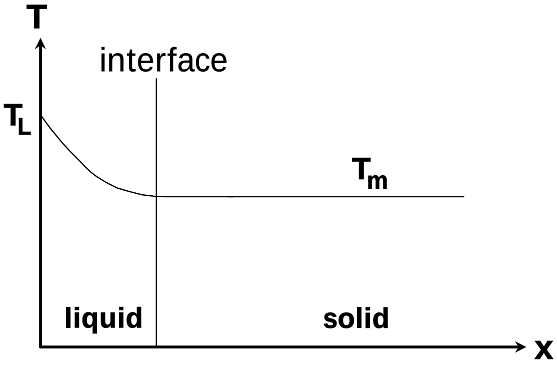

Figure 7.2. Schematic diagram of one-phase Stefan problem

Governing equation

$$\rho c_l \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k_l \frac{\partial T}{\partial x} \right),\ 0 < x < S(t),\ t > 0 \qquad (7.4A)$$

Initial conditions

$$T_l(x,0) = T_m \qquad (7.4B)$$

$$S(0) = 0 \tag{7.4C}$$

Boundary conditions

$$T(S(t),t) = T_m\,, \quad t > 0 \tag{7.4D}$$

$$-k_l \frac{\partial T}{\partial x} = \rho L \frac{dS(t)}{dt}\,, \quad t > 0 \tag{7.4E}$$

$$T(0,t) = T_L > T_m, \quad t > 0 \tag{7.4F}$$

Solution procedure

The similarity variable is introduced as:

$$\xi = \frac{x}{\sqrt{t}} \tag{7.4G}$$

and the solution in the form of an unknown function, $F(\xi)$, is sought:

$$T(x,t) = F(\xi) \tag{7.4H}$$

It is natural that the interface location *S(t)* is assumed to be proportional to $\sqrt{t}$. It is therefore desirable to search for a constant *A*, for which:

$$S(t) = A\sqrt{t} \tag{7.4I}$$

Substituting equation (7.4G) into equation (7.4A) and integrating yields:

$$F(\xi) = B\int_0^{\xi} e^{-\frac{s^2}{4\alpha_1}} ds + C = B\sqrt{\pi\alpha_l}\, erf\left(\frac{\xi}{2\sqrt{\alpha_l}}\right) + C \tag{7.4J}$$

in which *B* and *C* are constants of integration. The error function is defined as:

$$erf\,(z) = \frac{2}{\sqrt{\pi}} \int_0^{z} e^{-s^2} ds \tag{7.4K}$$

Using equations (7.4D) and (7.4F) resulting in the values of *B* and *C*:

$$B = \frac{T_m - T_L}{\sqrt{\pi\alpha_l}\, erf\left(\frac{A}{2\sqrt{\alpha_l}}\right)} \text{ and } C = T_L \tag{7.4L}$$

If the new variable is defined such that:

$$\lambda = \frac{A}{2\sqrt{\alpha_l}} \quad \text{and} \quad \Delta T_L = T_L - T_m \tag{7.4M}$$

and,

$$Ste_l \text{ (Stefan number)} = \frac{c_l \Delta T_L}{L} \tag{7.4N}$$

The Stefan condition, equation (7.4E), thus leads to an equation for λ:

$$\lambda e^{\lambda^2} erf(\lambda) = \frac{k_l}{\rho L}\frac{\Delta T_L}{\sqrt{\pi \alpha_l}} = \frac{c_l \Delta T_L}{\sqrt{\pi}\, L} = \frac{Ste_l}{\sqrt{\pi}} \tag{7.4O}$$

Hence, it is more convenient to express the solution in terms of λ. From equations (7.4I) and (7.4M):

$$S(t) = 2\lambda\sqrt{\alpha_l t} \tag{7.4P}$$

and from equations (7.4J) and (7.4L):

$$T(x,t) = T_L - \Delta T_L \frac{erf\left(\dfrac{x}{2\sqrt{\alpha_l t}}\right)}{erf(\lambda)} \tag{7.4Q}$$

with λ, a root of the transcendental equation (7.4O).

It can be easily shown that the quantity $f(\lambda) = \lambda e^{\lambda^2} erf(\lambda)$ is a strictly increasing function of $\lambda \geq 0$, $f(0) = 0$, $\lim_{\lambda \to \infty} f(\lambda) = +\infty$, and therefore, the graph of $y = f(\lambda)$ intersects any horizontal line $y = \frac{Ste_l}{\sqrt{\pi}}$ only once (Figure 7.3). In other words, for each value of $Ste_l > 0$, there exists a unique root λ of equation (7.4O). Once λ is determined by solving the transcendental equation (7.4O) the solution to the Stefan problem is given by equations (7.4P) and (7.4Q). This is the classical Neumann solution to the Stefan problem.

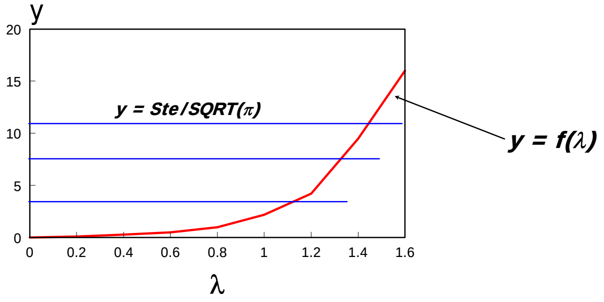

Figure 7.3. Demonstration of the uniqueness of the transcendental equation

7.5 Integral Solution

As it can be seen from the previous section solving partial differential equations analytically to obtain exact solutions is very cumbersome and time-consuming. For engineering purposes it is often more convenient if approximate analytical solutions can be obtained. One of the most useful approximate methods is the integral method. This method was firstly introduced by von Karman (1921) and Pohlhausen (1921) to solve boundary-layer momentum and energy equations. With this method a relatively simple and straightforward approach to the solution of one-dimensional transient phase change problems can be developed. In the following paragraphs the use of the integral method in finding the solution of a simple one-region melting problem is illustrated. For comparison the physical problem and thus the governing equation, initial and boundary conditions are kept the same as those in the previous section.

First we choose the region $0 \le x \le S(t)$ as the integral area and integrate the heat-conduction equation, equation (7.4A), from $x = 0$ to $x = S(t)$ to obtain:

$$\left.\frac{\partial T}{\partial x}\right|_{x=S(t)} - \left.\frac{\partial T}{\partial x}\right|_{x=0} = \frac{1}{\alpha_l}\int_{x=0}^{x=S(t)} \frac{\partial T}{\partial t} dx \tag{7.5A}$$

Equation (7.5A) can be rewritten as:

$$\left.\frac{\partial T}{\partial x}\right|_{x=S(t)} - \left.\frac{\partial T}{\partial x}\right|_{x=0} = \frac{1}{\alpha_l}\frac{d}{dt}\left[\int_{x=0}^{x=S(t)} T dx - T\right]_{x=S(t)} S(t) \tag{7.5B}$$

In terms of the interface boundary conditions, equations (7.4D) and (7.4E), equation (7.5B) reduces to:

$$-\frac{\rho L}{k_l}\frac{dS(t)}{dt} - \left.\frac{\partial T}{\partial x}\right|_{x=0} = \frac{1}{\alpha_l}\frac{d}{dt}\left[\Theta - T_m S(t)\right] \tag{7.5C}$$

where,

$$\Theta \equiv \int_{x=o}^{x=S(t)} T(x,t) dx \tag{7.5D}$$

Equation (7.5C) is the energy integral equation for this melting problem. Here a second-degree polynomial approximation for the temperature is chosen:

$$T(x,t) = a + b(x-S) + c(x-S)^2 \tag{7.5E}$$

where $S \equiv S(t)$. Three conditions are needed to determine the three unknowns, a, b and c. Equations (7.4D) and (7.4E) provide two necessary conditions. Unfortunately, the relation given by equation (7.4E) is not suitable for this purpose because; if it is used, the resulting temperature profile will involve the $dS(t)/dt$ term. When such a profile is substituted into the energy integral equation a second-order ordinary differential equation will be resulted in lieu of the usual first-order equation. To alleviate this difficulty, an alternative relation is developed. Differentiating the boundary condition (7.4D) yields:

$$dT = \left[\frac{\partial T}{\partial x} dx + \frac{\partial T}{\partial t} dt \right]_{x=S(t)} = 0 \tag{7.5F}$$

or,

$$\frac{\partial T}{\partial x} \frac{dS(t)}{dt} + \frac{\partial T}{\partial t} = 0, \quad \text{at } x = S(t) \tag{7.5G}$$

The term *dS(t)/dt* is now eliminated by substituting equation (7.4E) into equation (7.5G):

$$\left(\frac{\partial T}{\partial x} \right)^2 = \frac{\rho L}{k_l} \frac{\partial T}{\partial t}, \quad \text{at } x = S(t) \tag{7.5H}$$

and eliminating $\partial T / \partial t$ between equations (7.4A) and (7.5H):

$$\left(\frac{\partial T}{\partial x} \right)^2 = \frac{\alpha_l \rho L}{k_l} \frac{\partial^2 T}{\partial x^2}, \text{ at } x = S(t) \tag{7.5I}$$

With this relation, together with the boundary conditions at $x = 0$ and $x = S(t)$, i.e., equations (7.4D) and (7.4F), the assumed temperature profile (7.5E) becomes:

$$T(x,t) = T_m + b(x - S) + c(x - S)^2 \tag{7.5J}$$

where,

$$b = \frac{\alpha_l \rho L}{k_l S} \left[1 - (1 + \mu)^{1/2} \right] \tag{7.5K}$$

$$c = \frac{bS + (T_L - T_m)}{S^2} \tag{7.5L}$$

$$\mu = 2 \frac{k_l}{\alpha_l \rho L} (T_L - T_m) = \frac{2c_l (T_L - T_m)}{L} \tag{7.5M}$$

Substituting the temperature profile (7.5J) into the energy integral equation (7.5C) leads to the ordinary differential equation that can be solved for the location of the liquid-solid interface *S (t)*:

$$S \frac{dS}{dt} = 6\alpha_l \frac{1 - (1 + \mu)^{1/2} + \mu}{5 + (1 + \mu)^{1/2} + \mu} \tag{7.5N}$$

subject to:

$$S = 0 \text{ at } t = 0 \tag{7.5O}$$

The solution of equation (7.5N) is:

$$S(t) = 2\lambda\sqrt{\alpha_l t} \tag{7.5P}$$

where

$$\lambda = \left[3\frac{1-(1+\mu)^{1/2}+\mu}{5+(1+\mu)^{1/2}+\mu} \right]^{1/2} \tag{7.5Q}$$

Note that the approximate solution (7.5P) for *S(t)* is of the same form as the exact solution given in section 7.4. The parameter λ, however, is given by equation (7.5Q) for the approximate solution, whereas it is the root of the transcendental equation (7.4O) for the exact solution.

7.6 Brief Introduction to the Free Convection-Dominated Melting

While most freezing situations are controlled by conduction through a frozen layer that builds up during energy discharge cycle the melting process is, in reality, governed by free convective heat transfer process. This phenomenon arises whenever the density of the liquid is not constant and it is most notable when melting is induced by heating from below of a liquid of relatively low viscosity. Only a small portion of the literature on free convection-dominated melting is reported partly due to the serious mathematical and computational difficulties encountered when dealing with convective flows.

Gartling (1980) was the first to model convection-dominated melting with the standard Galerkin finite element technique. Since then a number of investigations have been performed on this class of problem (e.g., Gong and Mujumdar, 1998a,b). Free convective currents develop during melting are shown to enhance the energy transfer rate considerably (Gong and Mujumdar, 1998a). Keeping this advantage in mind several investigators (e.g., Devahastin et al., 1998, Mujumdar et al., 1998; Gong et al., 1999) come up with novel and yet simple methods to enhance the heat storage rate in a PCM. The techniques are proved numerically to enhance up to 50%, in some specific cases (Devahastin et al., 1998), in the energy storage rate.

For more information on free convection-dominated melting please refer to Gong and Mujumdar (1998a,b).

Acknowledgments

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9. von Karman, T., 1921, Uber Laminare und Turbulente Reibung, **Zeitschrift fur angewandte Mathematic und Mechanic, Vol. 1**, pp. 233-252.

Suggested Readings

1. Crank, J., 1984, **Free and Moving Boundary Problems**, Oxford University Press, London.

Problem

1. Melting of a semi-infinite slab, $0 \le x \le \infty$, initially solid at a uniform temperature $T_s \le T_m$, is accomplished by imposing a constant temperature $T_L > T_m$ on the face $x = 0$. Thermophysical properties, ρ, c_l, c_s, k_l, k_s, L, are all constant. Obtain the following:

   (a) The governing equations and the associated initial and boundary conditions for the problem.
   (b) A complete derivation of the Neumann solution of the two-phase Stefan problem.
   (a) Equation for the location of the phase change front. Also present, in graphical form, the temperature profiles at t = 10, 30 and 60 hours and the temperature histories at x = 5, 10 and 15 cm. Use the following thermophysical properties: $\rho_s = \rho_l = 1460$ kg m^{-3},

T_m = 32^o C, L = 251.21 kJ kg^{-1}, c_l = 3.31 kJ kg^{-1} K^{-1}, c_s = 1.76 kJ kg^{-1} K^{-1}, k_l = 0.59 W m^{-1} K^{-1}, k_s = 2.16 W m^{-1} K^{-1}, T_L = 90^o C. Assume T_s = 25^o C.

You may use any mathematical software (e.g., Matlab™ or Mathcad™) to solve this problem.