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FDE618: Transport Phenomena in Food Processing
ChE610: Fundamentals of Transport Phenomena

Chapter 6
Analytical Solution Techniques for Transport Problems II: Integral Solution

In the previous chapter the approximate method, i.e., the scale analysis or scaling, for obtaining the order-of-magnitude solution of various transport problems has been outlined. Scale analysis indicates the manner in which various parameters affect the quantities of interest, e.g., friction and heat transfer rate, but makes no prediction of the numerical coefficients, which are needed if the less approximate (or more exact) solution is needed.

In this chapter another "approximate" method for obtaining the solution of various transport problems, i.e., the integral method, is discussed. The method was introduced by T. von Karman and developed by K. Pohlhausen in the early 1920's (Kakac and Yener, 1995). The integral method allows an engineer to obtain approximate analytical solutions, which are less crude than those obtained using the scale analysis when the complete equations cannot be solved exactly by analytical means.

6.1 Integral Method and Heat Conduction

The basic idea of the integral method is to convert a partial differential equation (PDE), which may take the form $F_1(x,y)=0$, into an ordinary differential equation (ODE) $F_2(x)$ or even an algebraic equation. Note again that the solution obtained is just an approximate solution – not an exact solution.[1]

Consider, for example, a transient, one-dimensional heat conduction of Figure 6.1. Initially, the temperature is everywhere equals to T_i. At t = 0, the wall temperature has been raised and maintained at T_0.

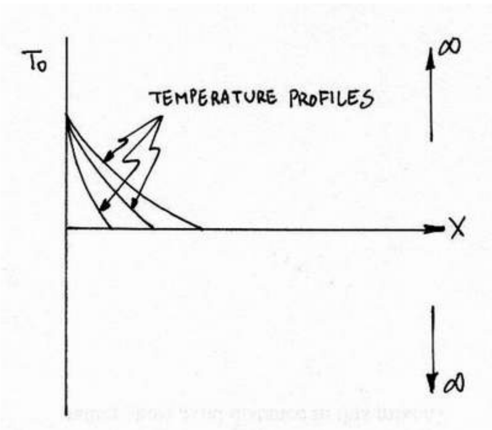

Figure 6.1. An illustrated example of the use of the integral method

[1] There are indeed two basic ways to covert a PDE into an ODE, i.e., the similarity method and the integral method. The former gives an exact solution while the latter gives an approximate solution.

In this case, the energy equation can be written as:

$$\frac{1}{\alpha}\frac{\partial T(x,t)}{\partial t} = \frac{\partial^2 T}{\partial x^2} \tag{6.1A}$$

subject to the following initial and boundary conditions:

$$T(x,t) = T_i\,,\ t = 0\,,\ x \ge 0 \tag{6.1B}$$

$$T(x,t) = T_0\,, x = 0,\ t > 0 \tag{6.1C}$$

$$T(x,t) = T_i\,,\ x \to \infty\,,\ t > 0 \tag{6.1D}$$

The following steps are followed in order to obtain an integral solution of the above equation.

1. Integrate the PDE over the whole domain thus obtaining the energy integral equation.
2. Select a suitable temperature profile that fits the actual given boundary conditions.

If the first step is followed, equation (6.1A) can be integrated and becomes:

$$\int_0^{\delta(t)} \frac{\partial^2 T}{\partial x^2} dx = \frac{1}{\alpha}\int_0^{\delta(t)} \frac{\partial T}{\partial t} dx \tag{6.1E}$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=\delta(t)} - \left.\frac{\partial T}{\partial x}\right|_{x=0} = \frac{1}{\alpha}\int_0^{\delta(t)} \frac{\partial T}{\partial t} dx \tag{6.1F}$$

The following rule is found useful in integrating the last term in equation (6.1F):

$$\frac{d}{dy}\int_{\alpha(y)}^{\beta(y)} f(x,y)dx = \int_{\alpha(y)}^{\beta(y)} \frac{\partial f}{\partial y} dx + \beta^{/}(y) f(\beta(y),y) - \alpha^{/}(y) f(\alpha(y),y) \tag{6.1G}$$

In this case $\alpha = 0$, $\beta = \delta$ and $y = t$. Thus,

$$\frac{d}{dt}\int_0^{\delta(t)} T dx = \int_0^{\delta(t)} \frac{\partial T}{\partial t} dx + \frac{d\delta}{dt} T(\delta,t) \tag{6.1H}$$

or,

$$\int_0^{\delta(t)} \frac{\partial T}{\partial t} dx = \frac{d}{dt}\int_0^{\delta(t)} T dx - T_i \frac{d\delta}{dt} \tag{6.1I}$$

Substituting equation (6.1I) into (6.1F) yields:

$$\left.\frac{\partial T}{\partial x}\right|_{x=\delta(t)} - \left.\frac{\partial T}{\partial x}\right|_{x=0} = \frac{1}{\alpha}\left[\frac{d}{dt}\int_0^{\delta(t)} T dx - T_i \frac{d\delta}{dt}\right] \tag{6.1J}$$

Designating the first term in the bracket as θ yields:

$$-\alpha\left.\frac{\partial T}{\partial x}\right|_{x=0} = \frac{d}{dt}\left[\theta - T_i\delta\right] \tag{6.1K}$$

or,

$$\underbrace{-k\left.\frac{\partial T}{\partial x}\right|_{x=0}}_{heat\ flow} = \underbrace{\rho c_p \frac{d}{dt}\left[\theta - T_i\delta\right]}_{sensible\ heat\ change} \tag{6.1L}$$

Equation (6.1K) is the energy integral equation. The problem is now to find the appropriate expression of the temperature profile. Assuming, for example, the cubic (3^{rd} order polynomial) temperature profile:

$$T(x,t) = a + bx + cx^2 + dx^3 \tag{6.1M}$$

Four boundary conditions are needed to solve for all constants in (6.1M). The following conditions are then listed:

$$T = T_0 \text{ at } x = 0;\ T = T_i \text{ at } x = \delta(t) \tag{6.1N}$$

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{x=0} = 0;\ \left.\frac{\partial T}{\partial x}\right|_{x=\delta(t)} = 0 \tag{6.1O}$$

Using these four conditions, the temperature profile (6.1M) becomes:

$$\frac{T(x,t) - T_i}{T_0 - T_i} = 1 - \frac{3}{2}\left(\frac{x}{\delta}\right) + \frac{1}{2}\left(\frac{x}{\delta}\right)^3 \tag{6.1P}$$

which satisfies the four boundary conditions given. Substituting this profile into the energy integral equation (6.1K) yields a simple ODE:

$$4\alpha = \delta\frac{d\delta}{dt} \text{ (at } t = 0,\ \delta = 0) \tag{6.1Q}$$

or,

$$\delta = \sqrt{8\alpha t} \tag{6.1R}$$

The final (approximate) temperature profile is obtained by substituting this result into the assumed temperature profile (6.1M).

If higher degree of accuracy is sought, a temperature profile of higher degree can be used. For example, if a 4^{th} order temperature profile was used the final profile would take the following form:

$$\frac{T(x,t)-T_i}{T_0-T_i}=1-2\left(\frac{x}{\delta}\right)+2\left(\frac{x}{\delta}\right)^3-\left(\frac{x}{\delta}\right)^4 \tag{6.1S}$$

and,

$$\delta=\sqrt{\frac{40}{3}\alpha t} \tag{6.1T}$$

Comparing these simple solutions with the following exact solution:

$$\frac{T(x,t)-T_i}{T_0-T_i}=1-erf\left(\frac{x}{\sqrt{4\alpha t}}\right) \tag{6.1U}$$

The cubic profile yields a solution with 6% error compared with the exact solution while the 4^{th} order profile yields the solution with only 3% error.

If a problem of interest is in cylindrical coordinates, the assumed profile should take the form:

$$T(r,t)=\frac{Polynomial\ in\ r}{ln(r)} \tag{6.1V}$$

If spherical coordinates are used, the assumed profile should take the following form:

$$T(r,t)=\frac{Polynomial\ in\ r}{r} \tag{6.1W}$$

Considering next a steady state, two-dimensional heat conduction in a semi-infinite slab shown in Figure 6.2.

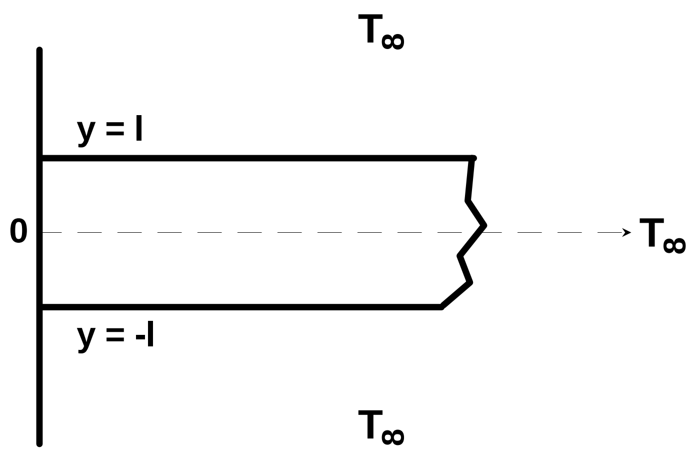

Figure 6.2. Steady state, two-dimensional heat conduction in a slab

Let $\theta = T - T_\infty$ and $\theta = \theta_{\max}\left(1 - \frac{y^2}{l^2}\right)$, where $\theta_{\max} = T_{\max} - T_\infty$. The energy equation, in terms of θ, for this case takes the form of the Laplace equation:

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0 \tag{6.1X}$$

subject to the following boundary conditions:

$$\theta(x, \pm l) = 0 \tag{6.1Y}$$

$$\theta(\infty, y) = 0 \tag{6.1Z}$$

$$\theta(0, y) = \theta_{\max}\left(1 - \frac{y^2}{l^2}\right) \tag{6.1AA}$$

The energy integral equation is obtained by integrating equation (6.1X) over the whole domain:

$$\int_0^l \int_0^\infty \left(\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2}\right) dx dy = 0 \tag{6.1AB}$$

The next step is to assume a temperature profile. Here, the Ritz profile is assumed:

$$\theta(x, y) = A\left(l^2 - y^2\right)e^{-Bx} \tag{6.1AC}$$

At $x = 0$, $\theta = \theta_{\max}\left(1 - \frac{y^2}{l^2}\right)$. From boundary condition (6.1AA), the value of the coefficient A is:

$$A = \frac{\theta_{\max}}{l^2} \tag{6.1AD}$$

To find the value of B, substitute A from equation (6.1AD) into equation (6.1AB) and integrate it:

$$\int_0^l \left.\frac{\partial\theta}{\partial x}\right|_0^\infty dy + \int_0^\infty \left.\frac{\partial\theta}{\partial y}\right|_0^l dx = 0 \tag{6.1AE}$$

or,

$$\frac{2}{3}\theta_{\max} Bl - \frac{2\theta_{\max}}{Bl} = 0 \tag{6.1AF}$$

So, the value of B is:

$$B = \frac{\sqrt{3}}{l} \tag{6.1AG}$$

Finally,

$$\theta = \frac{\theta_{\max}}{l^2}\left(l^2 - y^2\right)e^{-\frac{\sqrt{3}x}{l}} \tag{6.1AH}$$

If how the temperature variation in x (or in y) is not known, the following profile may be chosen:

$$\theta(x, y) = \left(l^2 - y^2\right)X(x) \tag{6.1AI}$$

or,

$$\theta(x, y) = e^{-Bx}Y(x) \tag{6.1AJ}$$

The derivation of the final temperature profiles is left to the student as an exercise.

6.2 Integral Solution of Convection-Type Problem

Consider, for example, a laminar flow over a flat plate of Figure 6.3. For simplicity, assuming first that the fluid has very low Prandtl number, hence $\delta << \delta_t$ and only the energy equation is required since the velocity is uniform at U_∞ (Arpaci and Larsen, 1984).

In this case the energy integral equation will be derived using a control volume analysis. For the derivation of the energy integral equation by direct integration of the governing conservation equation, see Bejan (1984) or Kakac and Yener (1995).

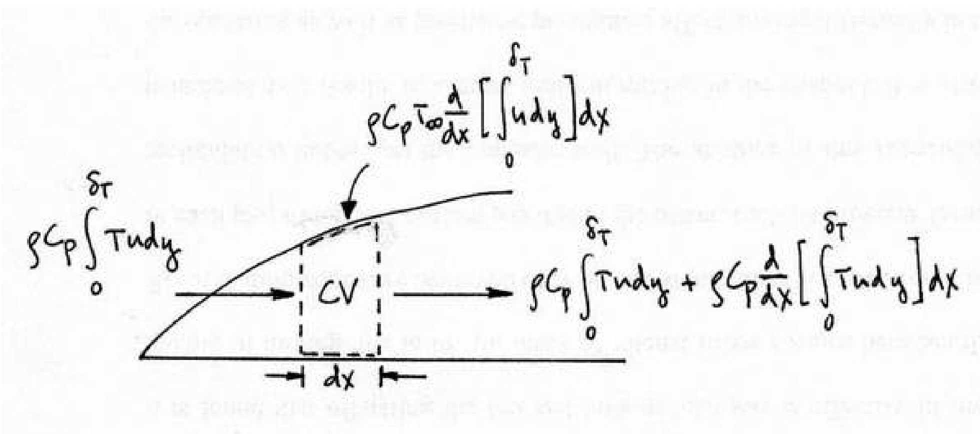

Figure 6.3. Control volume used to derive the energy integral equation

An energy balance of a control volume yields the required energy integral equation:

$$\rho c_p \frac{d}{dx}\int_0^{\delta_T} u(T - T_\infty)dy = -k\frac{\partial T}{\partial y}\bigg|_{y=0} \tag{6.2A}$$

where T_∞ is the free-stream temperature. The next step is, again, to assume a temperature profile. For simplicity, consider a simple quadratic profile:

$$\frac{T - T_0}{T_\infty - T_0} = a + b\left(\frac{y}{\delta_T}\right) + c\left(\frac{y}{\delta_T}\right)^2 \tag{6.2B}$$

Three boundary conditions are needed to obtain the values of *a*, *b* and *c*:

$$T(x,0) = T_0 \tag{6.2C}$$

$$T(x,\delta_T) = T_\infty \tag{6.2D}$$

$$\frac{\partial T(x,\delta_T)}{\partial y} = 0 \tag{6.2E}$$

Here, the values of *a*, *b* and *c* are: *a* = 0, *b* = 2 and *c* = -1. The assumed temperature profile thus becomes:

$$\frac{T - T_0}{T_\infty - T_0} = 2\left(\frac{y}{\delta_T}\right) - \left(\frac{y}{\delta_T}\right)^2 \tag{6.2F}$$

The final temperature profile is obtained once δ_T is known. This is obtained by inserting the obtained temperature profile back into the energy integral equation (6.2A), which becomes:

$$d\left(\delta_T^2\right) = 12\left(\frac{\alpha}{U_\infty}\right)dx \tag{6.2G}$$

where α is the thermal diffusivity. Equation (6.2G) is solved subject to the boundary condition $x = 0, \delta_T = 0$, which yields:

$$\delta_T = \left(\frac{12\alpha x}{U_\infty}\right)^{1/2} \tag{6.2H}$$[2]

The next step in the analysis of the convection-type problem is to derive a momentum integral equation. Recall that for steady, two-dimensional, incompressible, laminar boundary layer flow of constant physical property and constant free-stream temperature fluid, the energy integral equation is:

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[2] Keeping in mind that the solution obtained is only an approximate solution that fits the given boundary conditions.

$$\rho c_p \frac{d}{dx}\int_0^{\delta_T} u(T - T_\infty)dy = -k\left.\frac{\partial T}{\partial y}\right|_{y=0} \tag{6.2A}$$

To derive the momentum integral equation for the same type of problem, the law of conservation of linear momentum is written for the control volume of Figure 6.4a.[3] The net efflux of momentum in x-direction is:

$$\frac{d}{dx}\left[\int_0^{\delta} \rho u^2 dy\right]dx - U_\infty \frac{d}{dx}\left[\int_0^{\delta} \rho u dy\right]dx \tag{6.2I}$$

If all body forces are neglected, the forces acting on the control volume are just pressure forces and shear forces (Figure 6.4b). Net forces acting in x-direction is thus:

$$-\frac{dp}{dx}\delta dx - \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} dx \tag{6.2J}$$

The law of conservation of linear momentum (for steady state process) requires that the net rate of change of momentum in one coordinate direction be equal to the summation of all forces in the same coordinate direction. Thus,

$$\frac{d}{dx}\left[\int_0^{\delta} \rho u^2 dy\right] - U_\infty \frac{d}{dx}\left[\int_0^{\delta} \rho u dy\right] = -\frac{dp}{dx}\delta - \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} \tag{6.2K}$$

Since the flow is incompressible and

$$\frac{dp}{dx}\delta = -\rho U_\infty \frac{dU_\infty}{dx}\delta = -\rho U_\infty \frac{dU_\infty}{dx}\int_0^{\delta} dy \tag{6.2L}$$

Equation (6.2K) may be rearranged as:

$$\frac{d}{dx}\int_0^{\delta} u(U_\infty - u)dy - \frac{dU_\infty}{dx}\int_0^{\delta}(U_\infty - u)dy = \nu\left(\frac{\partial u}{\partial y}\right)_{y=0} \tag{6.2M}$$

Equation (6.2M) is the momentum integral equation in x-direction. Momentum integral equations in y- and z-direction can be derived using similar momentum and force balances in respective coordinate directions.

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[3] For the derivation of the momentum integral equation by direct integration of the momentum equation see, for example, Bejan (1984).

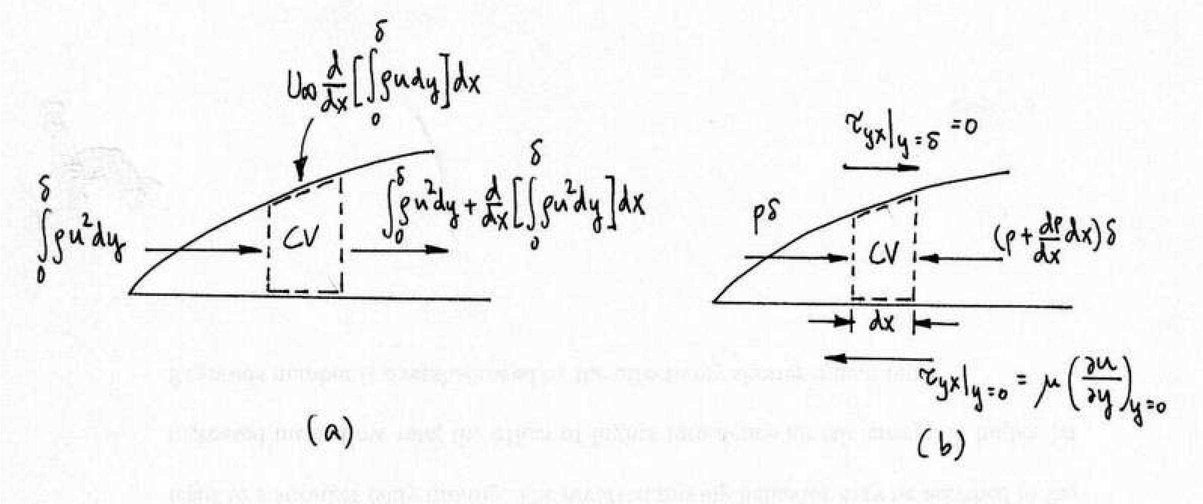

Figure 6.4. Control volume used to derive the momentum integral equation

6.3 Applications of the Integral Equations

First, consider a problem of the steady, two-dimensional laminar boundary layer flow over a flat plate. Assuming that the pressure gradient is zero and that U_∞ = constant. In this case the momentum integral equation simplifies to:

$$\frac{d}{dx}\int_0^\delta u\left(U_\infty - u\right)dy = \nu\left(\frac{\partial u}{\partial y}\right)_{y=0} \tag{6.3A}$$

Letting $\eta = \frac{y}{\delta}$ and $\bar{u} = \frac{u}{U_\infty}$ yields:

$$\frac{d}{dx}\left[\delta\int_0^1 \bar{u}\left(1-\bar{u}\right)d\eta\right] = \frac{\nu}{U_\infty \delta}\left.\frac{\partial \bar{u}}{\partial \eta}\right|_{\eta=0} \tag{6.3B}$$

Assuming that a linear velocity profile is used:

$$u = a + by \tag{6.3C}$$

Two boundary conditions are needed to obtain the values of a and b: at y = 0, u = 0; and at y = δ, u = U_∞. The final profile is thus:

$$\bar{u} = \eta \tag{6.3D}$$

Substituting the above profile into the momentum integral equation (6.3B) yields:

$$\frac{d}{dx}\left[\delta\int_0^1 \eta(1-\eta)\,d\eta\right]=\frac{\nu}{U_\infty\delta} \tag{6.3E}$$

which reduces to:

$$\delta\frac{d\delta}{dx}=\frac{6\nu}{U_\infty} \tag{6.2F}$$

which can be solved, subject to the following boundary condition: $\delta(0) = 0$, to yield the following solution:

$$\delta=\sqrt{\frac{12\nu x}{U_\infty}} \text{ or } \delta=3.47\sqrt{\frac{\nu x}{U_\infty}} \text{ or } \frac{\delta}{x}=\frac{3.47}{\sqrt{\mathrm{Re}_x}} \tag{6.2G}$$

where $\mathrm{Re}_x=\frac{U_\infty x}{\nu}$. The wall shear stress is:

$$\tau_w(x)=\frac{\mu U_\infty}{\delta}=0.288\mu U_\infty\sqrt{\frac{U_\infty}{\nu x}} \tag{6.3H}$$

and the friction coefficient, C_f, is given by:

$$C_f=\frac{\tau_w(x)}{\frac{1}{2}\rho U_\infty^2}=\frac{0.576}{\sqrt{\mathrm{Re}_x}} \tag{6.3I}$$

which can be averaged over the whole length of the plate L:

$$\overline{C}_{f_L}=\frac{1.152}{\sqrt{\mathrm{Re}_L}} \tag{6.3J}$$

The value of δ in equation (6.2G) is about 30% lower than that of the exact solution.

If the following cubic (3rd order) velocity profile is used:

$$\frac{u}{U_\infty}=a+b\left(\frac{y}{\delta}\right)+c\left(\frac{y}{\delta}\right)^2+d\left(\frac{y}{\delta}\right)^3 \tag{6.3K}$$

subject to the following boundary conditions: at $y = 0$, $u = 0$; at $y = \delta$, $u = U_\infty$; at $y = \delta$, $\frac{\partial u}{\partial y}=0$; and at $y = 0$, $\frac{\partial^2 u}{\partial y^2}=0$. The final profile is thus:

$$\frac{u}{U_\infty} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3 \tag{6.3L}$$

Let $\eta = \dfrac{y}{\delta}$ and $\bar{u} = \dfrac{u}{U_\infty}$, the momentum integral equation becomes:

$$\frac{d}{dx}\left[\delta \int_0^1 \left(1 - \frac{3}{2}\eta + \frac{\eta^3}{2}\right)\left(\frac{3}{2}\eta - \frac{\eta^3}{2}\right) d\eta\right] = \frac{3}{2}\frac{\nu}{U_\infty \delta} \tag{6.3M}$$

Finally,

$$\frac{\delta}{x} = \frac{4.64}{\sqrt{\mathrm{Re}_x}} \tag{6.3N}$$

which is only 7% lower than that of the exact solution. The wall shear stress and the average friction coefficient are:

$$\tau_w(x) = 0.323\mu U_\infty \sqrt{\frac{U_\infty}{\nu x}} \tag{6.3O}$$

$$\overline{C}_{f_L} = \frac{1.292}{\sqrt{\mathrm{Re}_L}} \text{ (3\% error)} \tag{6.3P}$$

For the next example, consider thermal boundary layer over a flat plate shown in Figure 6.5. The plate is heated and maintained at a temperature T_o only beyond the distance x_0 from the leading edge. The flow is assumed to be steady, two-dimensional, laminar and that the fluid has constant thermophysical properties. Viscous dissipation is also neglected. Since the velocity distribution becomes independent of the temperature distribution when the fluid properties are constant the velocity profile obtained earlier can be readily applied here. Starting from the energy integral equation:

$$\rho c_p \frac{d}{dx}\int_0^{\delta_T} u(T - T_\infty)dy = -k\left.\frac{\partial T}{\partial y}\right|_{y=0} \tag{6.2A}$$

Let $\theta = \dfrac{T - T_0}{T_\infty - T_0}$ and $\eta_T = \dfrac{y}{\delta_T}$, equation (6.2A) becomes:

$$\frac{d}{dx}\left[\delta_T \int_0^1 \bar{u}(1-\theta)d\eta_T\right] = \frac{\alpha}{U_\infty \delta_T}\left(\frac{\partial \theta}{\partial \eta_T}\right)_{\eta_T = 0} \tag{6.3Q}$$

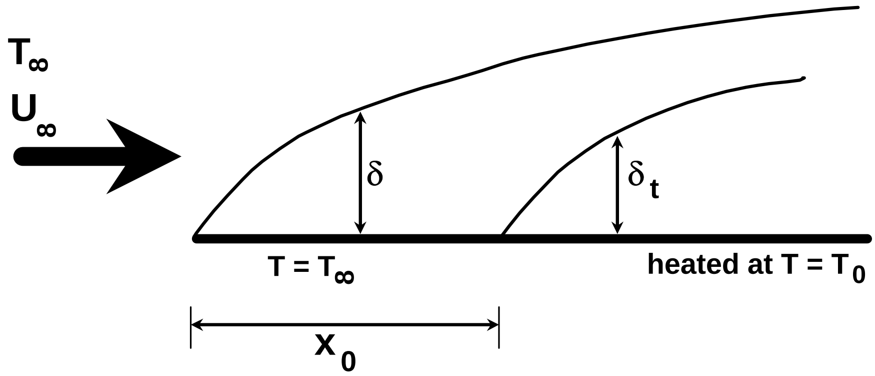

Figure 6.5. A flat plate with an unheated section

where $\bar{u} = \dfrac{u}{U_\infty}$. A cubic temperature profile can be assumed subject to the following boundary conditions: at y = 0, $T = T_0$, $\dfrac{\partial^2 T}{\partial y^2} = 0$; at $y = \delta_t$, $T = T_\infty$, $\dfrac{\partial T}{\partial y} = 0$ and $\dfrac{\partial^2 T}{\partial y^2} = 0$. The final temperature profile is thus:

$$\frac{T - T_0}{T_\infty - T_0} = \frac{3}{2}\left(\frac{y}{\delta_T}\right) - \frac{1}{2}\left(\frac{y}{\delta_T}\right)^3 \tag{6.3R}$$

Substituting this profile and a cubic velocity profile (6.3L) into the energy integral equation (6.3Q) yields:

$$\frac{d}{dx}\left[\delta_T \int_0^1 \left(1 - \frac{3}{2}\eta_T + \frac{1}{2}\eta_T^3\right)\left(\frac{3}{2}\eta - \frac{1}{2}\eta^3\right) d\eta_T\right] = \frac{3}{2}\frac{\alpha}{U_\infty \delta_T} \tag{6.3S}$$

Defining $\xi = \dfrac{\delta_T}{\delta} = \dfrac{\eta}{\eta_T}$ and assuming that $\delta_T << \delta$ yield:

$$\frac{d}{dx}\left[\xi \delta_T \left(\frac{3}{20} - \frac{3}{280}\xi^2\right)\right] = \frac{3}{2}\frac{\alpha}{U_\infty \delta_T} \tag{6.3T}$$

Since $\xi << 1$ (since $\delta_T << \delta$), the second term in the bracket can be neglected. Thus,

$$\frac{d}{dx}\left(\xi^2 \delta\right) = \frac{10\alpha}{U_\infty \xi \delta} \tag{6.3U}$$

and since $\delta = 4.64\sqrt{\dfrac{\nu x}{U_\infty}}$, equation (6.3U) becomes:

$$\xi^3 + \frac{4}{3}x\frac{d\xi^3}{dx} = \frac{0.929}{\Pr} \tag{6.3V}$$

subject to the boundary condition at $x_0 = 0, \xi = 0$ ($\delta_T = 0$). The solution of equation (6.3V) is thus:

$$\xi = \frac{0.976}{\sqrt[3]{\Pr}}\left[1 - \left(\frac{x}{x_0}\right)^{-3/4}\right]^{1/3} \tag{6.3W}$$

If $x_0 = 0$, i.e., there is no unheated section, equation (6.3W) reduces to:

$$\xi = \frac{0.976}{\sqrt[3]{\Pr}} = \text{constant} \tag{6.3X}$$

The local heat flux at the surface of the plate can be determined as:

$$q_w^{//} = -k\left(\frac{\partial T}{\partial y}\right)_{y=0} = \frac{3}{2}\frac{k}{\delta_T}(T_0 - T_\infty) = \frac{3}{2}\frac{k(T_0 - T_\infty)}{\xi\delta} \tag{6.3Y}$$

or,

$$q_w^{//} = 0.331\frac{k(T_0 - T_\infty)}{x}\Pr^{1/3}\mathrm{Re}_x^{1/2}\left[1 - \left(\frac{x}{x_0}\right)^{-3/4}\right]^{-1/3} \tag{6.3Z}$$

The local heat transfer coefficient is thus:

$$h_x = \frac{q_w^{//}}{T_0 - T_\infty} = 0.331\frac{k}{x}\Pr^{1/3}\mathrm{Re}_x^{1/2}\left[1 - \left(\frac{x}{x_0}\right)^{-3/4}\right]^{-1/3} \tag{6.3AA}$$

and the Nusselt number is:

$$\mathrm{Nu}_x = 0.331\Pr^{1/3}\mathrm{Re}_x^{1/2}\left[1 - \left(\frac{x}{x_0}\right)^{-3/4}\right]^{-1/3} \tag{6.3AB}$$

In the case where $\delta_T > \delta$, it is necessary to write the energy integral equation in two parts viz. one from $y = 0 \rightarrow y = \delta$ and the other from $y = \delta \rightarrow y = \delta_T$:

$$\frac{d}{dx}\int_0^\delta u(T_\infty - T)dy + U_\infty\frac{d}{dx}\int_\delta^{\delta_T}(T_\infty - T)dy = \alpha\left.\frac{\partial T}{\partial y}\right|_{y=0} \tag{6.3AC}$$

If the plate is heated at a constant heat flux the energy integral equation can be written as (assuming that $\delta_T << \delta$):

$$\frac{d}{dx}\int_0^{\delta_T} u(T - T_\infty)dy = \frac{1}{\rho c_p} q''_w \tag{6.3AD}$$

Assuming a cubic temperature profile:

$$T - T_\infty = a + b\left(\frac{y}{\delta_T}\right) + c\left(\frac{y}{\delta_T}\right)^2 + d\left(\frac{y}{\delta_T}\right)^3 \tag{6.3AE}$$

subject to the following boundary conditions: at y = 0, $\frac{\partial T}{\partial y} = -\frac{q''_w}{k}$ and $\frac{\partial^2 T}{\partial y^2} = 0$; at y = δ_t, $T = T_\infty$ and $\frac{\partial T}{\partial y} = 0$. Thus,

$$T - T_\infty = \frac{q''_w \delta_T}{3k}\left[2 - 3\left(\frac{y}{\delta_T}\right) + \left(\frac{y}{\delta_T}\right)^3\right] \tag{6.3AF}$$

and using also the cubic velocity profile $\frac{u}{U_\infty} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$. Substituting both velocity and temperature profiles into the energy integral equation (6.3AD) yields:

$$\frac{d}{dx}\left[\delta_T^2\left(\xi - \frac{1}{14}\xi^3\right)\right] = \frac{10\alpha}{U_\infty} \tag{6.3AG}$$

where $\eta = \frac{y}{\delta_T}$, $\xi = \frac{\delta_T}{\delta}$, and $\alpha = \frac{k}{\rho c_p}$. Since $\xi << 1$ ($\delta_T << \delta$), it is reasonable to neglect $\frac{1}{14}\xi^3$ as compared to ξ. Thus,

$$\frac{d}{dx}\left[\xi \delta_T^2\right] = \frac{10\alpha}{U_\infty} \tag{6.3AH}$$

or,

$$\frac{d}{dx}\left[\delta^2 \xi^3\right] = \frac{10\alpha}{U_\infty} \tag{6.3AI}$$

and since $\delta = 4.64\sqrt{\frac{\nu x}{U_\infty}}$,

$$\xi = \frac{0.774}{\sqrt[3]{\Pr}}\left[1-\frac{x_0}{x}\right]^{1/3} \quad (6.3AJ)$$

The local surface temperature can be calculated by setting $y = 0$ in equation (6.3AF):

$$T_0 - T_\infty = \frac{2}{3}\frac{q_w^{//}\delta_T}{k} \quad (6.3AK)$$

and since

$$\delta_T = \xi\delta = 3.591\Pr^{-1/3}\mathrm{Re}_x^{-1/2}\left[1-\frac{x_0}{x}\right]^{1/3} \quad (6.3AL)$$

Thus,

$$T_0 - T_\infty = 2.394\frac{q_w''}{k}x\Pr^{-1/3}\mathrm{Re}_x^{-1/2}\left[1-\frac{x_0}{x}\right]^{1/3} \quad (6.3AM)$$

The local heat transfer coefficient along the plate is calculated from:

$$h = \frac{q_w^{//}}{T_0 - T_\infty} = 0.418\frac{k}{x}\Pr^{1/3}\mathrm{Re}_x^{1/2}\left[1-\frac{x_0}{x}\right]^{-1/3} \quad (6.3AN)$$

and the local Nusselt number is:

$$\mathrm{Nu}_x = 0.418\Pr^{1/3}\mathrm{Re}_x^{1/2}\left[1-\frac{x_0}{x}\right]^{-1/3} \quad (6.3AO)$$

The above result for the local Nusselt number is about 10% lower than that of exact result. It is interesting to note that the Nusselt number for a plate with uniform heat flux exceeds the Nusselt number for an isothermal plate by about 25%.

For a plate with varying temperature profiles the principles of superposition can be applied (but only for constant-property fluids).

<u>Example 1</u> A flat plate, 100 cm in length and 30 cm in width, is used as a heating element. Air at 20°C and atmospheric pressure flows over the plate with a velocity of 6 m/s. The plate has unheated sections at both ends, each 25 cm in length, and the heated section is maintained at 140°C. Calculate the total heat transfer rate from the plate to the air.

<u>Solution</u> First, the properties of air should be evaluated at the film temperature, which is:

$$T_f = \frac{20+140}{2} = 80\,^{\mathrm{o}}\mathrm{C}$$

At this temperature the air properties are as follows: k = 0.0299 W m^{-1} K^{-1}, ν = 20.94× 10^{-6} m^2 s^{-1}, Pr = 0.708. At $x = 0.75$ m:

$$\text{Re}_L = \frac{U_\infty L}{\nu} = \frac{6 \times 0.75}{20.94 \times 10^{-6}} = 2.15 \times 10^5 < 5 \times 10^5 \text{ (laminar BL flow)}$$

and,

$$\xi = \frac{\delta_T}{\delta} = \frac{0.976}{\sqrt[3]{\text{Pr}}}\left[1 - \left(\frac{x}{x_0}\right)^{-3/4}\right]^{1/3} \tag{E61A}$$

$$= \frac{0.976}{\sqrt[3]{0.708}}\left[1 - \left(\frac{0.75}{0.25}\right)^{-3/4}\right]^{1/3} = 0.903 < 1$$

Thus, the flow is laminar and $\delta_T < \delta$ at $x = 0.75$ m. The local heat transfer coefficient in the region where $0.25 \text{ m} < x < 0.75$ m is then given by:

$$h_x = 0.331\frac{k}{x}\text{Pr}^{1/3}\,\text{Re}_x^{1/2}\left[1 - \left(\frac{x}{x_0}\right)^{-3/4}\right]^{-1/3} \tag{E61B}$$

or,

$$h_x = 4.72x^{-1/2}\left[1 - \left(\frac{x}{x_0}\right)^{-3/4}\right]^{-1/3} \tag{E61C}$$

where $x_0 = 0.25$ m. The average heat transfer coefficient can now be calculated as:

$$\bar{h} = \frac{1}{0.5}\int_{0.25}^{0.75} h_x dx = \frac{4.72}{0.5}\int_{0.25}^{0.75} x^{-1/2}\left[1 - \left(\frac{x}{x_0}\right)^{-3/4}\right]^{-1/3} dx \tag{E61D}$$

Let $z = x^{3/4} - x_0^{3/4}$, then $dz = \frac{3}{4}x^{-1/4}dx$. Therefore, the average heat transfer coefficient becomes:

$$\bar{h} = 12.59\int_0^{0.452} z^{-1/3}dz = 18.89 \times (0.452)^{2/3} = 11.13 \text{ W m}^{-2}\text{ K}^{-1} \tag{E61E}$$

Hence, the total rate of heat transfer is:

$$q = \bar{h}A(T_w - T_\infty) = 200.34 \text{ W} \tag{E61F}$$

The final example of the use of the integral method is the analysis of the flow in parallel plates channel. Consider a laminar flow of a constant physical properties fluid through a channel shown in Figure 6.6.

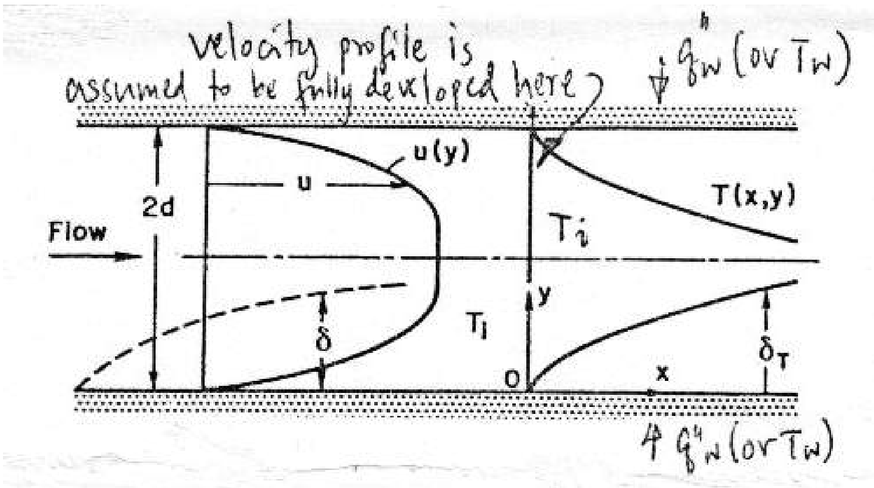

Figure 6.6. Laminar flow in parallel plates channel

The following conditions are assumed: the flow is laminar, all fluid properties are constant, viscous dissipation is neglected, temperature of the fluid outside the thermal boundary layer is at T_i and $\delta_T = 0$ at $x = 0$. The velocity profile is assumed to be fully developed at x = 0. Therefore, the heated (or cooled) section starts from x = 0 and the thermal boundary layer grows in thickness along the length of the duct until it reaches the center line where it meets the boundary layer from the other wall of the channel.

The analysis starts with the energy integral equation (written in terms of the heat flux):

$$q''_w = \frac{d}{dx}\int_0^{\delta_T} \rho c_p u(T_i - T)dy \tag{6.3AP}$$

where the parabolic velocity profile is an obvious choice here. A cubic temperature profile is again chosen:

$$\frac{T - T_i}{q''_w \delta_T / k} = a + b\left(\frac{y}{\delta_T}\right) + c\left(\frac{y}{\delta_T}\right)^2 + d\left(\frac{y}{\delta_T}\right)^3 \tag{6.3AQ}$$

Four boundary conditions are thus needed: at $y = 0, \frac{\partial T}{\partial y} = -\frac{q_w}{k}, \frac{\partial^2 T}{\partial y^2} = 0$; at $y = \delta_T$, $T = T_i$, $\frac{\partial T}{\partial y} = 0$. The resulting temperature profile is therefore:

$$T - T_i = \frac{q_w'' \delta_T}{3k}\left[2 - 3\left(\frac{y}{\delta_T}\right) + \left(\frac{y}{\delta_T}\right)^3\right] \text{ for } 0 \le y \le \delta_T \tag{6.3AR}$$

Substituting the velocity and temperature profiles into the energy integral equation yields:

$$q_w'' = \frac{d}{dx}\left[\frac{\rho c_p U_c q_w''}{3k}\delta_T \int_0^1 \left(2\eta\xi - \eta^2\xi^2\right)\left(2 - 3\eta + \eta^3\right) d\eta\right] \tag{6.3AS}$$

where $\eta = \frac{y}{\delta_T}$ and $\xi = \frac{\delta_T}{d}$. Performing the integral in equation (6.3AS) yields:

$$\frac{d\xi^3}{dx} = \frac{15}{2}\frac{k}{\rho c_p U_c d^2} \tag{6.3AT}$$

where the higher-order terms in ξ have been neglected since $\xi < 1$. Solving this equation leads to:

$$\xi^3 = \frac{15}{2}\frac{k}{\rho c_p U_c d^2} x + C \tag{6.3AU}$$

Since $\xi = 0$ at $x = 0$, the solution for ξ becomes:

$$\xi = \sqrt[3]{\frac{80x/d_H}{\text{Re Pr}}} \tag{6.3AV}$$

where $d_H = 4d$, $\text{Re} = \frac{U_m d_H}{\nu}$ and $U_m = \frac{2}{3}U_c$. From equation (6.3AR) the temperature profile is:

$$T_w - T_i = \frac{2}{3}\frac{q_w'' \delta_T}{k} \tag{6.3AW}$$

or,

$$T_w - T_i = \frac{2q''_w d}{3k} \sqrt[3]{\frac{80x / d_H}{\text{Re Pr}}} \tag{6.3AX}$$

The heat transfer coefficient based on $T_w - T_i$ is given by:

$$h_x = \frac{6k}{d_H} \sqrt[3]{\frac{\text{Re Pr}}{80x / d_H}} \tag{6.3AY}$$

and the local Nusselt number is given by:

$$\text{Nu}_x = 6\sqrt[3]{\frac{\text{Re Pr}}{80x / d_H}} \quad \text{(based on } T_w - T_i\text{)} \tag{6.3AZ}$$

If the bulk temperature, T_m, is used instead of the wall temperature, T_w, the temperature profile becomes:[4]

$$T_w - T_m = \frac{2q''_w d_H}{k} \left[\frac{\xi}{3} - \frac{\xi^3}{10} + \frac{\xi^4}{48} \right] \tag{6.3BA}$$

Hence, the heat transfer coefficient based on $T_w - T_m$ can be written as:

$$h_x = \frac{k}{d_H} \frac{2}{\left[\frac{\xi}{3} - \frac{\xi^3}{10} + \frac{\xi^4}{48} \right]} \tag{6.3BB}$$

and the expression for the local Nusselt number is thus:

$$\text{Nu}_x = \frac{2}{\left[\frac{\xi}{3} - \frac{\xi^3}{10} + \frac{\xi^4}{48} \right]} \quad \text{(based on } T_w - T_m\text{)} \tag{6.3BC}$$

For the case of the constant wall temperature, the temperature profile can be expressed as:

$$\frac{T - T_i}{T_w - T_i} = 1 - \frac{1}{2}\left(\frac{y}{\delta_T}\right)\left[3 - \left(\frac{y}{\delta_T}\right)^2\right] \quad \text{for } 0 \le y \le \delta_T \tag{6.3BD}$$

Substituting the velocity and temperature profiles into the energy integral equation and performing the required integral yields:

[4] The derivation of the subsequent equations is left as an exercise to the student.

$$\xi = \sqrt[3]{\frac{120x/d_H}{\text{Re Pr}}} \tag{6.3BE}$$

The local Nusselt numbers based on $T_w - T_i$ and $T_w - T_m$ are:

$$\text{Nu}_x = 6\sqrt[3]{\frac{\text{Re Pr}}{120x/d_H}} \quad (\text{based on } T_w - T_i) \tag{6.3BF}$$

$$\text{Nu}_x = \frac{6}{\xi\left[1 - \frac{3}{2}\left(\frac{\xi^2}{5} + \frac{\xi^3}{24}\right)\right]} \quad (\text{based on } T_w - T_m) \tag{6.3BG}$$

References

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Problems

1. In a food processing operation, 100^{o}F oil flows over a wide, 20-ft long, heated plate at 0.2 ft/s. For a surface temperature of 200^{o}F, determine:

   (a) Hydrodynamic boundary layer thickness δ at the end of the plate.
   (b) Thermal boundary layer thickness δ_T at the end of the plate.
   (c) Local heat transfer coefficient h_x at the end of the plate.
   (d) Total heat flux from the surface per unit width.

   Hint: The average heat transfer coefficient required in evaluating the total heat flux equals to two times the local heat transfer coefficient at $x = L$.

   Assume the thermal diffusivity α of the oil to be 2.8×10^{-3} ft^2 h^{-1}, the kinematic viscosity ν to be 7×10^{-4} ft^2 s^{-1}, and the thermal conductivity k to be 0.123 Btu h^{-1} ft^{-1} ^{o}F^{-1} at the film temperature.

2. Consider a two-dimensional steady and laminar boundary-layer flow of a constant property fluid over a flat plate with a constant and uniform free-stream velocity U_∞ and temperature T_∞. Let the heat flux from the plate to the fluid be prescribed as:

$$q_w'' = 0, \text{ when } 0 < x < x_0$$

$$q_w'' = mx, \text{ when } x > x_0$$

where m is a given constant and x denotes the distance from the leading edge of the plate. Assuming linear velocity and temperature profiles in the respective boundary layers and neglecting viscous dissipation, develop an expression for the local heat transfer coefficient applicable for those $x > x_0$ where $\delta_T > \delta$.

3. With a smooth horizontal flow of air over a flat plate of length L and width W, a laminar boundary layer is developed on the plate surface. The expression for the velocity profile in the boundary layer is:

$$\frac{u}{U_\infty} = \sin \pi \left(\frac{y}{\delta} \right)$$

Assuming an incompressible flow of air and that the free stream velocity U_∞ is constant, develop relationships for:

(a) The boundary layer thickness δ in terms of the coordinate location x and the Reynolds number at any point x downstream of the leading edge.
(b) The resistance to the flow of air offered by the plate surface.

4. Consider the steady and two-dimensional laminar boundary-layer flow of a constant-property fluid over a flat plate with a constant and uniform free-stream velocity U_∞ and temperature T_∞. The plate is maintained isothermally at T_w. Assuming that the velocity and temperature profiles in the respective boundary layers at any x from the leading edge of the plate can be approximated as a function of the distance y normal to the plate by the following relations:

$$\frac{u}{U_\infty} = \frac{y}{\delta} \text{ and } \frac{T - T_w}{T_\infty - T_w} = \frac{y}{\delta_T}$$

where δ and δ_T are the velocity and thermal boundary layer thicknesses, respectively.

(a) Develop an expression for δ.
(b) It is estimated that $\delta / \delta_T = 10$. Develop an expression for the local heat transfer coefficient, h_x.
(c) Estimate the Prandtl number of the fluid.

5. Consider the laminar flow of a two-dimensional liquid film on a flat wall inclined at an angle α relative to the horizontal direction. The film flow is driven by the gravitational acceleration component $g\sin\alpha$ acting parallel to the wall. Attach the Cartesian system of coordinates (x,y) and (u,v) to the wall, such that x and u point in the flow direction. In this notation derive the terminal velocity distribution in the liquid film $u(y)$; in other words, determine the flow in the limit where the film inertia is negligible and the x-momentum

equation expresses a balance between the film weight and the wall friction. Let U be the undetermined free-surface velocity at $y = \delta$, where δ is the film thickness.

6. From Problem 5, consider next the heat transfer from the wall to the liquid film, in the case where the film and wall temperature is T_0 everywhere upstream of $x = 0$, and where the wall temperature alone is raised to $T_0 + \Delta T$ downstream of $x = 0$. Let δ_T be the thermal boundary layer thickness of the thin liquid region in which the wall heating effect is felt. Based on scale analysis, demonstrate that immediately downstream from $x = 0$ (where δ_T is much smaller than δ), the thermal boundary layer thickness δ_T scales as $\left(\frac{\alpha \delta x}{U}\right)^{1/3}$. Also determine the temperature distribution in the film based on an integral method, assuming the following temperature profile:

$$\frac{T(x,y)-T_0}{\Delta T} = 1 - 2\left(\frac{y}{\delta_T}\right) + \left(\frac{y}{\delta_T}\right)^2 \; ; \; 0 \le y \le \delta_T$$
$$T(x,y) = T_0 \; ; \; \delta_T \le y \le \delta$$

Note that this integral analysis is valid as long as $\delta_T(x) \le \delta$. At what distance $x = x_1$ will the free surface feel the heating effect of the wall (i.e., at what x will δ_T equal δ)? Devise also an integral analysis to determine the film temperature field $T(x,y)$ downstream from the point $x = x_1$.

7 Consider hydrodynamically fully developed steady laminar flow of a constant-property fluid between two parallel plates separated by a distance L. Let the plates at $y = 0$ and $y = L$ be maintained at constant temperatures T_{w1} and T_{w2}, respectively, for $x > 0$, and the fluid be at a constant temperature T_i at $x = 0$. Assuming uniform velocity profile across the entire flow area (i.e., slug flow) and neglecting axial conduction of heat. Obtain an expression for the temperature distribution $T(x,y)$ in the thermal entrance region

8. Consider the laminar boundary layer flow where the mass is being swept away from a wall surface of concentration C_0 by a uniform stream containing none of the species released by the wall (or $C_\infty = 0$). Analyze this problem according to the following steps:

- Start with a concentration integral equation:

$$\frac{d}{dx}\int_0^{\delta_c} uC dy = -D\left(\frac{\partial C}{\partial y}\right)_{y=o} = k_c C_0$$

or, in dimensionless form,

$$\frac{d}{dx}\left[\delta_c \int_0^1 \bar{u}\omega d\eta_c\right] = -\frac{D}{U_\infty \delta_c}\left(\frac{\partial \omega}{\partial \eta_c}\right)_{\eta_c=0}$$

where δ_c is the concentration boundary layer thickness, D is the mass diffusivity, k_c is the mass transfer coefficient, $\bar{u} = \frac{u}{U_\infty}, \omega = \frac{C}{C_0}$ and $\eta_c = \frac{y}{\delta_c}$.

- Assume the following velocity and concentration profiles:

$$u = U_\infty \text{ and } C = C_0\left[1 - \frac{y}{\delta_c(x)}\right] \quad 0 \le y \le \delta_c$$

These choices imply that the fluid of interest has low Schmidt number (Sc = $\frac{\nu}{D}$) or $\delta << \delta_c$.

Determine the following:

(a) The concentration boundary layer thickness, in terms of Re_x and Sc.
(b) The local Sherwood number, defined as $Sh_x = \frac{k_c x}{D}$, in terms of Re_x and Sc.

9. A water stream is heated in fully developed flow through a parallel-plates channel with uniform heat flux at the walls. The mass flowrate of water is $\dot{m}$ = 10 g/s, the heat flux $q^{//}$ = 0.1 W/cm^2, and the spacing between the two plates is 1 cm. Based on the results of an integral analysis of this type of problem (using a parabolic velocity profile and a cubic temperature profile), calculate:

   (a) The Reynolds number based on the channel hydraulic diameter and the mean fluid velocity.
   (b) The difference between the wall temperature and the mean (bulk) fluid temperature.
   (c) The heat transfer coefficient based on the difference between the wall temperature and the mean fluid temperature.

   Use the following fluid properties in your calculation: ρ = 0.9971 g/cm^3, μ = 0.00891 g/cm s, ν = 0.00894 cm^2/s, k = 0.60 W/m K, C_p = 4.179 kJ/kg K.

10. A kitchen in a restaurant has a large, flat burner plate for frying. Since a great deal of heat rises from the plate the chef decides to let a small fan blow over the burner, which is 1.2 m long and positioned some 1.5 m down a flat smooth counter from the fan (the total length is 2.7 m). If 30^o C air blows at 2.0 m/s and the plate is at 120^o C what is the heat transfer rate per square meter at x = 2.7 m? Use the following properties of air in the calculation: $\nu = 2.079 \times 10^{-5}$ m^2 s^{-1}, k = 0.030 W m^{-1}K^{-1}, and Pr = 0.697.

11. Using the parabolic velocity profile in the momentum integral equation when assuming that the pressure gradient is zero and the free stream velocity is constant, determine the following quantities for steady, two-dimensional laminar boundary layer flow over a flat plate:

(a) Boundary layer thickness, in terms of the local Reynolds number, Re_x.

(b) Local skin friction coefficient C_f, which is defined as $C_f = \dfrac{\tau_w(x)}{\frac{1}{2}\rho U_\infty^2}$, in terms of Re_x,

where $\tau_w(x)$ is the wall shear stress and is determined from the Newton's law of viscosity, $\tau_w(x) = \mu \left.\dfrac{\partial u}{\partial y}\right|_{y=0}$.