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FDE618: Transport Phenomena in Food Processing
ChE610: Fundamentals of Transport Phenomena

Chapter 3 Interface and Continua

The interface visible between two phases is actually a thin region over which the properties of the materials change in a continuous manner from the values in one bulk phase to those in the others. When using the continuum assumption for the bulk phases interfaces require careful treatment.

3.1 The Interface between Motionless Fluids

Figure 3.1 shows the variation of the density between two isothermal, motionless phases, (+) and (-). The upward direction is positive; gravity acts downward. The density varies continuously over a thin region, perhaps 1-100 nm thick, from $\rho^{(-)}$ in the lower phase to $\rho^{(+)}$ in the upper phase. There is no density variation in the plane normal to x_3 . In each bulk phase the stress is isotropic:

$$\underline{\sigma} = - p \underline{I} \quad (3.1A)$$

This stress originates in the motion of the molecules and in intermolecular forces. In the bulk phases, where the density is uniform, these motions and forces are the same in all directions; the stress is isotropic and equal to the negative of the pressure.

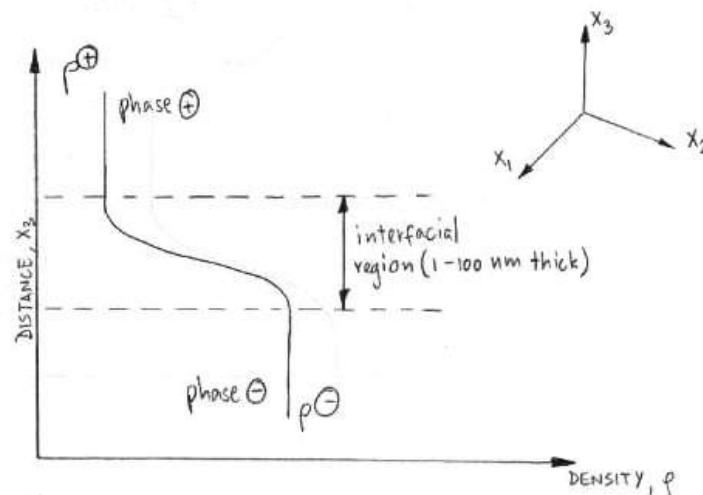


Figure 3.1. Density variation between phases

In the interfacial region, however, the concentrations of molecules vary and the intermolecular forces are not the same in all directions. Since the density varies in the x_3

direction, the stress is not isotropic; it is the same in any direction lying in the $x_1 - x_2$ plane, but it is different in the x_3 direction. The stress tensor in the interfacial region may be written as:

$$\underline{\sigma} = \begin{vmatrix} \sigma_t & 0 & 0 \\ 0 & \sigma_t & 0 \\ 0 & 0 & \sigma_n \end{vmatrix} \quad (3.1B)$$

where σ_n is the stress in the x_3 direction and σ_t is the stress in any direction normal to x_3 , i.e., tangential to the interface. The values of these stresses vary with x_3 in the interfacial region.

The variation of the density results in variations of other physical properties throughout the interfacial region. For example, the thermal conductivity tensor in the interfacial region is:

$$\underline{k} = \begin{vmatrix} k_t & 0 & 0 \\ 0 & k_t & 0 \\ 0 & 0 & k_n \end{vmatrix} \quad (3.1C)$$

where k_n is the thermal conductivity normal to the interface and k_t is the conductivity tangential to the interface. The values of the normal and tangential conductivities vary with x_3 . Other scalar quantities, such as the heat capacity, also vary normal to the interface in the interfacial region.

3.2 The Discontinuity Model of the Interface

The bulk phases are assumed to be separated by a mathematical discontinuity, i.e., the model interface has no thickness. The continuum bulk phases extend up to the discontinuity and there are step function changes in properties at the interface. This treatment can be described as the continuum/discontinuity model.

The model configuration is illustrated in Figure 3.2. The position of the interface (the discontinuity) is denoted $x_3 = 0$. The precise location of the interface is unknown, however. This is of no importance unless one of the phase volumes is very small.¹ In order to preserve the mass balance and Newton's second law across the interface it is endowed with special properties.

3.2.1 Mass and the Interface

The location of the interface can be determined by requiring that the total mass in the interfacial region be equal to the mass in the model configuration. To apply this condition, consider the "pillbox" control volume sketched in Figure 3.3. The pillbox is a right circular cylinder extending a distance λ into each phase. The intersection of the interface and the

¹The continuum/discontinuity model will be a poor approximation if a characteristic dimension of one of the phases is of the same order of magnitude as the thickness of the interfacial region.

control volume is a circle, C . The area of the interface inside the control volume is A . In the model the volume below the interface, $V^{(-)}$, contains bulk phase (-) with density $\rho^{(-)}$ while the volume above the interface, $V^{(+)}$, contains bulk phase (+) with density $\rho^{(+)}$.

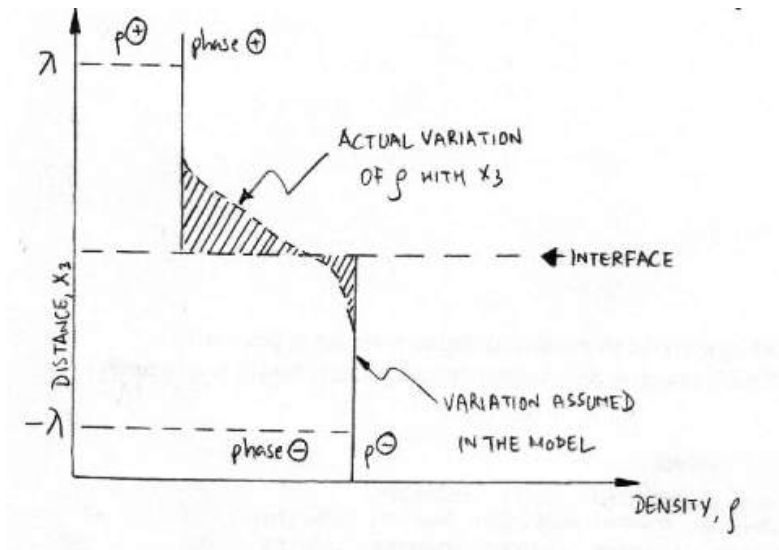


Figure 3.2. Model of interface

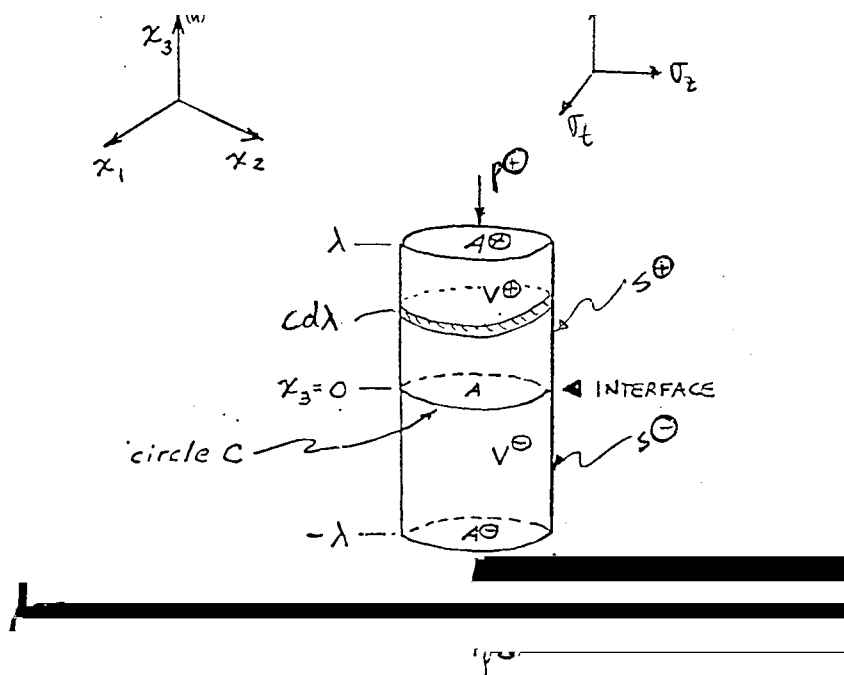


Figure 3.3. Pillbox control volume, $A^{(-)} = A = A^{(+)}$

The height of the pillbox is fixed by choosing λ large enough so that the surfaces $A^{(+)}$ and $A^{(-)}$ lie in the bulk phase. Recall that the interfacial region is less than 100 nm thick, so λ is indeed very small. A vertical cross-section through the pillbox is shown in Figure 3.2. The actual variation of density within the control volume is shown by the dash-dot curve while

the lines at $\rho^{(-)}$ and $\rho^{(+)}$ show the density variation assumed in the continuum/discontinuity model.

As mentioned earlier it is required that the mass within the control volume in the model equal the actual mass contained within the control volume. The actual mass in the control volume is obtained by integrating the density over the volume:

$$\text{Actual mass} = A \int_{-\lambda}^{\lambda} \rho dx_3 \quad (3.2A)$$

The model mass is:

$$\text{Model mass} = A \int_{-\lambda}^0 \rho^{(-)} dx_3 + A \int_0^{\lambda} \rho^{(+)} dx_3 \quad (3.2B)$$

Equating these masses yields:

$$\int_{-\lambda}^0 (\rho - \rho^{(-)}) dx_3 + \int_0^{\lambda} (\rho - \rho^{(+)}) dx_3 = 0 \quad (3.2C)$$

Referring to Figure 3.2, equation (3.2C) indicates that the interface is fixed so that the shaded areas above and below the plane $x_3 = 0$ are equal.

Now consider each phase to be a solution and compute the mass of species i contained within the pillbox control volume:

$$\text{Actual mass of species } i = A \int_{-\lambda}^{\lambda} \rho \omega_i dx_3 \quad (3.2D)$$

where ω_i is the mass fraction of species i . The model is required to have the same mass of this species. Since the location of the interface is fixed by equation (3.2C), an additional term is needed in the model to ensure that the masses will balance. The model mass is written as:

$$A \int_{-\lambda}^0 \rho^{(-)} \omega_i^{(-)} dx_3 + A \int_0^{\lambda} \rho^{(+)} \omega_i^{(+)} dx_3 + A \Gamma_{mi} \quad (3.2E)$$

where $\omega_i^{(-)}$ and $\omega_i^{(+)}$ are the mass fractions of species i in the bulk phases and Γ_{mi} is the extra mass per unit area of the interface. Equating the actual and model masses gives the quantity Γ_{mi} , “surface mass excess,” which may be either positive or negative:

$$\Gamma_{mi} = \int_{-\lambda}^0 (\rho \omega_i - \rho^{(-)} \omega_i^{(-)}) dx_3 + \int_0^{\lambda} (\rho \omega_i - \rho^{(+)} \omega_i^{(+)}) dx_3 \quad (3.2F)$$

Equation (3.2F) can be written for each species. Since the summation of all such equations yields the overall mass balance, equation (3.2C), then:

$$\sum_{i=1}^N \Gamma_{mi} = 0 \quad (3.2G)$$

where N is the total number of species. The surface mass excess expressed in molar units, Γ_i , is:

$$\Gamma_i = \frac{\Gamma_{mi}}{M_i} \quad (3.2H)$$

where M_i is the molecular weight of species i . Combining equations (3.2F) and (3.2H) yields:

$$\Gamma_i = \int_{-\lambda}^0 (c_i - c_i^{(-)}) dx_3 + \int_0^{\lambda} (c_i - c_i^{(+)}) dx_3 \quad (3.2I)$$

where c_i is the molar concentration of species i and $c_i^{(-)}$ and $c_i^{(+)}$ are the bulk phase molar concentrations.

3.2.2 Hydrostatics and the Interface

Newton's second law is now applied to the pillbox control volume. Since the fluids are motionless:

$$[\text{Body forces}] + [\text{Surface forces}] = 0 \quad (3.2J)$$

In the direction normal to the interface:

$$-gA \int_{-\lambda}^{\lambda} \rho dx_3 + (\sigma_{n,\lambda}^{(+)} - \sigma_{n,-\lambda}^{(-)})A = 0 \quad (3.2K)$$

where the first term is the body force and the second term is the net force acting on the upper and lower surfaces of the control volume. The quantities $\sigma_{n,\lambda}^{(+)}$ and $\sigma_{n,-\lambda}^{(-)}$ are the stresses in the direction normal to the interface at $x_3 = \lambda$ and $x_3 = -\lambda$, respectively. Since $A^{(+)}$ and $A^{(-)}$ are in the bulk phases, these stresses are equal to the pressures:

$$-\sigma_{n,-\lambda}^{(-)} = P_{-\lambda}^{(-)} \text{ and } -\sigma_{n,\lambda}^{(+)} = P_{\lambda}^{(+)} \quad (3.2L)$$

Combination of equations (3.2K) and (3.2L) yields:

$$g \int_{-\lambda}^{\lambda} \rho dx_3 = P_{-\lambda}^{(-)} - P_{\lambda}^{(+)} \quad (3.2M)$$

For the model, the sum of the body and surface forces is:

$$\text{Sum of forces} = -gA \left[\int_{-\lambda}^0 \rho^{(-)} dx_3 + \int_0^{\lambda} \rho^{(+)} dx_3 \right] + (p_{-\lambda}^{(-)} - p_{\lambda}^{(+)})A \quad (3.2N)$$

Substitution of equation (3.2M) into equation (3.2N) and use of equation (3.2C) shows that, for the model, the sum of forces equals zero. The placement of the interface in the model so that the total mass is conserved also satisfies Newton's second law in the normal direction.

In the direction tangential to the interface there is no body force, hence:

$$C \int_{-\lambda}^{\lambda} \sigma_t dx_3 = 0 \quad (3.2O)$$

where C is the circumference of the circle of intersection of the control volume with the interface. For the model:

$$\text{Sum of forces} = -C \left[\int_{-\lambda}^0 p^{(-)} dx_3 + \int_0^{\lambda} p^{(+)} dx_3 \right] + C\gamma = 0 \quad (3.2P)$$

The quantity γ is an extra force, which is required in the model to make the sum of forces equal zero. This force acts in the plane of the interface. Adding the last two equations gives the following expression for γ :

$$\gamma = \int_{-\lambda}^0 (\sigma_t + p^{(-)}) dx_3 + \int_0^{\lambda} (\sigma_t + p^{(+)}) dx_3 \quad (3.2Q)$$

This force, called the interfacial tension, arises from the difference between the tangential stress in the interfacial region and the isotropic stress assumed in the model.

3.3 Derivation of Interface Boundary Conditions

In the continuum/discontinuity model the interface has two properties: it exhibits a tension, which acts in the plane of the interface and it may contain excess masses if the phases are solutions. To derive boundary conditions a pillbox control volume is used as shown in Figure 3.4. The conservation of mass, Newton's second law and the first law of thermodynamics are applied to this control volume, which extends a distance ε into each phase. The final expressions are obtained by letting $\varepsilon \rightarrow 0$.

By writing a mass balance over a control volume of Figure 3.4 (see also Figure 3.5 for additional notations used):

$$\{\text{Rate of mass accumulation}\} = \{\text{Rate of mass input}\} - \{\text{Rate of mass output}\} \quad (3.3A)$$

$$\text{or,} \quad \{\text{Rate of mass accumulation}\} + \{\text{Net rate of mass efflux}\} = 0 \quad (3.3B)$$

$$\frac{D}{Dt} \left[\int_{V^{(+)}} \rho^{(+)} dV + \int_{V^{(-)}} \rho^{(-)} dV \right] = \int_{A^{(-)}} \rho^{(-)} (\vec{v}^{(-)} - \vec{v}_s) \cdot \vec{n} dA - \int_{A^{(+)}} \rho^{(+)} (\vec{v}^{(+)} - \vec{v}_s) \cdot \vec{n} dA - \int_{S^{(-)}} \rho^{(-)} (\vec{v}^{(-)} - \vec{v}_s) \cdot \vec{t} dS + \int_{S^{(+)}} \rho^{(+)} (\vec{v}^{(+)} - \vec{v}_s) \cdot \vec{t} dS \quad (3.3C)$$

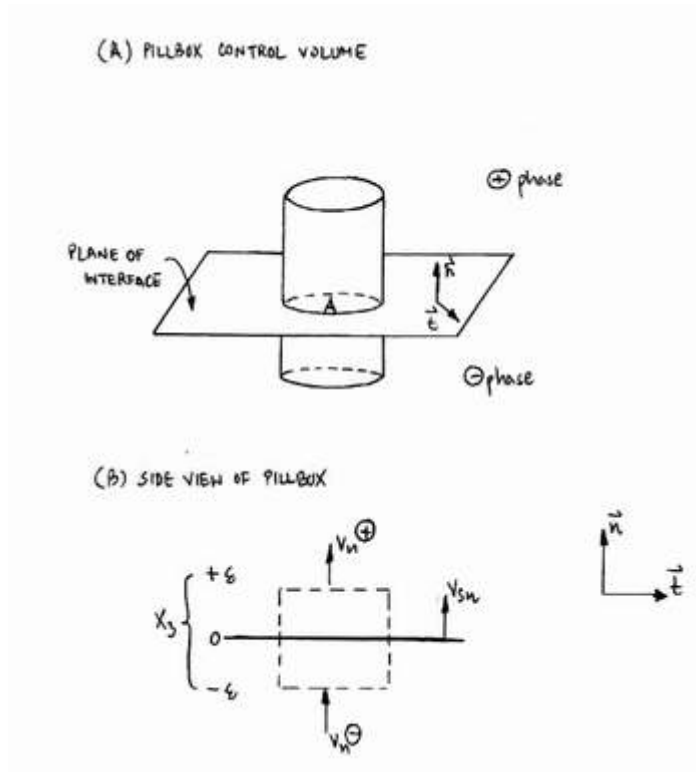


Figure 3.4. Control volume for boundary conditions in continuum/discontinuity model

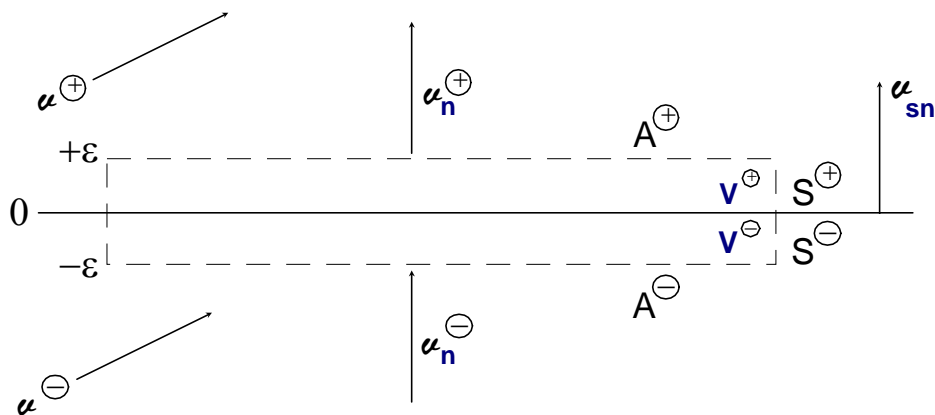


Figure 3.5. Control volume used to derive interface boundary conditions for mass

As mentioned earlier the final expressions for interface boundary conditions are obtained by letting $\varepsilon \rightarrow 0$, i.e., shrink the control volume thickness to zero ($V^{(+)}, V^{(-)}, S^{(+)}, S^{(-)} \rightarrow 0$). This implies that the {Rate of mass accumulation} and the third and fourth terms on the right hand side of equation (3.3C) all vanish.

In normal direction, equation (3.3C) then becomes:

$$\rho^{(+)}(v_n^{(+)} - v_s) = \rho^{(-)}(v_n^{(-)} - v_s) = \dot{m} \quad (3.3D)$$

where \dot{m} is the mass flux across the interface. In tangential direction, interface mass balance is written as:

$$v_t^{(+)} = v_t^{(-)} \quad (3.3E)$$

which is indeed the no slip condition at the interface. If the new operator is defined such that:

$$\|X\| = X^{(+)} - X^{(-)} \quad (3.3F)$$

where $X^{(+)}$ and $X^{(-)}$ denote the values of X (any quantity of interest) in the (+) and (-) phases, respectively. The operator defined in equation (3.3F) is the so-called “jump operator.”

Equations (3.3D) and (3.3E) then become:

$$\|\rho(v_n - v_s)\| = 0 \quad (3.3G)$$

$$\|v_t\| = 0 \quad (3.3H)$$

Equation (3.3G) is *the jump (or interface) mass balance* and is similar to equation (5-6) of Slattery (1972, p. 24). This may be used to derive the interface boundary conditions for the continuity equation.

For momentum balance, consider the control volume shown in Figure 3.6.

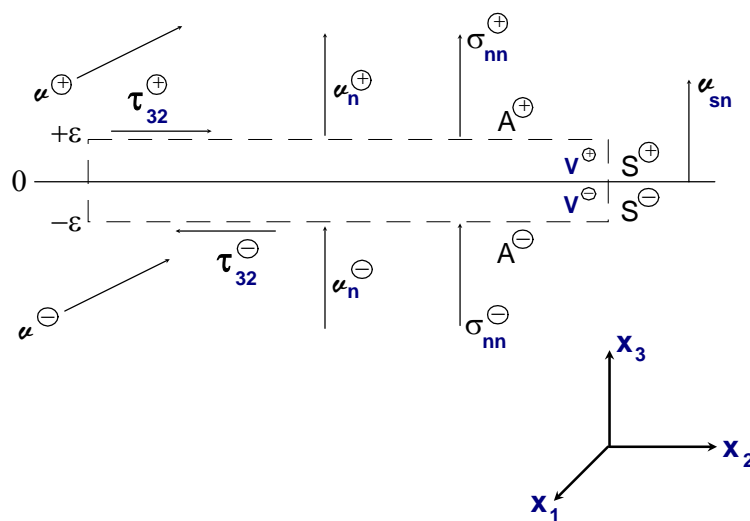


Figure 3.6. Control volume used to derive interface boundary conditions for momentum

Newton's 2nd law of motion states that:

$$\{\text{Rate of change of momentum}\} + \{\text{Net rate of efflux of momentum}\} = \{\text{Body forces}\} + \{\text{Surface forces}\} \quad (3.3I)$$

If the control volume is shrunk such that $\varepsilon \rightarrow 0$, the first term on the left as well as the first term on the right would both equal to zero.

In normal direction equation (3.3I) becomes:

$$\int_{A^{(+)}} \dot{m} v_n^{(+)} dA - \int_{A^{(-)}} \dot{m} v_n^{(-)} dA = \int_{A^{(+)}} \sigma_{nn}^{(+)} dA - \int_{A^{(-)}} \sigma_{nn}^{(-)} dA \quad (3.3J)$$

or,

$$\dot{m}(v_n^{(+)} - v_n^{(-)}) = \sigma_{nn}^{(+)} - \sigma_{nn}^{(-)} \quad (3.3K)$$

or in terms of pressure and viscous stresses,

$$\dot{m}(v_n^{(+)} - v_n^{(-)}) + (p^{(+)} - p^{(-)}) - (\tau_{nn}^{(+)} - \tau_{nn}^{(-)}) = 0 \quad (3.3L)$$

or using the jump operator,

$$\left\| \dot{m} v_n + p - \tau_{nn} \right\| = 0 \quad (3.3M)$$

Equation (3.3M) is the *jump (interface) momentum balance* in normal direction (see Exercise 2.2.3-1 in Slattery (1972, p. 40)). In tangential direction (say, for "32" component), equation (3.3I) reduces to:

$$0 = \{\text{Surface forces}\} \quad (3.3N)$$

or,

$$\int_{A^{(+)}} \tau_{32}^{(+)} dA - \int_{A^{(-)}} \tau_{32}^{(-)} dA \quad (3.3O)$$

or,

$$\left\| \tau_{32} \right\| = 0 \quad (3.3P)$$

For all components of shear stresses equation (3.3P) can be written in terms of shear stress vector as:

$$\left\| \vec{t}_t \right\| = 0 \quad (3.3Q)^2$$

To derive the interface boundary conditions for energy transfer a control volume shown in Figure 3.7 is used.

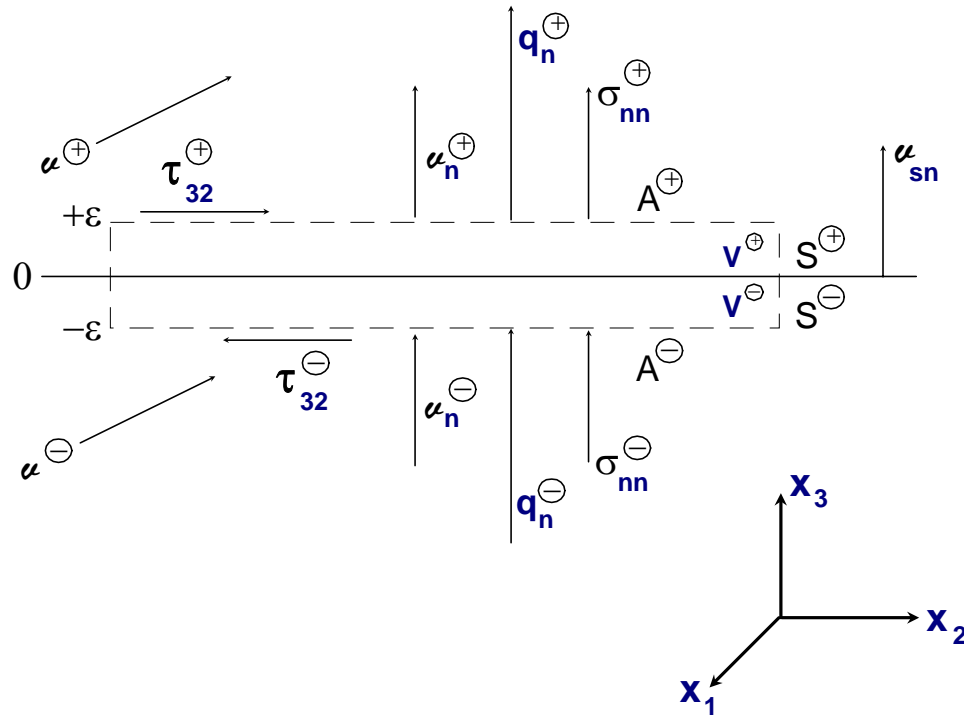


Figure 3.7. Control volume used to derive interface boundary conditions for energy

First law of thermodynamics states that:

$$\{\text{Rate of change of total energy}\} + \{\text{Net rate of efflux of energy}\} = \{\text{Net rate of heat addition}\} - \{\text{Rate at which work is done by system}\} \quad (3.3R)$$

The first term on the left again equals to zero if the control volume is shrunk such that $\varepsilon \rightarrow 0$. The remaining terms can be expressed as:

$$\int_{A^{(+)}} \dot{m} \left(u^{(+)} + \frac{v^{(+)^2}}{2} \right) dA - \int_{A^{(-)}} \dot{m} \left(u^{(-)} + \frac{v^{(-)^2}}{2} \right) dA = \int_{A^{(-)}} q_n^{(-)} dA - \int_{A^{(+)}} q_n^{(+)} dA + \int_{A^{(+)}} v_n^{(+)} \sigma_{nn}^{(+)} dA - \int_{A^{(-)}} v_n^{(-)} \sigma_{nn}^{(-)} dA \quad (3.3S)$$

or,

² If there is a variation in surface tension, the jump momentum balance in tangential direction will become

$$\left\| \vec{t}_t \right\| = -\nabla_s \gamma, \quad \text{where } \gamma \text{ is the surface tension. } \nabla_s \text{ denotes the gradient in the plane of interface.}$$

$$\dot{m} \left[u^{(+)} + \frac{v^{(+)^2}}{2} - u^{(-)} - \frac{v^{(-)^2}}{2} \right] = q_n^{(-)} - q_n^{(+)} + v_n^{(+)} \sigma_{nn}^{(+)} - v_n^{(-)} \sigma_{nn}^{(-)} \quad (3.3T)$$

by using a thermodynamic relationship, $u = h - \frac{p}{\rho}$, and write σ_{nn} as $\sigma_{nn} = -p + \tau_{nn}$, equation (3.3T) can be rewritten (in terms of jump operator) as:

$$\left\| \dot{m} \left(h + \frac{v^2}{2} \right) + v_s p + q_n - v_n \tau_{nn} \right\| = 0 \quad (3.3U)$$

Two additional interface boundary conditions arise from the equilibrium at the interface:

$$T^{(+)} = T^{(-)} \text{ or } \|T\| = 0 \quad (3.3V)$$

$$\mu_i^{(+)} = \mu_i^{(-)} \text{ or } \|\mu_i\| = 0 \quad (3.3W)$$

where μ_i is the chemical potential of component i .

Summary Interface boundary conditions³

Jump mass balance: $\|\rho(v_n - v_s)\| = 0$

No slip condition at the interface: $\|v_t\| = 0$

Jump momentum balance (normal direction): $\left\| \dot{m} v_n + p - \tau_{nn} \right\| = 0$

Simplified form of jump momentum balance in normal direction: $\|p\| = 0$

Jump momentum balance (tangential direction): $\left\| \vec{t}_t \right\| = 0$ ⁴

Jump energy balance: $\left\| \dot{m} \left(h + \frac{v^2}{2} \right) + v_s p + q_n - v_n \tau_{nn} \right\| = 0$

Simplified form of jump energy balance: $\left\| \dot{m} h + q_n \right\| = 0$

Equilibrium at the interface (1): $\|T\| = 0$

Equilibrium at the interface (2): $\|\mu_i\| = 0$

³See also Slattery (1972)

⁴See footnote on p. 44.

Example 1 Consider a plane Couette flow with vaporization as shown in Figure E3.1. The plates are assumed to be infinite (i.e., no variations with x , $\frac{\partial v_x}{\partial x} = \frac{\partial T}{\partial x} = 0$) and are porous at $z = 0$ and $z = h$. Write the conservation equations that govern this problem along with a complete set of boundary conditions required to solve the equations.

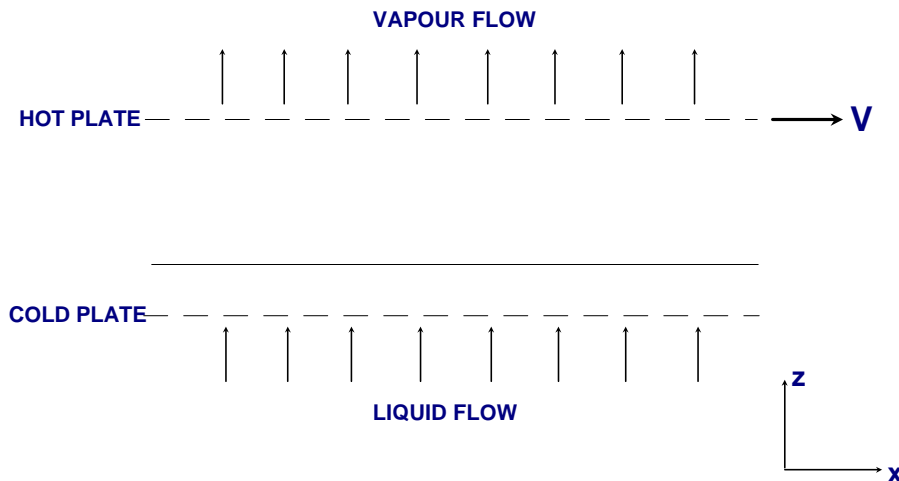


Figure E3.1. Couette flow with vaporization

Solution

The following assumptions can be made: the plate is infinite and the interface is stationary ($v_s = 0$); the flow is steady, two-dimensional; properties of both fluids are constant, but not the same; viscous dissipation is negligible. Based on the above assumptions the governing conservation equations can be written for the vapor phase and liquid phase as follows:

Vapour phase

$$\text{Continuity:} \quad \frac{\partial v_z}{\partial z} = 0 \quad (\text{E31A})$$

$$\text{x-momentum:} \quad \rho v_z \frac{\partial v_x}{\partial z} = \mu \frac{\partial^2 v_x}{\partial z^2} \quad (\text{E31B})$$

$$\text{z-momentum:} \quad \frac{\partial p}{\partial z} = 0 \quad (\text{E31C})$$

$$\text{Energy:} \quad \rho c_p v_z \frac{\partial T}{\partial z} = k \frac{\partial^2 T}{\partial z^2} \quad (\text{E31D})$$

Liquid phase (prime phase)

$$\text{Continuity:} \quad \frac{\partial v'_z}{\partial z} = 0 \quad (\text{E31E})$$

$$\text{x-momentum:} \quad \rho' v'_z \frac{\partial v'_x}{\partial z} = \mu' \frac{\partial^2 v'_x}{\partial z^2} \quad (\text{E31F})$$

$$\text{z-momentum:} \quad \frac{\partial p'}{\partial z} = 0 \quad (\text{E31G})$$

$$\text{Energy:} \quad \rho' c'_p v'_z \frac{\partial T'}{\partial z} = k' \frac{\partial^2 T'}{\partial z^2} \quad (\text{E31H})$$

Twelve boundary conditions are needed to solve the above sets of equations:

$$\text{at } z = 0, \quad v'_x = 0, \quad T' = T_0$$

$$\text{at } z = h, \quad v_x = V, \quad T = T_h$$

$$\text{at } z = \delta, \quad \rho' v'_z = \rho v_z, \quad v'_x = v_x, \quad \mu' \frac{\partial v'_x}{\partial z} = \mu \frac{\partial v_x}{\partial z}, \quad p' = p$$

$$\text{at } z = \delta, \quad \rho v_z \Delta h = k \frac{\partial T}{\partial z} - k' \frac{\partial T'}{\partial z}, \quad T' = T, \quad \mu = \mu'$$

For a pure component, relationship is usually presented for a vapor pressure as a function of temperature, or $p = f(T)$. Note that a pressure boundary condition is still needed.

3.4 Rigid Body

Consider a rigid (or solid) body (i.e., a body with no relative motion inside) of Figure 3.8. Let \vec{v} be the velocity of the center of mass of the body. Note that the velocity at the interface may not be equal to \vec{v} (like the case of solid coating or burning of solid fuels).

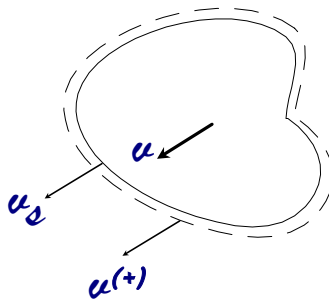


Figure 3.8. An arbitrary rigid body

If the system is made to cover all the solid body, it is possible to write Newton's second law of motion for the body as:

$$\frac{d}{dt}(m \vec{v}) + \int_{A_s} \dot{m} \vec{v}^{(+)} dA = \vec{F}_{body} + \vec{F}_{fluid} + \vec{F}_{other} \quad (3.4A)$$

where $\vec{F}_{body} = \int_{V_s} \rho_s \vec{G} dV$, $\vec{F}_{fluid} = \int_{A_s} \vec{v}_s \cdot \underline{\underline{\sigma}} dA = \int_{A_s} \vec{v}_s \cdot (-p \underline{I} + \underline{\underline{\tau}}) dA$

Special cases

1. Body in free falling, $\vec{F}_{other} = 0$.
2. If $\dot{m} = 0$ ($m = \text{constant}$), then:

$$m \frac{d \vec{v}}{dt} = \int_{V_s} \rho_s \vec{G} dV + \int_{A_s} \vec{v}_s \cdot (-p \underline{I} + \underline{\underline{\tau}}) dA \quad (3.4B)$$

The second term on the right is the drag force.

3. Motionless body, $\vec{v} = 0$.

Note that the continuity and momentum equations are not applicable for rigid body and the energy equation reduces to the following heat conduction equation:

$$\rho c_p \frac{\partial T}{\partial t} = k \nabla^2 T \quad (3.4C)$$

Example 2 A piece of ice is melted from below in an insulated cylindrical container. The ice can slide up and down freely. The ice is at 0°C while the lower surface of the cylinder is held at 4°C. In this range of temperature $\rho_{water} > \rho_{ice}$ and no natural convection will occur. The original configuration is the cylinder filled with 0°C ice to a height L_0 . At time zero the bottom of the cylinder has its temperature change from 0°C to 4°C and it is then held at the latter temperature. The top surface of the ice is insulated. Assume also that the fluid and solid properties are constant (but not equal) and that the melting point of ice is 0°C and is independent of pressure.

- (a) Set up the differential equations as well as the initial and boundary conditions, which will enable one to calculate the time when all of the ice melts.
- (b) Give the momentum equation for the liquid phase. Show that the pressure (in the liquid) at the ice-water interface, denoted by p_ξ is related to the pressure above the ice, denoted by p_a by $p_\xi - p_a = \rho_i (L - \xi) + \Psi$. Give an expression for the term denoted by Ψ .

Solution

(a) Assuming that the problem is one-dimensional the governing conservation equations (for the liquid phase) can be written as:⁵

$$\text{Continuity:} \quad \frac{\partial v_y}{\partial y} = 0 \quad (\text{E32A})$$

$$\text{Energy:} \quad \frac{\partial T}{\partial t} = \alpha_w \frac{\partial^2 T}{\partial y^2} \quad (\text{E32B})$$

The initial condition at $t = 0$ is $T = 0^\circ\text{C}$. The velocity boundary conditions is: at $y = 0$, $v_y = 0$. The two boundary conditions for the energy equation are derived as follows: at $y = 0$, $T = 4^\circ\text{C}$ and at $y = \xi$, $T = 0^\circ\text{C}$ (since $\|T\| = 0$). The problem is now to find the location of the interface, ξ . From the jump energy balance (at $y = \xi$):

$$\dot{m}(h_i - h_w) = q_w - q_i \quad (\text{E32C})$$

or,

$$-\dot{m}\Delta h_f = -k_w \left(\frac{\partial T}{\partial y} \right)_{y=\xi} \quad (\text{E32D})$$

where $\Delta h_f = h_w - h_i$. From the jump mass balance (at $y = \xi$):

$$\dot{m} = \rho_i (v_i - v_s) = \rho_w (v_w - v_s) \quad (\text{E32E})$$

Thus,

$$\dot{m} = -\rho_w \frac{d\xi}{dt} \quad (\text{E32F})$$

Equation (E32D) thus becomes:

$$\frac{d\xi}{dt} = \frac{-k_w}{\rho_w \Delta h_f} \left(\frac{\partial T}{\partial y} \right)_{y=\xi} \quad (\text{E32G})$$

With an additional initial condition that at $t = 0$, $\xi = 0$. The melting of ice ends when the mass of water is equal to the original mass of ice. In other words,

$$\rho_i AL_o = \rho_w A \xi_f \quad (\text{E32H})$$

or,

$$\xi_f = \frac{\rho_i L_o}{\rho_w} \quad (\text{E32I})$$

⁵ Why momentum equation is not needed here?

where A = cross-sectional area of cylinder and ξ_f = final height of water.

(b) To determine the pressure distribution in water, the momentum equation is written as:

$$-\frac{\partial p}{\partial y} = \rho_w g \quad (\text{E32J})$$

With the following boundary condition: at $y = \xi$, $p = p_\xi$. Thus,

$$p - p_\xi = \rho_w g (\xi - y) \quad (\text{E32K})$$

To determine p_ξ , write Newton's second law of motion for the ice as:

$$\frac{d}{dt}(m_i v_i) = -m_i g + p_\xi A - p_a A \quad (\text{E32L})$$

where m_i = mass of ice = $\rho_i A(L - \xi)$ and p_a = atmospheric pressure at $y = L$. Finally,

$$p_\xi - p_a = \rho_i (L - \xi)g + \rho_i \frac{d}{dt} \left[(L - \xi) \frac{dL}{dt} \right] \quad (\text{E32M})$$

$$\text{Thus, } \Psi = \rho_i \frac{d}{dt} \left[(L - \xi) \frac{dL}{dt} \right].$$

3.5 Dimensionless Form of the Interface Boundary Conditions

Consider, for example, the tangential mass balance at the interface:

$$\|v_t\| = 0 \quad (\text{3.5A})$$

or,

$$v_t^{(+)} - v_t^{(-)} = 0 \quad (\text{3.5B})$$

If U is chosen as the characteristic velocity, then:

$$\frac{v_t^{(+)}}{U} - \frac{v_t^{(-)}}{U} = 0 \quad \text{or} \quad \|v_t^*\| = 0 \quad (\text{3.5C})$$

Equation (3.5C) is the dimensionless no slip condition at the interface.

For the mass balance in normal direction:

$$\frac{\rho^{(+)}}{\rho^{(-)}} \left(\frac{v_n^{(+)}}{U} - \frac{v_s}{U} \right) - \left(\frac{v_n^{(-)}}{U} - \frac{v_s}{U} \right) = 0 \quad (3.5D)$$

For the tangential momentum balance at the interface:

$$\left\| \vec{t}_t \right\| = -\nabla_s \gamma \quad (= 0 \text{ if neglect all interfacial effects}) \quad (3.5E)$$

or,

$$\vec{t}_t^{(+)} - \vec{t}_t^{(-)} = -\nabla_s \gamma \quad (3.5F)$$

Let's pick one direction, say, 32:

$$t_{32}^{(+)} - t_{32}^{(-)} = -\frac{\partial \gamma}{\partial x_2} \quad (3.5G)$$

or,

$$\mu^{(+)} \left(\frac{\partial v_3^{(+)}}{\partial x_2} + \frac{\partial v_2^{(+)}}{\partial x_3} \right) - \mu^{(-)} \left(\frac{\partial v_3^{(-)}}{\partial x_2} + \frac{\partial v_2^{(-)}}{\partial x_3} \right) = -\frac{\partial \gamma}{\partial x_2} \quad (3.5H)$$

nondimensionalization of equation (3.5H) yields:

$$\frac{\mu^{(+)}}{\mu^{(-)}} \left[\frac{\partial v_3^{(+)*}}{\partial x_2^*} + \frac{\partial v_2^{(+)*}}{\partial x_3^*} \right] - \left[\frac{\partial v_3^{(-)*}}{\partial x_2^*} + \frac{\partial v_2^{(-)*}}{\partial x_3^*} \right] = -\frac{1}{\mu^{(-)}U} \frac{\partial \gamma}{\partial x_2^*} \quad (3.5I)$$

Consider only the term on the right hand side of equation (3.5I), $-\frac{1}{\mu^{(-)}U} \frac{\partial \gamma}{\partial x_2^*}$. In general $\gamma = f(T, p)$ but the effect of pressure is almost always neglected. Hence,

$$\frac{\partial \gamma}{\partial x_2^*} = \left(\frac{d\gamma}{dT} \right) \frac{\partial T}{\partial x_2^*} \quad (3.5J)$$

and,

$$-\frac{1}{\mu^{(-)}U} \frac{\partial \gamma}{\partial x_2^*} = -\frac{1}{\mu^{(-)}U} \left(-\frac{d\gamma}{dT} \right) \frac{\partial T}{\partial x_2^*} \quad (3.5K)$$

If temperature is made dimensionless by defining $T^* = \frac{T - T_R}{\Delta T}$, the right-hand-side term of equation (3.5K) transforms to:

$$\frac{\Delta T}{\underline{\mu^{(-)}U}} \left(-\frac{d\gamma}{dT} \right) \frac{\partial T^*}{\partial x_2^*} \quad (3.5L)$$

The underlined term can be expressed as:

$$\frac{\Delta T}{\underline{\mu^{(-)}U}} \left(-\frac{d\gamma}{dT} \right) = \left[\frac{\alpha^{(-)}}{UL} \right] \frac{L\Delta T}{\underline{\mu^{(-)}\alpha^{(-)}}} \left(-\frac{d\gamma}{dT} \right) \quad (3.5M)$$

The underlined term in equation (3.5M) is the so-called Marangoni number and is the indicator of how important the effect of surface tension is.

The student is encouraged to perform a similar operation as done above with the jump energy balance to obtain yet another dimensionless group called the Stefan number:

$$Ste = \frac{c\Delta T}{\Delta h} \quad (3.5N)$$

where c is the heat capacity and Δh is the enthalpy difference for phase change.

References

1. Bird, R.B., Stewart, W.E., Lightfoot, E.N., 1960, **Transport Phenomena**, Wiley, New York.
2. Slattery, J.C., 1972, **Momentum, Energy, and Mass Transfer in Continua**, McGraw-Hill, New York.

Problems

1. Consider the melting of a solid in a two-dimensional, rectangular container. The side of the container at $x = 0$ is held at a temperature of T_0 ; the side at $x = L$ is held at T_L , where $T_0 > T_m > T_L$ and T_m is the melting point of the solid (which is assumed to be independent of pressure). The bottom of the container ($y = 0$) is insulated. The top of the container is open, hence at $y = H$ it may be assumed that there is no heat loss (i.e., it behaves as if it were insulated) and that the liquid portion of the surface is a free surface. The pressure in the gas phase (at $y = H$) is P_a and the viscosity of the gas phase is negligible compared to that of the liquid (the melt) phase. There is natural convection in the melt, which is driven by the variation of the liquid density with temperature. The Boussinesq approximation is made to account for buoyancy. In addition, it is assumed that:

- The density of the solid is constant at ρ_s .
- The density of the liquid is constant (except in the buoyancy term in the momentum equation) at ρ and $\rho = \rho_s$. This will ensure that the liquid interface will remain at $y = H$.
- The remaining properties of the solid are constant and different from those of the liquid, which are also constant.
- Dissipation may be neglected.
- The process is at steady state.
- The simplified jump balances apply.

- (a) Give the differential equations describing the velocity and temperature fields in the liquid and solid.
 (b) Give the boundary conditions necessary to completely solve this problem.
 (c) Sketch the shape of the interface between the liquid and solid.
2. A continuous, solid tape of thickness t is pulled from a slot in a solid wall into a fluid. The tape moves as a rigid body with a velocity V . The wall is located at the plane $z = 0$ and there is symmetry about the central plane of the tape. The tape extends to $z \rightarrow \infty$. Far from the tape the fluid is stagnant and has a temperature of T_∞ . The die from which the tape emerges extends to $z \rightarrow -\infty$ and is perfectly insulated. In the analysis, assume that the material properties are constant, there is no dissipation, the process is at steady state and the simplified jump balances apply.
- (a) Give the equations governing the motion and temperature of the tape and the fluid.
 (b) Give the boundary conditions necessary to completely specify this problem.
 (c) Sketch the streamlines.
3. The flat end of a cylindrical heater of radius a is contacted with the surface of a semi-infinite solid, which is initially at a temperature of T_∞ as shown in Figure P3.3:

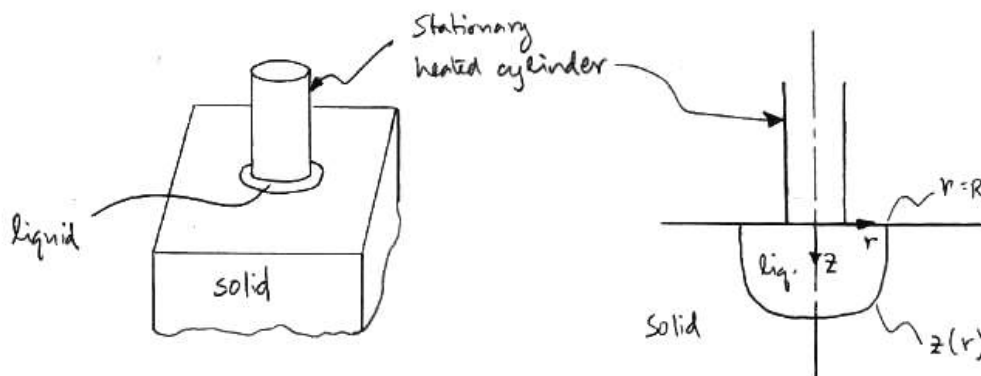


Figure P3.3. A schematic sketch of the heating system

The end of the cylinder is maintained at a constant temperature T_0 , which is above the melting temperature of the solid T_m ; thus, $T_0 > T_m > T_\infty$. After some time the interface between the liquid and the solid will reach a steady position where the depth of the liquid, Z , is a function only of the radial position. The liquid and solid are not volatile and the pressure in the gas phase at $z = 0$ is one atmosphere. The liquid/gas interface is a free surface, i.e., the shear stress on the gas side is negligible. In addition, no heat is transferred between the gas and the solid or the liquid. The liquid pool extends to $r = R$ at the surface ($z = 0$). The liquid in the pool circulates slowly, driven by the variation of the surface tension with the temperature.

Assuming that:

- All properties except surface tension are constant
- Solid and liquid densities are equal but not other properties

- There is no dissipation and no density-driven flow
- The melting temperature is independent of pressure
- Gravity acts in the z -direction

Assuming also the axial symmetry and that the simplified jump balances can be used to analyze the problem. The process is also at steady state.

- Give the differential equations governing the solid and liquid
- Give the boundary conditions needed to specify the problem

Hint: When there exists a variation in surface tension, the jump momentum balance in tangential direction becomes $\left\| \vec{t}_t \right\| = -\nabla_s \gamma = \left(-\frac{d\gamma}{dT} \right) \nabla_s T$, where γ is the surface tension.

∇_s denotes the gradient in the plane of interface. In the present case, this gradient is given as $\nabla_s T = \left(\frac{\partial T}{\partial r} \right) \vec{e}_r + \left(\frac{\partial T}{\partial \theta} \right) \vec{e}_\theta$. The stress vector (for the liquid) is $\vec{t}_t = \tau_{rz} \vec{e}_r + \tau_{z\theta} \vec{e}_\theta$.

Expressions for the stress components can be found in Bird et al. (1960, p. 89; 2002, p. 844).

- A small spherical droplet of water is suspended in superheated steam at the end of a very fine hypodermic needle. Far from the droplet the vapor is motionless and the pressure is distributed hydrostatically. The water is fed to the droplet at a rate that keeps the size of the droplet constant (i.e., the radius is constant at, say, a). Far from the droplet the vapor is at a temperature of 110°C . Assume that the pressure in the vapor phase at the droplet surface is one atmosphere and that the water fed to the droplet is at T_0 . Assuming also that the viscous dissipation in each phase is negligible and the properties in each phase are constant (but different) and both internal and external velocity and temperature fields are spherically symmetric.
 - Give the differential equations governing the velocity, pressure and temperature in each phase
 - Give the necessary boundary conditions. Use the simplified jump balances.

- Single crystals of semiconductors, compound semiconductors and oxides are very important as device materials in the electronic industry. When single crystals are grown from a melt, natural convection due to density differences and Marangoni convection due to an interfacial tension gradient at the free interface of the melt are formed spontaneously. Since these convective flows affect the quality of single crystals, it is necessary to clarify and control these melt flows with some adequate degree of accuracy.

The theoretical model is shown in Figure P3.5. The model considers a two-dimensional rectangular open container with a free interface, which is heated from one side (T_H) and cooled from the other side (T_C). The model includes the following assumptions: (1) steady state, (2) Newtonian fluid, (3) a flat interface, (4) an adiabatic bottom wall and free interface, and (5) constant values of all physical properties except the interfacial tension in the stress balance equation for the free melt interface and the density in the buoyancy force term. Although, in reality, the problem involves Marangoni convection, it is neglected here for simplicity.

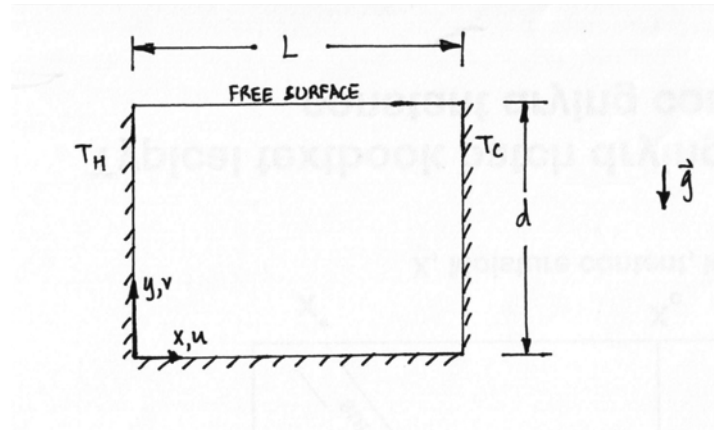


Figure P3.5. Open rectangular container for growing crystals

Continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Navier-Stokes equations:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + g\beta\Delta T$$

Energy equation:

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)$$

The notations used are as follows:

d = melt depth

g = gravitational acceleration

L = length of free interface

p = pressure

T = temperature

t = time

u = velocity parallel to interface

v = velocity normal to interface

x = coordinate parallel to interface

y = coordinate normal to interface

<Greek letters>

α = thermal diffusivity

β = thermal expansion coefficient

$\Delta T = T_H - T_C$

μ = viscosity

ν = kinematic viscosity

ρ = density

<Subscripts>

C = cold

H = hot

Assume that a two-dimensional treatment of the problem is adequate. In order to limit the problem to the liquid phase, assume that the viscosity of the gas above the free surface is zero and that there is no vaporization at the free surface (i.e., the free surface is adiabatic).

- (a) Are the differential equations listed correct? If not, what corrections should be made?
 - (b) How many boundary conditions are required to completely specify this problem? Give a complete set of conditions using the simplified jump balances.
 - (c) Sketch the steady state streamlines of the flow.
6. One major source of error in the prediction of freezing time is the imprecise knowledge of the heat transfer coefficient, especially when freezing is due to natural convection. One way to alleviate this problem is to replace the external convective boundary condition at the surface of the food sample undergoing freezing with the set of governing conservation equations that can be used to simulate the natural convection of the surrounding air in the freezer.

In the present situation a two-dimensional food sample, initially at a temperature T_{fo} , is placed inside a freezer initially at a temperature T_{ao} as shown in Figure P3.6. At $t = 0$ the refrigeration system is activated and the freezer walls start to cool down, along with the air inside the freezing chamber, and later the food located on the bottom wall of the chamber.

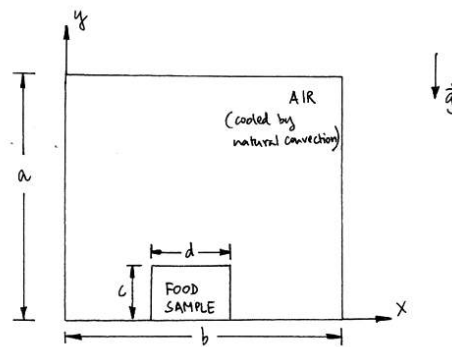


Figure P3.6. Physical situation of food inside the freezer

In order to develop the mathematical model of the aforementioned process the following assumptions are made: the air is assumed to be Newtonian fluid, and the flow is laminar; the food is assumed to be an homogeneous isotropic material; and mass transfer of water inside the food and the change of its volume are negligible. The Boussinesq's approximation and simplified jump balances can be used to analyze this problem. The wall temperature variations with time can be measured experimentally.

- (a) Give the differential equations that enable one to model the fluid mechanics and heat transfer in air and the heat transfer in the solid food.
 - (b) Give the initial and boundary conditions needed to specify the problem.
7. Consider the melting of a phase change material (PCM) around a horizontal circular cylinder of radius r_o illustrated in Figure P3.7. The cylinder is immersed in an infinite solid PCM, which is initially at the melting temperature T_m . When the cylinder is heated to T_o (which is higher than T_m) the melting process starts. To simplify the analysis the

following assumptions can be made: (1) the liquid PCM is Newtonian and incompressible; (2) all thermophysical properties of the PCM are constant except the density; (3) Boussinesq approximation is valid; (4) viscous dissipation is neglected; (5) pressure gradients are neglected. The problem can also be considered as a two-dimensional problem (only in r and θ directions).

Hint: The gravity components in r and θ directions are $g\sin\theta$ and $g\cos\theta$, respectively. Axial symmetry can also be assumed at $\theta = \pm 90^\circ$.

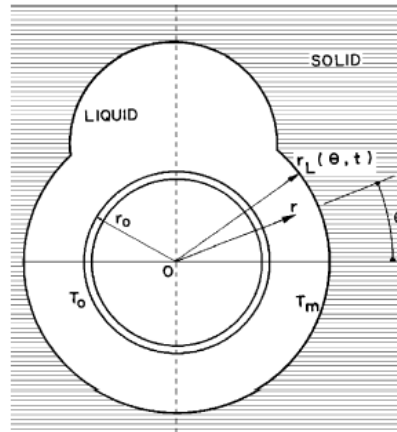


Figure P3.7. A schematic sketch of the melting problem

- (a) Give the differential equations describing the velocity and temperature fields in the liquid and solid.
 - (b) Give the initial and boundary conditions necessary to completely solve this problem.
 - (c) Explain briefly why there exists a hump in the upper part of the liquid.
8. A uniformly heated sphere of radius R is suspended in a large, motionless body of fluid. It is desired to study the heat transfer in the fluid surrounding the sphere. In this case it is assumed that the free convection effects can be neglected. Other appropriate assumptions may be made but must be stated clearly.
- (a) Set up the differential equation describing the steady-state temperature distributions in the surrounding fluid as a function of the distance from the center of the sphere, r . It is assumed that the thermal conductivity of the fluid is constant.
 - (b) Integrate the differential equation using the following boundary conditions to obtain the steady-state temperature distributions in the fluid: at $r = R$, $T = T_R$ and at $r \rightarrow \infty$, $T = T_\infty$.
 - (c) From the temperature distributions derived in (b), obtain an expression for the heat flux at the surface of the sphere. Equate this result to the heat flux written as Newton's law of cooling and show that the Nusselt number (based on the diameter of the sphere) is equal to 2. This result is indeed a classical result for heat transfer from spheres at low Reynolds or Grashof numbers.