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FDE618: Transport Phenomena in Food Processing
ChE610: Fundamentals of Transport Phenomena

Chapter 5
Analytical Solution Techniques for Transport Problems I: Scale Analysis

In previous chapters a complete set of the governing conservation equations as well as initial and boundary conditions needed in the modeling of momentum, heat and mass transfer processes are derived for both single- and multi-component systems. The steps in using these equations and initial/boundary conditions to solve various transport problems are outlined schematically in Figure 5.1. Generally, a complete set of equations and initial/boundary conditions are first written for a problem. Assumptions are then made in order to simplify the problem; a simplest possible model that focuses only on important aspects of the problem (but is still physically real) is almost always desirable at this point. Either analytical or numerical solution is then sought for the problem.

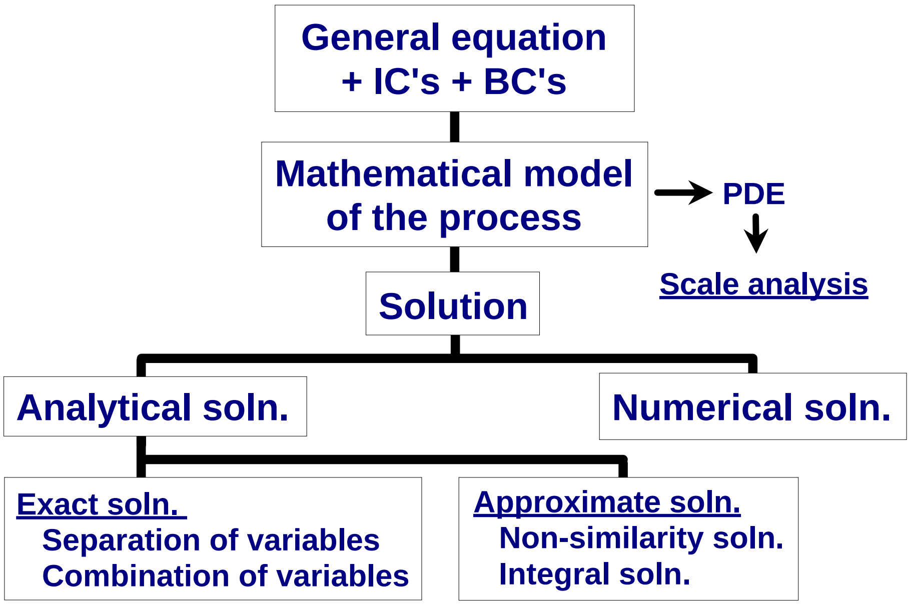

Figure 5.1. Steps in modeling transport problems

We will begin Part II with the simplest, but very powerful, techniques viz. scale analysis (or scaling).[1] This technique uses basic principles of transport phenomena to produce order-of-magnitude estimates for the quantity of interest. When done properly it gives the relative order of magnitude of terms in the governing conservation equations for a specific domain and over a specified period of time. It is recommended as the premier method when analyzing transport problems as it gives the most information per unit of intellectual effort (Bejan, 1984).

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[1] This technique should not be confused with the dimensional analysis or the non-dimensionalization of the governing conservation equations.

5.1 Basic Rules of Scale Analysis

To illustrate the technique consider the following example. In Figure 5.2 a metal plate is plunged into a highly conductive fluid (i.e., very small temperature gradients) at $t = 0$ such that the surfaces of the plate instantaneously assume the fluid temperature $T_\infty = T_0 + \Delta T$. Suppose that the quantity of interest here is the time needed by the thermal front to penetrate through the plate.

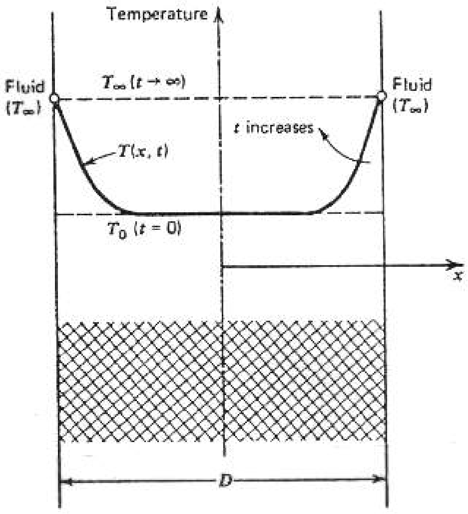

Figure 5.2. Transient heat conduction in a one-dimensional conducting slab with a sudden temperature change at the boundary

Focusing on the half-plate of thickness $D/2$ and writing the energy equation for pure conduction in one direction as:

$$\rho c_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} \tag{5.1A}$$

The order of magnitude of each of the terms in equation (5.1A) can be estimated as:

$$\rho c_p \frac{\partial T}{\partial t} \sim \rho c_p \frac{\Delta T}{t} \tag{5.1B}$$

and,

$$k \frac{\partial^2 T}{\partial x^2} = k \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) \sim \frac{k}{D/2}\frac{\Delta T}{D/2} = \frac{k\Delta T}{(D/2)^2} \tag{5.1C}$$

Equating the two orders of magnitude, equations (5.1B) and (5.1C), as required by the original equation (5.1A) yields:

$$t \sim \frac{(D/2)^2}{\alpha} \tag{5.1D}$$

where α is thermal diffusivity of the medium. The penetration time predicted by equation (5.1D) compares well with any interpretation of the exact solution to this problem. However, the time and effort required to derive this penetration time do not compare with the effort required by Fourier analysis needed in order to obtain the exact solution of the same problem.

Based on the above examples the following rules of scale analysis are observed (Bejan, 1984):

1. Define the spatial extent of the region in which the scale analysis is being performed. Any scale analysis of a region that is not uniquely defined is meaningless.
2. Any equation constitutes equivalence between the scales of two dominant terms appearing in the equation.
3. If $c = a+b$ and $O(a) > O(b)$ then $O(c) \sim O(a)$[2]
4. If $c = a+b$ and $O(a) \sim O(b)$ then $O(c) \sim O(a) \sim O(b)$
5. If $p = ab$ then $O(p) \sim O(a)O(b)$; if $q = a/b$ then $O(q) \sim O(a)/O(b)$

For brevity the notation *O( )* for order of magnitude will be drop from this point onwards.

Scale analysis will be first applied to the problem of laminar boundary layer flow due to its popularity among engineers who study transport phenomena and due to its being a "model" for other more complicated transport problems.

5.2 Laminar Boundary Layer Flow

Consider a flow situation in Figure 5.3. In this figure a flat plate that has a temperature of T_0 is suspended in a uniform (free) stream of velocity U_∞ and temperature T_∞. If this plate is the plate-fin protruding from a heat-exchanger surface into the stream that flows through it some basic question can be asked: what is the net force exerted by the fluid stream on the plate? What is the resistance to the transfer of heat from the plate to the fluid stream?

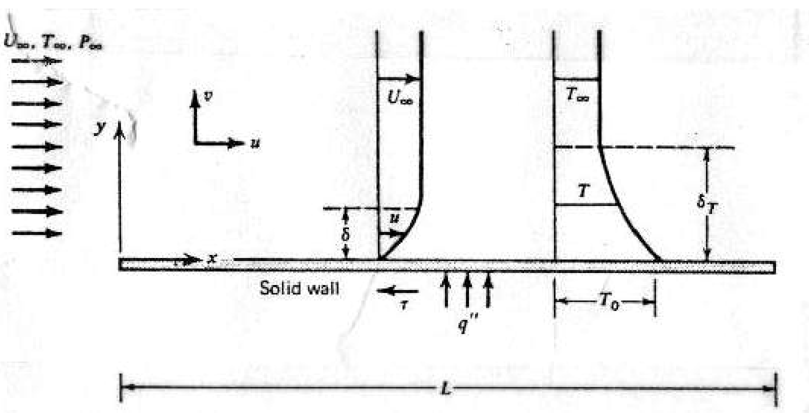

Figure 5.3. Boundary layer flow over a flat plate

[2] *O(a)* means "order of magnitude of a."

The first question above must be answered, for example, in order to predict the total drag force exerted by the fluid stream on the heat-exchanger surface. The second question must be answered to predict the rate of heat transfer from the solid surface to the fluid.

Assuming the steady-state laminar flow of Newtonian fluid of constant physical properties, the total force can be calculated from the following equation:

$$F = \int_0^L \tau w dx \tag{5.2A}$$

where $\tau = -\mu \left(\frac{\partial u}{\partial y} \right)_{y=0}$ is the shear stress experienced by the wall. The total heat transfer rate can be calculated from:

$$q = \int_0^L q'' w dx \tag{5.2B}$$

where $q'' = -k \left(\frac{\partial T}{\partial y} \right)_{y=0} = h(T_0 - T_\infty)$ is the wall heat flux; w is the width of the plate in the direction perpendicular to the plane of Figure 5.3.

It can be seen that most quantities of engineering interest relate only to the wall region. It is therefore interesting to see if it is possible to focus our attention only on the region close to the wall.[3] Based on the aforementioned assumptions a complete set of the governing conservation equations (for the whole flow region) can be first written as:

(continuity)
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{5.2C}$$

(x-momentum)
$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{\mu}{\rho} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \tag{5.2D}$$

(y-momentum)
$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \frac{\mu}{\rho} \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \tag{5.2E}$$

(energy)
$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{k}{\rho c_p} \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) \tag{5.2F}$$

The above equations are subjected to the following boundary conditions:

$$y = 0, \quad u = v = 0 \tag{5.2G}$$

$$y \to \infty, \quad u \to U_\infty, \quad v = 0 \tag{5.2H}$$

$$y = 0, \quad T = T_0 \tag{5.2I}$$

$$y \to \infty, \quad T \to T_\infty \tag{5.2J}$$

[3] This region is the so-called boundary layer region.

It is interesting to apply the scale analysis to this flow problem. Consider again Figure 5.3; in this figure:

$\delta \times L$ - Domain for fluid flow (hydrodynamic BL)
$\delta_T \times L$ - Domain for heat transfer (thermal BL)[4]

where δ is the "order of magnitude"[5] of the boundary layer thickness, which is the "order of magnitude" of the distance in which u changes from 0 to roughly U_∞ and δ_T is the "order of magnitude" of the thickness of the region in which T varies from T_0 at the wall to T_∞ in the free stream.

In the first domain $(\delta \times L)$: $x \sim L$, $u \sim U_\infty$, $y \sim \delta$. The x-momentum equation can then be scaled as:

$$U_\infty\left(\frac{U_\infty}{L}\right), v\left(\frac{U_\infty}{\delta}\right) \sim \left(\frac{p}{\rho L}\right), \nu\frac{U_\infty}{L^2}, \nu\frac{U_\infty}{\delta^2} \tag{5.2K}$$

The terms on the left represent the inertia force, while the second and third terms on the right represent the viscous force. If it is assumed that $\frac{\delta}{L} << 1$[6] or $(\delta << L)$, which is a very reasonable assumption for the boundary layer flow, the x-momentum equation can now be scaled as:

$$U_\infty\left(\frac{U_\infty}{L}\right) \sim \left(\frac{p}{\rho L}\right), \nu\frac{U_\infty}{\delta^2} \tag{5.2L}$$

A modified x-momentum equation is thus:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \tag{5.2M}$$

Conducting the same scale analysis for the y-momentum equation yields:

$$U_\infty\left(\frac{v}{L}\right), v\left(\frac{v}{\delta}\right) \sim \frac{p}{\rho\delta}, \nu\frac{v}{L^2}, \nu\frac{v}{\delta^2} \tag{5.2N}$$

With the aid of the scaled continuity equation and the slenderness assumption, relation (5.2N) reduces to:

[4] The domain used for fluid flow analysis is not the same (except in some cases) as the domain used for heat (or mass) transfer analysis. Some textbooks do not point out this difference clearly, however.
[5] Recall that the scale analysis is currently used to analyze the problem.
[6] This assumption is the so-called slenderness assumption.

$$\frac{U_\infty^2\delta}{L^2}, \frac{U_\infty^2\delta}{L^2} \sim \frac{p}{\rho\delta}, \nu\frac{v}{\delta^2} \tag{5.2O}$$

So the *y*-momentum equation becomes:

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\frac{\partial^2 v}{\partial y^2} \tag{5.2P}$$

Equation (5.2P) is not usually discussed in connection with the boundary layer analysis but it forms a basis for another important result viz. the replacement of $\dfrac{\partial p}{\partial x}$ in equation (5.2M). To see how this is done, consider first the general case where pressure is a function of both coordinates:

$$p = p(x, y) \tag{5.2Q}$$

and its total derivative is:

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy \tag{5.2R}$$

or,

$$\frac{dp}{dx} = \frac{\partial p}{\partial x} + \frac{\partial p}{\partial y}\frac{dy}{dx} \tag{5.2S}$$

The orders of magnitude of the two pressure gradients can be deduced from the *x*- and *y*-momentum equations by performing a balance between pressure forces and either the friction or inertia term. The choice is arbitrary as long as it is the same for both equations. For example, if the scaling pair is the pressure and friction terms:

$$\frac{\partial p}{\partial x} \sim \mu\frac{u_\infty}{\delta^2} \tag{5.2T}$$

$$\frac{\partial p}{\partial y} \sim \mu\frac{v}{\delta^2} \tag{5.2U}$$

and,

$$\frac{\dfrac{\partial p}{\partial y}\dfrac{dy}{dx}}{\dfrac{\partial p}{\partial x}} \sim \frac{\left(\dfrac{\mu v}{\delta^2}\right)\left(\dfrac{\delta}{L}\right)}{\dfrac{\mu U_\infty}{\delta^2}} \sim \left(\frac{\delta}{L}\right)^2 \quad << \quad 1 \tag{5.2V}$$

It is reasonable therefore to drop the term $\dfrac{\partial p}{\partial y}\dfrac{dy}{dx}$ and assume that, in the boundary layer:

$$\frac{dp}{dx}=\frac{\partial p}{\partial x} \text{ or } p=p(x) \text{ only} \tag{5.2W}$$

Equation (5.2W) implies that the pressure varies mainly in *x*-direction or longitudinal direction in the boundary-layer region. In other words, at any *x* position, the pressure inside the boundary layer is practically the same as the pressure immediately outside it. In mathematical term:

$$\frac{\partial p}{\partial x}=\frac{dp_\infty}{dx} \tag{5.2X}$$

Substituting this new information into equation (5.2M) yields:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{dp_\infty}{dx}+\nu\frac{\partial^2 u}{\partial y^2} \tag{5.2Y}$$

This is the *Prandtl's boundary layer equation* and is a statement for momentum conservation in both *x* and *y* directions.

For the energy conservation, the energy equation, equation (5.2F), is:

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right) \tag{5.2F}$$

which can be scaled as:

$$u\frac{\Delta T}{L},v\frac{\Delta T}{\delta_T}\sim\alpha\frac{\Delta T}{L^2},\alpha\frac{\Delta T}{\delta_T^2} \tag{5.2Z}$$

Assuming again the slenderness assumption for the thermal boundary layer, the energy equation reduces to:

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2} \tag{5.2AA}$$ [7]

If $u\sim U_\infty$ and assuming that $\dfrac{d\,p_\infty}{dx}=0$, which is the simplest form of the free stream possible, the boundary layer momentum equation implies:

[7] This equation is now solvable in many ways. We will continue solving this equation using the scale analysis for the moment.

$$\frac{U_\infty^2}{L}, \frac{\nu U_\infty}{\delta} \sim \nu \frac{U_\infty}{\delta^2} \tag{5.2AB}$$

or,

$$\frac{U_\infty^2}{L} \sim \nu \frac{U_\infty}{\delta^2} \tag{5.2AC}$$

hence,

$$\delta^2 \sim \frac{\nu L}{U_\infty} \text{ or } \delta \sim \sqrt{\frac{\nu L}{U_\infty}} \tag{5.2AD}$$

If the Reynolds number based on the longitudinal dimension of the boundary-layer region, Re_L, is defined as $\mathrm{Re}_L = \frac{L v_\infty}{\nu}$. Then,

$$\frac{\delta}{L} \sim \mathrm{Re}_L^{-1/2} \tag{5.2AE}$$

Here, $\mathrm{Re}_L^{1/2}$ has no physical meaning; it has only the geometrical meaning. It simply states that the slenderness postulate on which the boundary layer theory is based is valid provided that $\mathrm{Re}_L^{1/2} >> 1$. It is a test of whether the boundary-layer analysis can be used with a given external flow situation as Re_L can be easily calculated beforehand.

Returning to the problem at hand, the wall shear stress can now be scaled as:

$$\tau \sim \mu \frac{U_\infty}{L} \mathrm{Re}_L^{1/2} \sim \rho U_\infty^2 \, \mathrm{Re}_L^{-1/2} \tag{5.2AF}$$

If the skin friction coefficient C_f is defined as,

$$C_f = \frac{\tau}{\frac{1}{2}\rho U_\infty^2} \tag{5.2AG}$$

then,

$$C_f \sim \mathrm{Re}_L^{-1/2} \tag{5.2AH}$$

The question of wall friction has now been answered in an order of magnitude sense.

For the heat transfer problem, the heat transfer coefficient can be scaled as:

$$h \sim \frac{k(\Delta T/\delta_T)}{\Delta T} \sim \frac{k}{\delta_T} \tag{5.2AI}$$

where $\Delta T = T_0 - T_\infty$ is the temperature variation in the region $\delta_T \times L$. From equation (5.2AA), the scale analysis implies:

$$u\frac{\Delta T}{L}, v\frac{\Delta T}{\delta_T} \sim \alpha\frac{\Delta T}{\delta_T^2} \tag{5.2AJ}$$

and from the continuity equation,

$$v \sim u\frac{\delta_T}{L} \tag{5.2AK}$$

Thus,

$$u\frac{\Delta T}{L} \sim \alpha\frac{\Delta T}{\delta_T^2} \tag{5.2AL}$$

To evaluate δ_T it is erroneous, as pointed out by Bejan (1984), to directly scale u with U_∞. The actual velocity scale in the δ_T layer depends on the relative size of δ and δ_T as shown in Figure 5.4.

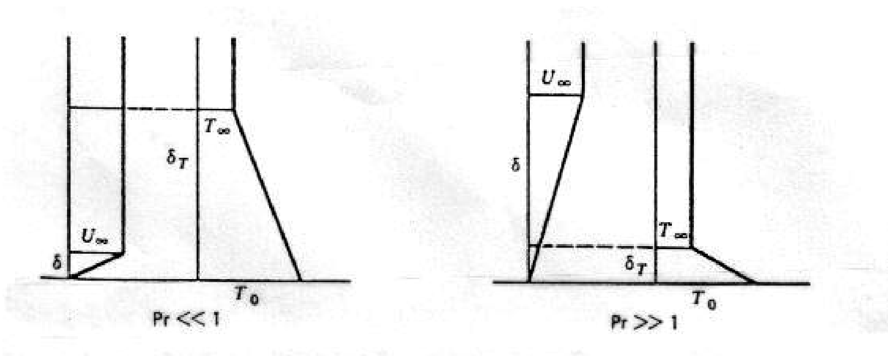

Figure 5.4. The Prandtl number effect on the joint development of velocity and temperature boundary layers

Consider the first case where $\delta << \delta_T$, hence $u \sim U_\infty$ everywhere. Equation (5.2AL) can then be scaled as:

$$\frac{U_\infty \Delta T}{L} \sim \alpha\frac{\Delta T}{\delta_T^2} \tag{5.2AM}$$

or,

$$\frac{\delta_T}{L} \sim \mathrm{Pe}_L^{-1/2} \sim \mathrm{Pr}^{-1/2}\,\mathrm{Re}_L^{-1/2} \tag{5.2AN}$$

or in terms of the hydrodynamic boundary-layer thickness,

$$\frac{\delta_T}{\delta} \sim \Pr^{-\frac{1}{2}} (>>1) \qquad (5.2AO)$$

where $\mathrm{Pe} = \dfrac{U_\infty L}{\alpha}$ is the Peclet number. It is very interesting to observe that the relative sizes of δ and δ_T depends only on the Prandtl number of the fluid of interest. The assumption that $\delta << \delta_T$ is therefore valid only when $\Pr^{\frac{1}{2}} << 1$, which represents the range occupied by liquid metals. Following this assumption and result, the heat transfer coefficient can be scaled as:

$$h \sim \frac{k}{\delta_T} \sim \frac{k}{L} \Pr^{\frac{1}{2}} \mathrm{Re}_L^{\frac{1}{2}} \qquad (5.2AP)$$

The Nusselt number is then:

$$\mathrm{Nu} \sim \Pr^{\frac{1}{2}} \mathrm{Re}_L^{\frac{1}{2}} \qquad (5.2AQ)$$

For the second case where $\delta >> \delta_T$, $u \sim \dfrac{\delta_T}{\delta} U_\infty$. Hence, from equation (5.2AL):

$$\frac{\delta_T}{\delta} \frac{U_\infty \Delta T}{L} \sim \alpha \frac{\Delta T}{\delta_T^2} \qquad (5.2AR)$$

or,

$$\frac{\delta_T}{L} \sim \Pr^{-\frac{1}{3}} \mathrm{Re}_L^{-\frac{1}{2}} \qquad (5.2AS)$$

or, again, in terms of the hydrodynamic boundary-layer thickness,

$$\frac{\delta_T}{\delta} \sim \Pr^{-\frac{1}{3}} (<<1) \qquad (5.2AT)$$

This assumption and result is therefore valid for high-Pr fluids such as oils, water, which are of practical interest in food engineering and processing. The heat transfer results are then:

$$h \sim \frac{k}{\delta_T} \sim \frac{k}{L} \Pr^{\frac{1}{3}} \mathrm{Re}^{\frac{1}{2}} \qquad (5.2AU)$$

and,

$$\mathrm{Nu} \sim \Pr^{\frac{1}{3}} \mathrm{Re}_L^{\frac{1}{2}} \qquad (5.2AV)$$

Example 1 Consider the development of a two-dimensional laminar jet discharging in the x-direction into a fluid reservoir that contains the same fluid as the jet. The reservoir pressure P_∞ is uniform. The jet is generated by a narrow slit of width D_0; the average fluid velocity through the slit is U_0.

Let $D(x)$ and $U(x)$ be the jet thickness scale and the centerline velocity scale at a sufficiently long distance x away from the nozzle (the slit). Relying on the mass and momentum conservation equations, on boundary layer theory ($D<<x$), and on scale analysis in a flow region of length x and thickness D, determine the order of magnitude of D and U in terms of (D_0, U_0, x, ν).

Solution Based on the problem statement and the assumptions that the process is at steady state and that the fluid is Newtonian and has constant physical properties, the (simplified) governing conservation equations for this flow situation can be written as:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad \text{(E51A)}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{dp_\infty}{dx}+\nu\frac{\partial^2 u}{\partial y^2} \qquad \text{(E51B)}$$

The following scaling balances can then be written:

$$\frac{U}{x}\sim\frac{v}{D} \text{ or } v\sim U\frac{D}{x} \qquad \text{(E51C)}$$

$$U\frac{U}{x}, v\frac{U}{D}\sim\nu\frac{U}{D^2} \text{ or } UD^2\sim\nu x \qquad \text{(E51D)}^8$$

To make it easier to proceed it is better to write the momentum equation as:

$$\frac{\partial}{\partial x}\left(u^2\right)+\frac{\partial}{\partial y}\left(uv\right)=\nu\frac{\partial^2 u}{\partial y^2} \qquad \text{(E51E)}$$

Integrating this equation at x = constant plane yields:

$$\frac{d}{dx}\int_{-\infty}^{\infty}u^2dy+u_\infty v_\infty-u_{-\infty}v_\infty=\nu\left.\frac{\partial u}{\partial y}\right|_{y=\infty}-\nu\left.\frac{\partial u}{\partial y}\right|_{y=-\infty} \qquad \text{(E51F)}$$

or,

$$\frac{d}{dx}\int_{-\infty}^{\infty}u^2dy=0 \text{ or } \int_{-\infty}^{\infty}u^2dy = \text{constant} \qquad \text{(E51G)}^9$$

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[8] This balance can be written since P_∞ = constant.

[9] This equation is obtained since the free stream is assumed to be uniform.

and the scale of (E51G) is U^2D (or $U_0^2 D_0$ or $D \sim D_0 \dfrac{U_0^2}{U^2}$) since the scale is independent of the distance in x-direction as shown in (E51G). Since, from the balance (E51D), $UD^2 \sim \nu x$ or $U^2 \sim \dfrac{\nu^2 x^2}{D^4}$, the scale of the jet thickness is $D \sim \dfrac{U_0^2 D_0 D^4}{\nu^2 x^2}$. Finally, the scales of the jet thickness and the centerline velocity at a sufficient long distance from the slit are:

$$D \sim U_0^{-2/3} D_0^{-1/3} \nu^{2/3} x^{2/3} \tag{E51H}$$

$$U \sim U_0^{4/3} D_0^{2/3} \nu^{-1/3} x^{-1/3} \tag{E51I}$$

5.3 Laminar Duct Flow

In this section the fluid friction and heat transfer between a fluid stream and a solid object in internal flow is considered. The fundamental questions are the same as in external flow viz. What is the friction force or the pressure drop in the flow direction? What is the heat transfer coefficient in the direction normal to the flow?

We begin our analysis with the simplest duct flow problem, i.e., fully developed flow in parallel-plate duct (Figure 5.5); this class of flow problem is introduced as a direct consequence of the concept of boundary layer encountered in external flow. The steady state continuity and momentum equations are:

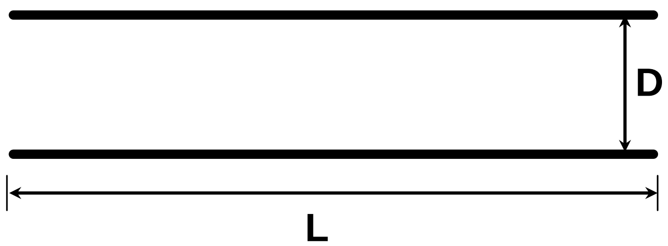

Figure 5.5. A two-dimensional parallel-plate duct

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{5.3A}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{5.3B}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{5.3C}$$

Far enough from the entrance (fully developed region), the following scales can be used: $x \sim L$, $y \sim D$ and $u \sim U$. The continuity equation thus implies:

$$v \sim U\left(\frac{D}{L}\right) \tag{5.3D}$$

In the fully developed region ($L >> D$), therefore, $v = 0$ and $\frac{\partial u}{\partial x} = 0$.[10] These two conditions indeed are the definition of the fully developed flow. Using these two conditions the *y*-momentum equation becomes:

$$\frac{\partial p}{\partial y} = 0 \quad \rightarrow \quad p = p(x) \text{ only!} \tag{5.3E}$$

and the *x*-momentum equation becomes:

$$\frac{dp}{dx} = \mu\frac{\partial^2 u}{\partial y^2} = \text{constant} \tag{5.3F}$$

Solving this equation subject to $u = 0$ at $y = \pm\frac{D}{2}$ yields,

$$u = \frac{3}{2}U\left[1 - \left(\frac{y}{(D/2)}\right)^2\right] \tag{5.3G}$$

where $U = \frac{D^2}{12\mu}\left(-\frac{dp}{dx}\right)$ and *y* is the distance measured from the centerline of the channel. The velocity profile is parabolic and the velocity is proportional to the pressure drop per unit duct length in the flow direction. This solution is the well-known Hagen-Poiseuille solution for fully developed flow between parallel plates.

For a duct of arbitrary cross section, equation (5.2F) is replaced by:

$$\frac{dp}{dx} = \mu\nabla^2 u = \text{constant} \tag{5.3H}$$

where, in the Laplacian operator, $\frac{\partial^2 u}{\partial x^2} = 0$. For example, the fully developed laminar flow in a round tube of radius r_0 can be represented as:

$$\frac{dp}{dx} = \mu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) \tag{5.3I}$$

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[10] In the entrance region, $y \sim \delta$ (not $y \sim D$) and $v \neq 0$, $\frac{\partial u}{\partial x} \neq 0$.

Solving this equation subject to $u = 0$ at $r = r_0$ yields:

$$u = 2U\left[1-\left(\frac{r}{r_o}\right)^2\right] \quad (5.3J)$$

where $U = \frac{r_o^2}{8\mu}\left(-\frac{dp}{dx}\right)$. These results were first reported by Hagen in 1839 and, independently, by Poiseuille in 1840 (Bejan, 1984). In general, the solution to the Poisson-type equation (5.3H) is considerably more difficult.

Consider next the basic heat transfer question of determining the heat transfer coefficient (i.e., the relationship between the heat transfer rate q'' and the wall-fluid temperature difference). If $\Delta T = T_0 - T_m$ is selected as the wall-fluid temperature difference, the heat transfer coefficient can be expressed as:

$$h = \frac{q''}{T_0 - T_m} = \frac{-k\left(\frac{\partial T}{\partial r}\right)_{r=r_o}}{T_0 - T_m} \quad (5.3K)$$

In order to determine the value of the so-called bulk temperature (or mixing cup temperature) T_m, a microscopic energy balance of a tube of radius r_0 in which the fluid with an average axial velocity U flows through (see Figure 5.6) is made:

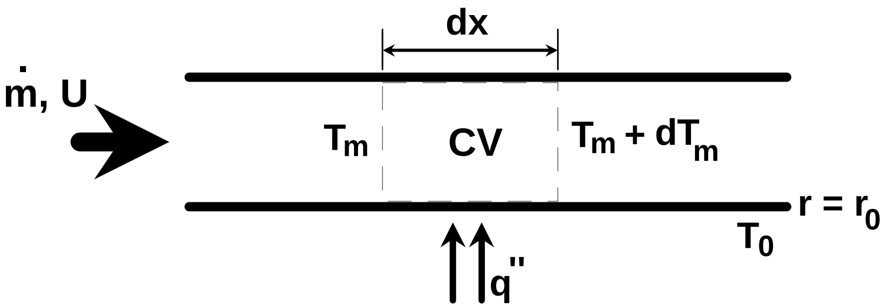

Figure 5.6. Microscopic energy balance in a duct

From the first law of Thermodynamics, the steady state energy balance can be written as:

$$c_p \dot{m}\, dT_m = q'' 2\pi r_0 dx \quad (5.3L)$$

or,

$$\frac{dT_m}{dx} = \frac{2}{r_0}\frac{q''}{\rho c_p U} \quad (5.3M)$$

where $\dot{m} = \pi r_0^2 \rho U$. Note that T_m is not just any mean or average temperature but is the mean temperature whose definition is derived from the first law analysis of bulk flow. If the first law is again written but for the bundle of mini streams $\rho u dA$ ($d\dot{m}$):

$$q'' 2\pi r_o dx = d \int_0^{2\pi} \int_0^{r_0} \rho c_p u T r \, dr d\theta \tag{5.3N}$$

or, with the aid of equation (5.3M),

$$T_m \rho c_p u A = \int_0^{2\pi} \int_0^{r_o} \rho c_p u T r \, dr d\theta \tag{5.3O}$$

In constant property tube flow, equation (5.3O) reduces to:

$$T_m = \frac{1}{\pi r_o^2 u} \int_0^{2\pi} \int_0^{r_o} u T r dr d\theta \tag{5.3P}$$

The problem is now to find the temperature at every point in the cross section *T(x,r)*. For steady, axisymmetric flow through a round tube, the energy equation can be written as:

$$\frac{1}{\alpha}\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial r} \right) = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial x^2} \tag{5.3Q}$$

In the hydrodynamic fully developed region ($v = 0$, $\frac{\partial u}{\partial x} = 0$), equation (5.3Q) reduces to:

$$\frac{u(r)}{\alpha}\frac{\partial T}{\partial x} = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial x^2} \tag{5.3R}$$

Equation (5.3R) expresses a balance between a maximum of three possible effects viz. axial convection, radial conduction[11] and axial conduction. The scales of these terms are:

$$\frac{U}{\alpha}\left(\frac{q''}{D\rho c_p u} \right) \sim \frac{\Delta T}{D^2}, \frac{1}{x}\left(\frac{q''}{D\rho c_p u} \right) \qquad (5.3S)^{12}$$

[11] This term must always be present or there will be no heat transfer from the wall to the fluid stream!

[12] In a thermal entrance region, the proper scale of $\frac{\partial^2 T}{\partial r^2}$ is $\frac{\Delta T}{\delta_T^2}$ with $\delta_T << D$.

Multiplying these scales by $\dfrac{D^2}{\Delta T}$ and using the definition of $h = \dfrac{q''}{\Delta T}$ yields:

$$\frac{hD}{k} \sim 1,\ \frac{1}{x}\left(\frac{hD}{k}\right)^2 \left(\frac{\alpha}{uD}\right)^2 \tag{5.3T}$$[13]

In the limit of large $\mathrm{Pe}\ (\mathrm{Pe} >> 1)$, it can be concluded therefore that the axial conduction is negligible and, from the convection and radial conduction balance, the Nusselt number is a constant of order one.

$$\frac{hD}{k} \sim 1 \quad [\,\mathrm{Nu} \sim O(1) \text{ in fully developed pipe flow}] \tag{5.3U}$$

In the same domain, the energy equation to solve for *T(x,r)* therefore reduces to:

$$\frac{u(r)}{\alpha}\frac{\partial T}{\partial x} = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} \tag{5.3V}$$[14]

5.4 Laminar Free (Natural) Convection

The flow and heat transfer problems of sections 5.2 and 5.3 are regarded as examples of forced convection, i.e., examples of flow and heat transfer that are forced to happen; the creation and maintenance of the flow requires a consistent input of mechanical power.

In this section another class of flow and heat transfer is focused. The flows of this section are not forced but happen freely (or naturally); these flows are driven by the buoyancy effect due to the presence of gravitational acceleration and density variations from one fluid layer to another. Mathematically, the flow field is coupled with the temperature field as temperature variations within the fluid can induce density variations and hence a buoyancy-driven flow (see Chapter 2, section 2.3).

Figure 5.7 shows the simplest configuration for the study of free convection. In this figure a body of temperature T_0 and height H immerses in a fluid of temperature T_∞. The fluid reservoir is assumed to be very large that the downward movement of the fluid is negligible. Thus, the vertical velocity of the fluid situated very far from the heated wall is zero.[15]

[13] Equation (47) of Chapter 3 in Bejan (1984) is incorrect; its correct form is equation (5.3T) presented here.
[14] The result is obtained as far as the scale analysis can go.
[15] The situation will be different, of course, if the circulation is in the enclosure.

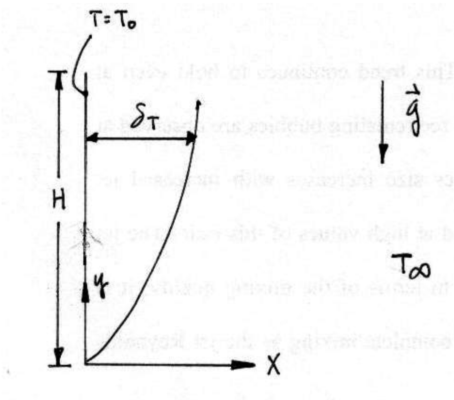

Figure 5.7. Free convection along a vertical wall

A set of simplified governing conservation equations can be written in the boundary layer region ($x \sim \delta_T$, $y \sim H$ and $\delta_T << H$) as:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{5.4A}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \nu\frac{\partial^2 v}{\partial x^2} + g\beta(T - T_\infty) \tag{5.4B}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\left(\frac{\partial^2 T}{\partial x^2}\right) \tag{5.4C}$$

Let $x \sim \delta_T$, $y \sim H$ and $T - T_\infty \sim \Delta T = T_o - T_\infty$. Scaling of the energy equation yields:

$$u\frac{\Delta T}{\delta_T}, v\frac{\Delta T}{H} \sim \alpha\frac{\Delta T}{\delta_T^2} \tag{5.4D}$$

From continuity equation:

$$\frac{u}{\delta_T} \sim \frac{v}{H} \text{ or } u \sim \frac{v\delta_T}{H} \tag{5.4E}$$

Since the two convection terms in equation (5.4D) are of the same order, the energy balance becomes:

$$\frac{v\Delta T}{H} \sim \alpha\frac{\Delta T}{\delta_T^2} \text{ or } v \sim \frac{\alpha H}{\delta_T^2} \tag{5.4F}$$ [16]

To find the scale of δ_T, consider the momentum equation in the $\delta_T \times H$ region. The momentum equation can be scaled as:

[16] This velocity is the so-called dominant velocity.

$$\underbrace{u\frac{v}{\delta_T}, v\frac{v}{H}}_{inertia} \text{ or } \underbrace{\frac{\nu}{\delta_T^2}}_{viscous} \sim \underbrace{g\beta\Delta T}_{buoyancy} \tag{5.4G}$$

The question now is which pair of balances to be chosen (or used) between the inertia ~ buoyancy and the viscous ~ buoyancy?

Dividing the scales (5.4G) by $g\beta\Delta T$ and using equation (5.4F) to eliminate the vertical velocity scale:

$$\frac{\frac{v^2}{H}}{g\beta\Delta T} \text{ or } \frac{\frac{\nu}{\delta_T^2}}{g\beta\Delta T} \sim 1 \tag{5.4H}$$

Define Rayleigh number as $\mathrm{Ra}_H = \dfrac{g\beta\Delta T H^3}{\alpha\nu}$ and write the scales (5.4H) as:

$$\underbrace{\left[\frac{H}{\delta_T}\right]^4 \mathrm{Ra}_H^{-1}\,\mathrm{Pr}^{-1}}_{inertia} \text{ or } \underbrace{\left[\frac{H}{\delta_T}\right]^4 \mathrm{Ra}_H^{-1}}_{viscous} \sim \underbrace{1}_{buoyancy} \tag{5.4I}$$

Therefore, the competition between inertia and viscous terms is decided by the Prandtl number, which represents the fluid property; high-Pr fluids (of interest to a food engineer) will form a thermal layer ruled by the viscous ~ buoyancy balance while low-Pr fluids will form a layer ruled by the inertia ~ buoyancy balance.

For high-Pr fluids (Pr >>1), the scales (5.4I) become:

$$\left[\frac{H}{\delta_T}\right]^4 \mathrm{Ra}_H^{-1} \sim 1 \text{ or } \left[\frac{H}{\delta_T}\right]^4 \sim \mathrm{Ra}_H \tag{5.4J}$$

or,

$$\delta_T \sim H\mathrm{Ra}_H^{-1/4} \tag{5.4K}$$

and from $v \sim \dfrac{\alpha H}{\delta_T^2}$,

$$v \sim \frac{\alpha}{H}\mathrm{Ra}_H^{1/2} \tag{5.4L}$$

Finally, since $h \sim \dfrac{k}{\delta_T}$, it follows that:

$$\mathrm{Nu} \sim \mathrm{Ra}_H^{1/4} \tag{5.4M}$$

For low-Pr fluids ($\mathrm{Pr} << 1$), the scales (5.4I) become:

$$\delta_T \sim H\left(\mathrm{Ra}_H \,\mathrm{Pr}\right)^{-1/4} \tag{5.4N}$$

and,

$$v \sim \frac{\alpha}{H}\left(\mathrm{Ra}_H \,\mathrm{Pr}\right)^{1/2} \tag{5.4O}$$

and,

$$\mathrm{Nu} \sim \mathrm{Ra}_H^{1/4}\,\mathrm{Pr}^{1/4} \tag{5.4P}$$

The Boussinesq number can be defined as $\mathrm{Bo}_H = \dfrac{g\beta \Delta T H^3}{\alpha^2}$ or $\mathrm{Bo}_H = \left(\mathrm{Ra}_H \,\mathrm{Pr}\right)$.

In the case of the constant heat flux q'' wall (instead of an isothermal wall):

$$h = \frac{q''}{\Delta T} \sim \frac{k}{\delta_T} \tag{5.4Q}$$

or,

$$\Delta T \sim \frac{q''}{k}\delta_T \tag{5.4R}$$

and the constant heat flux Rayleigh number is defined as:

$$\mathrm{Ra}_H^* = \frac{g\beta H^4 q''}{\alpha \nu k} \tag{5.4S}$$

Reference

1. Bejan, A., 1984, **Convection Heat Transfer**, Wiley, New York.

Problems

1. Consider a 0.6 m$\times$0.6 m thin square plate in a room at 30°C. One side of the plate is maintained at a temperature of 74°C, while the other side is insulated. Determine the order-of-magnitude rate of heat transfer from the plate by natural convection if the plate is placed vertically. Use the following properties: k = 0.0279 W/m °C, $\nu = 1.815\times10^{-5}$ m^2/s, Pr = 0.709 and β = 0.00308 K^{-1}.

2. A man has a bottle of beer at room temperature and would like to drink it cold and as soon as possible. The beer bottle has a height-to-diameter ratio of about 5. He places the bottle in the refrigerator; however, he has the option of positioning the bottle (1) vertically or (2) horizontally. The refrigerator cools by natural convection, i.e., it does not employ forced convection. Which way should the man position his bottle? Describe the goodness of his decision by calculating the ratio t_1/t_2, where t represents the order of magnitude of the time needed for the bottle to reach thermal equilibrium with the refrigerator chamber. Based this calculation on scale analysis.

3. Determine the velocity distribution for Hagen-Poiseuille flow through a duct whose cross-section has the shape of an extremely slender wedge (a triangle with tip angle $\varepsilon << 1$, and long sides equal to b). Neglect the friction effect introduced by the short wall opposing the tip angle, whose length is εb. Start with the following equation:

$$\frac{dP}{dx} = \mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \text{constant}$$

where y is the direction along b and z is the direction along εb. Check the relative order of magnitude of the two terms on the right hand side of the above equation and neglect the insignificant one.

4. A common procedure for increasing the humidity of air is to bubble it through a column of water. Assuming that the air bubbles are spheres of radius r_0 = 1 mm and are in thermal equilibrium with water at 25^{o}C. Write the transient mass diffusion equation and apply the scale analysis to determine the time the bubbles should remain in water to achieve a saturated (maximum) concentration. The air is dry when it enters the water column. Assuming also that the diffusion is only in radial direction and the external resistance to mass transfer is negligible. In addition, it is assumed that all physical properties involved in the calculation are constant. At 25^{o}C the binary diffusion coefficient of water in air is 0.26×10^{-4} m^2 s^{-1}.

5. Consider the laminar boundary layer flow where the mass is being swept away from a wall surface of concentration C_0 in the limit of $\text{Sc}\rightarrow 0$. Construct a scale analysis based on equation (P55A) and the assumption that the concentration boundary layer is much thicker than the velocity boundary layer, i.e., $u = U_\infty$ and $v = 0$. It is assumed here that the pressure gradient is zero and there is no wall-stream temperature difference. The free stream concentration is C_∞.

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D\frac{\partial^2 C}{\partial y^2} \qquad \text{(P55A)}$$

Determine the order-of-magnitude of the Sherwood number, which is defined as Sh = $\dfrac{k_c L}{D}$ in terms of the Reynolds number based on the longitudinal dimension of the boundary layer region, Re_L, and the Schmidt number, Sc. Here k_c is the mass transfer coefficient whose definition can be obtained from the heat transfer/mass transfer analogy (i.e., analogy from the Newton's law of cooling for the definition of heat transfer coefficient) and L is the length of the flat plate.

Hint: The definition of the Peclet number for mass transfer can also be easily obtained from the heat transfer/mass transfer analogy and can be used here.

6. A block of ice at the melting point (T_m) is pushed downward against another solid that moves to the right with the velocity U. The sliding solid plate is warmer (its temperature is maintained at $T_m + \Delta T$) and causes steady melting at the lowest surface of the melting block.

The liquid that is generated by this melting process fills the gap δ. A schematic sketch of this melting situation is shown in Figure P5.6.

(a) Write the energy equation for the liquid film δ (with reference to the x-y coordinates shown in Figure P5.6 in which u and v are the corresponding liquid velocity components) noting that the effect of viscous dissipation cannot be neglected in this case. The liquid film is assumed to be very thin, i.e., $\delta << L$. The fluid thermophysical properties may also be assumed to be constant.
(b) Apply scale analysis to the energy equation written in (a) to arrive at the criterion when convection in film is negligible relative to conduction.
(c) Repeat the steps in (b) to arrive at the criterion when the viscous heating is negligible with respect to conduction.

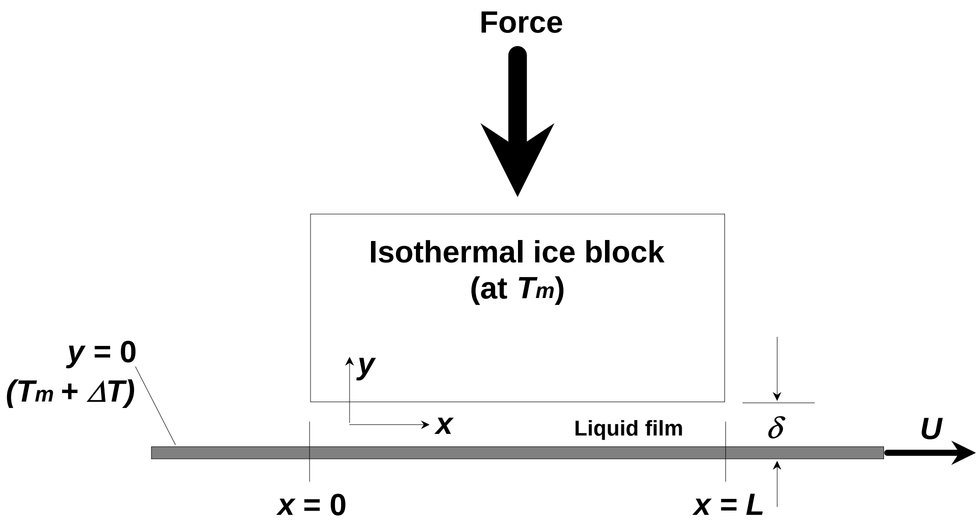

Figure P5.6. Block of ice pressed against a hot sliding plate

7. Consider another case of Problem 5.6 (see Figure P5.7) where the plane slider is not heated externally but the viscous heating of the liquid film is responsible for the steady melting of the ice block; this situation occurs at sufficiently high slider velocities U. In this case the conduction heat transfer absorbed by the melting front is balanced by the viscous heating effect for which the characteristic temperature difference ΔT can be obtained.

Write the continuity, x-momentum and energy equations (with dissipation term) along with the following energy balance equation at the melting front ($y = \delta$),

$$q = Lk\left(-\frac{\partial T}{\partial y}\right)_{y=\delta} = L\rho_{ice}V\lambda$$

where V and λ are the melting speed and the latent heat of melting, respectively; this equation represents the balance between the conduction heat transfer from below and the melting speed of the ice.

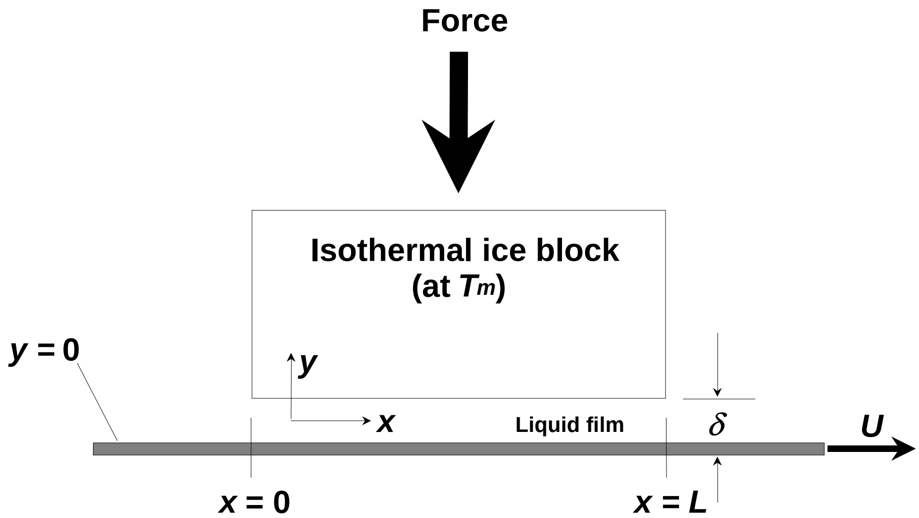

Figure P5.7. Block of ice pressed against a sliding plate

Show, via scale analysis of the above equations, that the melting speed can be displayed in the form of the Peclet number based on L as:

$$\frac{VL}{\alpha} \sim \left(\frac{\rho}{\rho_{ice}} \mathrm{Ste}_{\mu} \right)^{3/4} \left(\frac{\Delta P L^2}{\mu \alpha} \right)^{1/4}$$

where Ste_{μ} is the Stefan number based on the viscous heating temperature rise:

$$\mathrm{Ste}_{\mu} = \frac{C_p}{\lambda} \left(\frac{\mu}{k} U^2 \right)$$

and ΔP (which can be used as a pressure scale in the x-momentum equation) is the excess pressure, which is defined as the ratio of the net weight of the object surrounded by the liquid to the horizontal projected area of that object. When performing the scale analysis the effects of inertia and convection can be neglected.