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FDE618: Transport Phenomena in Food Processing
ChE610: Fundamentals of Transport Phenomena

Chapter 2
Derivation of the Governing Conservation Equations

In this chapter governing conservation equations of momentum, energy and mass are derived for continuum using the assumptions described in Chapter 1. In general four basic laws are used when analyzing transport problems: conservation of mass, Newton's second law of motion, conservation of energy (the first law of thermodynamics) and the second law of thermodynamics. Since analyzing the efficiency and availability of the system is not within the scope of the present notes, the second law of thermodynamics will not be discussed further.

2.1 The Continuity Equation

The law of mass conservation requires that the mass of a system cannot change with time. Taking the Lagrangian point of view this law can be written as follows for a flowing fluid:

$$\frac{d}{dt}\int_{V_s(t)} \rho dV = 0 \tag{2.1A}$$

The integral is taken over the system volume, which is a function of time due to the motion of the fluid.

If ρ is substituted for α in the Reynolds Transport Theorem it is possible to transform the above equation to the one that only field variables appear and the integration is over a fixed volume of space:

$$\frac{d}{dt}\int_{V_s(t)} \rho dV = \int_{V_c}\left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho v_i)\right] dV = 0 \tag{2.1B}$$

Since this equation is valid for any volume within the fluid, the integrand is everywhere zero:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho v_i) = 0 \tag{2.1C}$$

Equation (2.1C) is the continuity equation.[1] For a fluid with constant density equation (2.1C) simplifies to:

[1] A vectorial form of (2.1C) is left to the student as an exercise.

$$\frac{\partial v_i}{\partial x_i} = 0 \tag{2.1D}$$

For steady flow it reduces to:

$$\frac{\partial}{\partial x_i}(\rho v_i) = 0 \tag{2.1E}$$

Note the difference between (2.1D) and (2.1E).

The continuity equation in various coordinate systems is summarized in Bird et al. (1960, p. 83) or Bird et al. (2002, p. 846).

2.2 The Momentum Equation

Starting from the Newton's second law of motion:

$$\frac{d(m\vec{v})}{dt} = \vec{F} \tag{2.2A}$$

or, using index notation:

$$\frac{d(mv_i)}{dt} = F_i \tag{2.2B}$$

Applying this law to a fluid system yields an expression of the form:

$$\frac{d}{dt}\int_{V_s(t)} \rho v_i dV = F_i \tag{2.2C}$$

Equation (2.2C) is based on a Lagrangian description of the flow. To convert it into the equation based on an Eulerian description of the flow the RTT is used:

$$\frac{d}{dt}\int_{V_s(t)} \rho v_i dV = \int_{V_c}\left[\frac{\partial(\rho v_i)}{\partial t} + \frac{\partial}{\partial x_j}(\rho v_i v_j)\right]dV \tag{2.2D}$$

or,

$$\frac{d}{dt}\int_{V_s(t)} \rho v_i dV = \int_{V_c}\left[\rho\frac{\partial v_i}{\partial t} + v_i\frac{\partial \rho}{\partial t} + v_i\frac{\partial(\rho v_j)}{\partial x_j} + \rho v_j\frac{\partial v_i}{\partial x_j}\right]dV \tag{2.2E}$$

By using the continuity equation and the material derivative the term in the bracket in equation (2.2E) becomes:

$$\rho \frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial v_i}{\partial x_j} = \rho \frac{Dv_i}{Dt} \qquad (2.2F)$$

Then, Newton's second law in the Eulerian frame is:

$$\int_{V_c} \rho \frac{D\vec{v}}{Dt} dV = \vec{F} \qquad (2.2G)$$

As for the force term in equation (2.2G) there are two types of force that can act on the system: body forces and surface forces. It is natural therefore to decompose the above force term into two terms:

$$\vec{F} = \vec{F}_b + \vec{F}_s \qquad (2.2H)$$

where $\vec{F}_b$ is the total body force on the system (only gravitational force will be discussed in these notes). $\vec{F}_s$ is the total surface force on the system.

Assuming that the gravitational force per unit mass is derivable from a potential *gh*, where *g* is the acceleration due to gravity and *h* is the vertical distance above the reference plane. Assuming also that *g* is constant:

$$\vec{G} = -g\nabla h \qquad (2.2I)$$

and,

$$\vec{F}_b = \int_V \rho \vec{G}\, dV = -\int_V \rho g \nabla h dV \qquad (2.2J)$$

The attention now turns to the surface force $\vec{F}_s$ in equation (2.2H). It is desirable to represent this force in terms of a volume integral over some function of a field variable. Obviously, this field must describe the state of stress (force per unit area) at a point in the fluid.

Consider a surface element of the fluid system shown in Figure 2.1 and let:

$\vec{f}$ = surface force vector acting on the surface element

$\vec{n}$ = outward drawn unit normal vector at the center of the element

The surface force per unit area $\vec{f}/\delta A$ tends toward a limit as $\delta A \to 0$. This limit is called the stress vector. In addition to its dependence on space and time it also depends on the orientation of the surface that it acts on. Thus,

$$\lim_{\delta A \to 0}\left(\frac{\vec{f}}{\delta A}\right) = \vec{t}^{(\vec{n})} \qquad (2.2K)$$

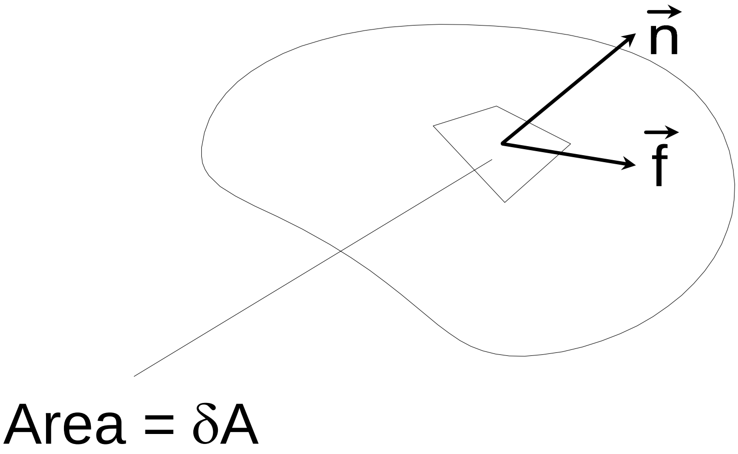

Figure 2.1. Surface element of a fluid system

The orientation of the surface is specified by giving the outward drawn unit normal vector for that surface. The total surface force acting on the fluid element can now be written as:

$$\vec{F}_s = \int_A \vec{t}^{(\vec{n})} dA \qquad (2.2L)$$

Without proof[2] equation (2.2L) can be rewritten as:

$$\vec{F}_s = \int_A (\vec{n} \cdot \underline{\sigma}) dA \qquad (2.2M)$$

Using Gauss's theorem equation (2.2M) can be written as:

$$\vec{F}_s = \int_A (\vec{n} \cdot \underline{\sigma}) dA = \int_V (\nabla \cdot \underline{\sigma}) dV \qquad (2.2N)$$

where the divergence of the stress tensor is a vector having components as follows:

$$\nabla \cdot \underline{\sigma} = \frac{\partial \sigma_{ij}}{\partial x_i} \qquad (2.2O)$$

[2] Interested student is referred to some advanced fluid mechanics textbooks, e.g., Panton (1995), for detail.

Using the results obtained for the body force and the surface force Newton's second law of motion for the fluid system becomes:

$$\int_V \rho \frac{D\vec{v}}{Dt} dV = -\int_V \rho g \nabla h dV + \int_V (\nabla \cdot \underline{\sigma}) dV \tag{2.2P}$$

or,

$$\int_V \left(\rho \frac{D\vec{v}}{Dt} + \rho g \nabla h - \nabla \cdot \underline{\sigma} \right) dV = 0 \tag{2.2Q}$$

But since this equation is valid for any arbitrary volume of space, the integrand must be identically zero everywhere. Hence,

$$\rho \frac{D\vec{v}}{Dt} = -\rho g \nabla h + \nabla \cdot \underline{\sigma} \tag{2.2R}$$

Equation (2.2R) is called the Cauchy's equation. It can also be written using index notation:

$$\rho \frac{Dv_j}{Dt} = -\rho g \frac{\partial h}{\partial x_j} + \frac{\partial}{\partial x_i}(\sigma_{ij}) \tag{2.2S}$$

When a fluid is at rest all the components of the stress tensor are not zero (it is expected that the normal stress be the pressure). When it moves the situation is quite different.

The thermodynamic pressure has a different conceptual origin than that of the surface forces. Surface forces are mechanical force concepts. Because of this ambiguity it is necessary to relate the normal surface stress to the thermodynamic pressure.

First, the stress tensor is broken into two parts by subtracting out the thermodynamic pressure p_t. In essence the viscous tensor τ_{ij} is defined:

$$\sigma_{ij} = -p_t \delta_{ij} + \tau_{ij} \tag{2.2T}$$

where δ_{ij} is the Kronecker delta. When a substance is not moving the normal stress is the same as the thermodynamic pressure. This requirement implies that the viscous stress must vanish when there is no motion. In general the normal stress is the sum of the pressure and a normal viscous stress. The normal stress, unlike the pressure, can have different values for different directions of the vector $\vec{n}$. It is possible to average the normal surface force (in three coordinate directions) and call this average the mechanical pressure p_m. This average gives:

$$p_m \equiv -\frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) = -\frac{1}{3}\sigma_{ii} \tag{2.2U}$$

An incompressible fluid does not have a thermodynamic pressure but it does have a mechanical pressure. When dealing with an incompressible fluid the pressure variable is always interpreted as the mechanical pressure.[3]

It is now possible to write the Cauchy's equation, equation (2.2S) in terms of the viscous stress:

$$\rho \frac{D\vec{v}}{Dt} = -\rho g \nabla h - \nabla p + \nabla \cdot \underline{\tau} \tag{2.2V}$$

An inviscid fluid is the one in which $\underline{\tau} = 0$. For such a material the last term in (2.2V) disappears and the result is the Euler's equation. Equation (2.2V) in various coordinate systems is listed in Bird et al. (1960, pp. 84-86) or Bird et al. (2002, p. 847). Note the difference in minus signs between the ones given here and the ones listed in Bird et al.

For an incompressible fluid the expression for the viscous stress is given, without proof, by the following definition. Note that it is more convenient to write this expression in index notation.

$$\tau_{ij} = \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \qquad (2.2W)^4$$

Combining (2.2V) and (2.2W) to eliminate τ_{ij}:

$$\rho \frac{Dv_j}{Dt} = -\rho g \frac{\partial h}{\partial x_j} - \frac{\partial p}{\partial x_j} + \frac{\partial}{\partial x_i}\left[\mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\right] \tag{2.2X}$$

If the viscosity is constant equation (2.2X) becomes:

$$\rho \frac{Dv_j}{Dt} = -\rho g \frac{\partial h}{\partial x_j} - \frac{\partial p}{\partial x_j} + \mu\left(\frac{\partial^2 v_i}{\partial x_i \partial x_j} + \frac{\partial^2 v_j}{\partial x_i \partial x_i}\right) \tag{2.2Y}$$

For an incompressible fluid, this equation becomes:

$$\rho \frac{Dv_j}{Dt} = -\rho g \frac{\partial h}{\partial x_j} - \frac{\partial p}{\partial x_j} + \mu \frac{\partial}{\partial x_i}\left(\frac{\partial v_j}{\partial x_i}\right) \tag{2.2Z}$$

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[3] For common fluids it is nearly always assumed that the two pressures are equivalent and there is no need to distinguish between mechanical and thermodynamic pressure. This is called Stoke's assumption.

[4] In tensor form this equation can be written as: $\underline{\tau} = \mu \underline{\dot{\gamma}}$, where $\underline{\dot{\gamma}}$ is the rate of deformation tensor, $\underline{\dot{\gamma}} = \nabla \vec{v} + \nabla \vec{v}^T$. Note that the velocity gradient can be decomposed into rate of deformation tensor and rate of rotation tensor.

These three equations are the Navier-Stokes equations for a fluid with constant density. Using vector notation, these become:

$$\rho \frac{D\vec{v}}{Dt} = -\rho g \nabla h - \nabla p + \mu \nabla^2 \vec{v} \tag{2.2AA}$$

The Navier-Stokes equations in various coordinate systems are listed in Bird et al. (1960, pp. 84-86) or Bird et al. (2002, p. 848). Note again the difference in minus sign of the two sets of equations.

Example 1 Determine $v_\theta(r)$ between two coaxial cylinders of radii R and κR rotating at angular velocities Ω_o and Ω_i, respectively. Assume that the space between cylinders is filled with an incompressible isothermal fluid in laminar flow.

Solution

First of all, the following simplifying assumptions can be made: the fluid is Newtonian and incompressible, the flow is laminar and steady, there is no end effect, and $V_r = V_z = 0$.

Based on the above-mentioned assumptions, the (simplified) continuity and momentum equations can be written as:

Continuity: $$\frac{\partial v_\theta}{\partial \theta} = 0 \tag{E21A}$$

r-momentum: $$-\rho \frac{v_\theta^2}{r} = -\frac{\partial p}{\partial r} \tag{E21B}$$

z-momentum: $$-\frac{\partial p}{\partial z} + \rho g_z = 0 \tag{E21C}$$

θ-momentum $$\mu \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right) = 0 \tag{E21D}$$

Integrating equation (E21D) twice yields:

$$v_\theta = C_1 r + \frac{C_2}{r} \tag{E21E}$$

Two boundary conditions are needed to solve equation (E21E) and hence the flow problem at hand: at $r = \kappa R$, $v_\theta = \kappa R \Omega_i$ and at $r = R$, $v_\theta = R\Omega_0$. The values of constants C_1 and C_2 are therefore:

$$C_1 = \frac{(\kappa R)^2 \Omega_i - \Omega_0 R^2}{(\kappa R)^2 - R^2} \text{ and } C_2 = \frac{(\Omega_0 - \Omega_i)\kappa^2 R^4}{(\kappa R)^2 - R^2}.$$

Inserting these constants into the velocity profile (E21E) yields:

$$v_\theta = \frac{1}{R^2(1-\kappa^2)}\left[R^2 r(\Omega_0 - \kappa^2\Omega_i) - \frac{\kappa^2 R^4}{r}(\Omega_0 - \Omega_i)\right] \tag{E21F}$$

2.3 The Energy Equation

Using the results obtained in Chapter 1 (section 1.7) the first law of thermodynamics is:

$$\int_{V_c} \rho\left[\frac{D}{Dt}\left(u + \frac{v^2}{2}\right)\right] dV = -\int_{V_s} \nabla \cdot \vec{q}\, dV + \int_{V_s} \rho \vec{G} \cdot \vec{v}\, dV + \int_{V_s} (\vec{v} \cdot \nabla \cdot \underline{\sigma} + \underline{\sigma} : \nabla \vec{v}) dV \tag{2.3A}$$

or,

$$\rho \frac{D}{Dt}\left(u + \frac{v^2}{2}\right) = -\nabla \cdot \vec{q} + \rho \vec{G} \cdot \vec{v} + \vec{v} \cdot \nabla \cdot \underline{\sigma} + \underline{\sigma} : \nabla \vec{v} \tag{2.3B}$$

Note that equation (2.3B) is a scalar equation. Equation (2.3B) is simplified by taking the dot product of the velocity with Cauchy's equation of the previous section:

$$\rho \vec{v} \cdot \frac{D\vec{v}}{Dt} = \rho \vec{v} \cdot \vec{G} + \vec{v} \cdot \nabla \cdot \underline{\sigma} \tag{2.3C}$$

or,

$$\rho \frac{D}{Dt}\left(\frac{v^2}{2}\right) = \rho \vec{v} \cdot \vec{G} + \vec{v} \cdot \nabla \cdot \underline{\sigma} \tag{2.3D}$$

This equation, sometimes called the mechanical energy balance (see Bird et al. (1960, p. 314) or Bird et al. (2002, p. 81). Note again the difference in sign), is subtracted from equation (2.3B) to give the energy equation:

$$\rho \frac{Du}{Dt} = -\nabla \cdot \vec{q} + \underline{\sigma} : \nabla \vec{v} \tag{2.3E}$$

The internal energy of an element of fluid can be changed by heat crossing the boundary or by work done against surface forces. Further modification of the above equation can be done by substitution of (2.2T) into the above equation:

$$\rho \frac{Du}{Dt} = -\nabla \cdot \vec{q} - p\nabla \cdot \vec{v} + \underline{\tau} : \nabla \vec{v} \tag{2.3F}$$

The last term, often called the dissipation term, originates in work done against viscous forces. The expression for the dissipation term in various coordinate systems for Newtonian fluids is listed in Bird et al. (1960, p. 91) or Bird et al. (2002, p. 849). Note that for an incompressible fluid the last term in equations (A), (B) and (C) in Table 3.4-8 of Bird et al. (1960) or the last terms of equations (B3.7-1) to (B3.7-3) of section B.7 of Bird et al. (2002) equal to zero from the continuity equation.

From the results of Chapter 1 Equation (2.3F) can be written in terms of enthalpy h as:

$$\rho \frac{Dh}{Dt} = -\nabla \cdot \vec{q} + \frac{Dp}{Dt} + \underline{\tau} : \nabla \vec{v} \qquad (2.3G)$$

Equations (2.3F) and (2.3G) are the most widely quoted form of the energy equation.

From equation (1.7L) equation (2.3G) can be rewritten as:

$$\rho c_p \frac{DT}{Dt} = -\nabla \cdot \vec{q} + T\beta \frac{Dp}{Dt} + \underline{\tau} : \nabla \vec{v} \qquad (2.3H)$$

Equation (2.3F) can also be rewritten as:

$$\rho c_v \frac{DT}{Dt} = -\nabla \cdot \vec{q} + T\left(\frac{\partial p}{\partial T}\right)_\rho \nabla \cdot \vec{v} + \underline{\tau} : \nabla \vec{v} \qquad (2.3I)$$

Note the difference in the sign of the above equation and equation (10.1-19) in Bird et al. (1960) or equation (I) in Table 11.4-1 of Bird et al. (2002).

It is usual to express the heat flux $\vec{q}$ in terms of the temperature gradient (via the use of Fourier's law of heat conduction) and the viscous stress tensor $\underline{\tau}$ in terms of the velocity gradient. For Newtonian fluids with constant thermal conductivity k, equation (2.3I) can be rewritten as:

$$\rho c_v \frac{DT}{Dt} = k\nabla^2 T - T\left(\frac{\partial p}{\partial T}\right)_\rho (\nabla \cdot \vec{v}) + \mu \Phi_v \qquad (2.3J)$$

This equation states that the temperature change is due to the heat conduction (1st term on the right), an expansion effect and the viscous heating (2nd and 3rd terms on the right, respectively).

Four more simplifications are widely used and are summarized in Bird et al. (1960, p. 316) or Bird et al. (2002, pp. 337-338). Note also the statement regarding various source terms. The equation of energy both in terms of energy and momentum fluxes and in terms of the transport properties are summarized in Bird et al. (1960, pp. 318-319) or Bird et al. (2002, p. 340). Note again the difference in the sign.

Five governing conservation equations are now available for the five variables of interest (3 velocity components, temperature and pressure). The sensible question to ask at this point is how many boundary and initial conditions are needed to solve the full set of equations?

Example 2 Consider two impinging water streams in a two-dimensional T-junction (see Figure E2.2). The two streams enter the two legs of the junction at different temperatures and mix with each other while flowing toward the exit at the other end. Write a complete set of the governing conservation equations along with boundary conditions for this flow situation if velocity and temperature profiles in this junction are of interest. Assume that the flow is at steady state. Make other assumptions where appropriate.

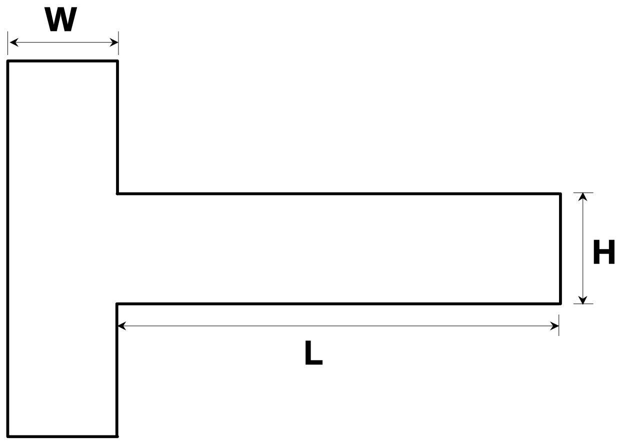

Figure E2.2. A two-dimensional T-junction

Solution

The following simplifying assumptions can be made in this problem: the fluid is Newtonian and incompressible and has constant physical properties, the flow is laminar and steady, viscous dissipation is negligible and the flow is fully developed at the outlet of the junction.

Based on the above-mentioned assumptions, the (simplified) governing conservation equations can be written as:

Continuity: $$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \quad \text{(E22A)}$$

x-momentum: $$\rho\left(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2}\right) \quad \text{(E22B)}$$

y-momentum: $$\rho\left(v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2}\right) - \rho g_y \quad \text{(E22C)}$$

Energy: $$\rho c_p\left(v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y}\right) = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \quad \text{(E22D)}$$

Since the process is at steady state, there is no need to specify the initial condition. Fourteen boundary conditions are needed to solve the above 4 conservation equations as follows:

at $x = 0$, $-\frac{H}{2} - w \le y \le \frac{H}{2} + w$, $v_x = 0$, $v_y = 0$, $\frac{\partial T}{\partial x} = 0$

at $x = w + L$, $-\frac{H}{2} \le y \le \frac{H}{2}$, $\frac{\partial v_x}{\partial x} = 0$, $v_y = 0$, $\frac{\partial T}{\partial x} = 0$

at $0 \le x \le w$, $y = \frac{H}{2} + w$, $v_x = 0$, $v_y = -v_{in1}$, $T = T_{in1}$

at $w \le x \le w + L$, $y = \frac{H}{2}$, $v_x = 0$, $v_y = 0$, $\frac{\partial T}{\partial y} = 0$

at $0 \le x \le w$, $y = -\frac{H}{2} - w$, $v_x = 0$, $v_y = v_{in2}$, $T = T_{in2}$

at $w \le x \le w + L$, $y = -\frac{H}{2}$, $v_x = 0$, $v_y = 0$, $\frac{\partial T}{\partial y} = 0$

at $x = w + L$, $\frac{\partial p}{\partial x} = 0$ and at inlets p = finite.

The momentum equation developed earlier is valid for non-isothermal flow. In using it, however, it is important to consider ρ and μ to be functions of temperature as well as pressure. In forced convection problems it is customary to use the momentum equation in the form given in section 2.2.[5] In free convection problems, where the temperature dependence of ρ is of critical importance, it is more convenient to modify the momentum equation to account automatically for buoyancy effects.

The density is a function of temperature and pressure through the equation of state of the fluid:

$$\rho = \rho(T, p) \tag{2.3K}$$

This function is linearized by using a Taylor series expansion around a reference temperature and pressure (denoted by T_R and p_R, respectively). Retaining only the first order terms, the series is:

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[5] When constant properties are assumed the continuity and momentum equations are independent of the energy equation. The flow problem is solved first to obtain v_1, v_2, v_3 and p. The velocity components then become known functions in the energy equation, which is solved for the temperature field.

$$\rho \;=\; \rho_R + \left(\frac{\partial \rho}{\partial T}\right)_{p_R, T_R} (T - T_R) \;+\; \left(\frac{\partial \rho}{\partial p}\right)_{p_R, T_R} (p - p_R) \tag{2.3L}$$

With the isothermal compressibility, κ, defined as:

$$\kappa \;= -\frac{1}{\tilde{v}}\left(\frac{\partial \tilde{v}}{\partial p}\right)_T \;= \frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_T \tag{2.3M}$$

and using β, the coefficient of thermal expansion, equation (2.3L) can be written as:

$$\rho \;=\; \rho_R\left[1 - \beta(T - T_R) + \kappa(p - p_R)\right] \tag{2.3N}$$

The parameters β and κ are evaluated at the reference temperature. The value of β for gases is only 3 to 10 times larger than the value for liquids, while κ is 10^4 to 10^5 times larger. It is expected that equation (2.3N) be a reasonable approximation if:

$$\left|- \beta(T - T_R) + \kappa(p - p_R)\right| \le 0.3 \tag{2.3O}$$

At ambient temperatures the linearization should be acceptable for temperature differences up to about 100 K for gases and even more for liquids. The pressure difference is up to about 30 kPa for gases and much more for liquids. Most flows in which density variations are important are those that involve small pressure differences, e.g. external flows or internal flows with moderate pressure gradients. Since the β-term in equation (2.3N) is usually much larger than the κ-term, the latter is dropped:

$$\rho = \rho_R\left[1 - \beta(T - T_R)\right] \tag{2.3P}$$

Equation (2.3P) forms the basis of the Boussinesq approximation. This approximation involves two assumptions: (1) the density varies linearly with temperature as described by equation (2.3P) and (2) the density is constant at ρ_R in the continuity and energy equations and in each term of the momentum equation except in the gravity term.

Consider Cauchy's equation written for the viscous fluid with $\vec{G}$ as the body force per unit mass:

$$\rho\frac{D\vec{v}}{Dt} = \rho\vec{G} - \nabla p + \nabla\cdot\underline{\tau} \tag{2.3Q}$$

Using the Boussinesq approximation this equation becomes:

$$\rho\frac{D\vec{v}}{Dt} \;=\; \rho\vec{G} - \;\nabla p \;- \rho\beta(T - T_R)\vec{G} \;\; + \nabla\cdot\underline{\tau} \tag{2.3R}$$

where the subscript R has been dropped from the density with the understanding that it is constant at the reference condition.

Under the Boussinesq approximation the only effect of variable density is the introduction of the third term on the right hand side of equation (2.3R). Although the energy equation is unchanged, it is coupled to the momentum equation through the $\rho\beta(T-T_R)\vec{G}$ term. Simultaneous solution of the thermal and flow problems is thus required. Variations of other properties with temperature, especially viscosity, often must be taken into account as well.

For situations **without** free surfaces equation (2.3R) may be rewritten by considering the pressure gradient in a motionless fluid having uniform density equal to the reference condition. The pressure gradient in this case is given by:

$$\nabla p^o = \rho \vec{G} \tag{2.3S}$$

Where the superscript *0* denotes the situation for uniform density and the subscript R has been deleted as above. Define P, the hydrodynamic pressure, as the difference between the actual pressure and the hydrostatic pressure in the reference condition:

$$P = p - p^o \tag{2.3T}$$

Substitution of equations (2.3S) and (2.3T) into equation (2.3R) yields:

$$\rho \frac{D\vec{v}}{Dt} = -\nabla P - \rho\beta(T - T_R)\vec{G} + \nabla \cdot \underline{\tau} \tag{2.3U}$$

2.4 Dimensionless Form of the Governing Conservation Equations

Consider the governing conservation equations for an incompressible Newtonian fluid under the Boussinesq approximation with the further assumption that c_p, k and μ are constant:

$$\nabla \cdot \vec{v} = 0 \tag{2.4A}$$

$$\rho \frac{D\vec{v}}{Dt} = -\nabla P - \rho\beta(T - T_R)\vec{G} + \mu\nabla \cdot \underline{\dot{\gamma}} \tag{2.4B}$$

$$\rho c_p \frac{DT}{Dt} = k\nabla^2 T + \mu\Phi_v \tag{2.4C}$$

Let L = characteristic length; U = characteristic velocity; and ΔT = characteristic temperature difference and define the following dimensionless variables: $v_i^* = v_i / U$; $x_i^* = x_i / L$; t^* = tU/L; $P^* = P/\rho U^2$; $G_i^* = G_i / G$; $T^* = T - T_R / \Delta T$; $\nabla^* = L\nabla$; $\nabla^{2^*} = L^2\nabla^2$; $\underline{\dot{\gamma}}^* = \frac{L}{U}\underline{\dot{\gamma}}$; $\Phi_v^* = \frac{L^2}{U^2}\Phi_v$. It is a normal practice to make these dimensionless variables fall in the range of 0 to 1.

Using these newly defined variables the governing equations become:

$$\nabla^* \cdot \vec{v}^* = 0 \tag{2.4D}$$

$$\frac{D\vec{v}^*}{Dt^*} = -\nabla^* P^* - \left(\frac{\beta \Delta T G L}{U^2}\right)\vec{G}^* T^* + \left(\frac{\mu}{\rho U L}\right)\nabla^* \cdot \underline{\dot{\gamma}}^* \tag{2.4E}$$

$$\frac{DT^*}{Dt^*} = \frac{k}{\rho c_p L U}\nabla^{2^*} T^* + \frac{\mu U}{\rho c_p \Delta T L}\Phi_v^* \tag{2.4F}$$

Various dimensionless groups arise when putting the governing equations in dimensionless form: $\mathrm{Re} = \dfrac{\rho U L}{\mu}$; $\mathrm{Gr} = \dfrac{\beta \Delta T G L^3}{\nu^2}$; $\mathrm{Pe} = \dfrac{UL}{\alpha}$; $\mathrm{Br} = \dfrac{\mu U^2}{k \Delta T}$

The Peclet number can also be written in terms of the Reynolds number, Re, and the Prandtl number, Pr:

$$\mathrm{Pe} = (\mathrm{Re})(\mathrm{Pr}) \tag{2.4G}$$

where

$$\mathrm{Pr} = \frac{c_p \mu}{k} \tag{2.4H}$$

The Prandtl number is defined in terms of the heat capacity at constant pressure whether properties are constant or not.

The dimensionless form of the momentum equation (with the presence of the free surface) is:

$$\frac{D\vec{v}^*}{Dt^*} = \frac{1}{\mathrm{Fr}}\left(\frac{\vec{G}}{\mathrm{G}}\right) - \nabla^* P^* - \frac{\mathrm{Gr}}{\mathrm{Re}^2}\left(\frac{\vec{G}}{\mathrm{G}}\right)\mathrm{T}^* + \frac{1}{\mathrm{Re}}(\nabla^* \cdot \underline{\dot{\gamma}}^*) \tag{2.4I}$$

Where G is the magnitude of $\vec{G}$ and $\underline{\dot{\gamma}}^*$ is the dimensionless rate of deformation tensor. The unit vector $\vec{G}/G$ appears in equation (2.4I) so that the dimensionless groups can be written as scalars following the usual practice. The new dimensionless group is the Grashof number:

$$\mathrm{Gr} = \frac{\beta \Delta T G L^3}{\nu^2} \tag{2.4J}$$

where ν is the kinematic viscosity, μ/ρ. No new groups are created when the continuity equation is put into dimensionless form. As shall be seen later the boundary conditions may introduce additional dimensionless groups.

The importance of the density variation in equation (2.4I) is indicated by the ration $\mathrm{Gr/Re}^2$. The two extreme cases are: forced convection when $\mathrm{Gr/Re}^2 \rightarrow 0$; and natural or free convection when $\mathrm{Gr/Re}^2 \rightarrow \infty$. In the first case there is an externally imposed velocity, which is used as the characteristic velocity in the definition of the Reynolds number. In the second case there is no externally imposed velocity, i.e., the velocity is generated as a consequence of the variation of the density.

The dimensionless energy equation is:

$$\frac{DT^*}{Dt^*} = \frac{1}{\text{Pe}}\left[\nabla^{2^*} T^* + \text{Br}\,\Phi_v^*\right] \tag{2.4K}$$

The Brinkman number is Br = $\dfrac{\mu U^2}{k\Delta T}$.

2.5 Heat Transfer Rates

Consider a single fluid phase and a solid surface with heat transfer between them. The rate of heat transfer at the surface is:

$$\vec{q}_s = -k(\nabla T)_s \tag{2.5A}$$

where the subscript *s* denotes the value in the fluid at the surface. In dimensionless form:

$$\frac{q_s L}{k\Delta T} = \left|\nabla^* T^*\right|_s \tag{2.5B}$$

where q_s is the magnitude of $\vec{q}$. The heat transfer coefficient, *h*, is defined by:

$$h = \frac{q_s}{\Delta T} \tag{2.5C}$$

where ΔT is the characteristic temperature difference. The Nusselt number, which is widely used in correlations of heat transfer data, is defined as:

$$\text{Nu} = \frac{hL}{k} \tag{2.5D}$$

Combining equations (2.5B) to (2.5D) yields:

$$\text{Nu} = \left|\nabla^* T^*\right|_s \tag{2.5E}$$

The dimensionless temperature *T** depends upon the dimensionless velocity $\vec{v}^*$, which depends upon Gr, hence:

$$T^* = f(\text{Fr, Re, Gr, Pr, Br}, t^*, x_s^*) \tag{2.5F}$$

The dimensionless velocity is a function of the same variables. From equations (2.5E) and (2.5F) it is concluded that:

$$\mathrm{Nu} = f(\mathrm{Fr},\ \mathrm{Re},\ \mathrm{Gr},\ \mathrm{Pr},\ \mathrm{Br},\ t^*,\ x_s^*) \qquad (2.5G)$$

Where x_s^* denotes the dimensionless position on the surface.
Some common situations involve steady state, no free surfaces and no dissipation, for which:

$$\mathrm{Nu} = f(\mathrm{Re},\ \mathrm{Gr},\ \mathrm{Pr},\ x_s^*) \qquad (2.5H)$$

For forced convection:

$$\mathrm{Nu} = f(\mathrm{Re}, \mathrm{Pr},\ x_s^*) \qquad (2.5I)$$

For natural convection:

$$\mathrm{Nu} = f(\mathrm{Gr}, \mathrm{Pr},\ x_s^*) \qquad (2.5J)$$

Equations (2.5G) to (2.5J) give the relationships for the local values of the Nusselt number. The surface-averaged value of Nu may be obtained by integrating equation (2.5E) over the surface area:

$$\mathrm{Nu}_{avg} = \frac{1}{\mathrm{A_s}} \int_{A_s} \left| \nabla^* T^* \right|_s dA \qquad (2.5K)$$

The integration removes the dependence on the position on the surface, i.e., the analog of equation (2.5H) is:

$$\mathrm{Nu}_{\mathrm{avg}} = f(\mathrm{Re}, \mathrm{Gr},\ \mathrm{Pr}) \qquad (2.5L)$$

References

1. Bird, R.B., Stewart, W.E., Lightfoot, E.N., 1960, **Transport Phenomena**, Wiley, New York.
2. Bird, R.B., Stewart, W.E., Lightfoot, E.N., 2002, **Transport Phenomena**, **2nd Edition**, Wiley, New York.
3. Panton, R.L., 1995, **Incompressible Flow, 2nd Edition**, Wiley, New York.

Problems

1. From the Reynolds Transport Theorem (RTT):

$$\frac{D}{Dt}\left[\int_{V_s} \alpha dV \right] = \int_{V_c} \left[\frac{\partial \alpha}{\partial t} + \nabla \cdot (\alpha \vec{v}) \right] dV$$

Let β be a scalar quantity per unit mass. Using the continuity equation show that RTT can be written as:

$$\frac{D}{Dt}\left[\int_{V_s} \rho\beta dV\right] = \int_{V_c}\left[\rho\frac{D\beta}{Dt}\right]dV$$

Since $V_s = V_c$, the subscripts could be dropped.

2. Does the velocity distributions shown below satisfy the continuity equation?

$$v_1 = \frac{a}{x_1}\cos(bx_2)$$
$$v_2 = -\frac{a}{x_1^2 b}\sin(bx_2)$$
$$v_3 = 0$$

3. Show that if a particle moves with a constant speed (magnitude of velocity) its acceleration vector is normal to its velocity vector. (<u>Hint:</u> (speed)^2 = $\vec{x}\cdot\vec{x}$).

4. A part of a lubrication system (see Figure P2.4) consists of two circular disks between which a lubricant flows radially. The flow takes place because of a modified pressure difference P_1-P_2 between the inner and outer radii r_1 and r_2, respectively.

 (a) Write the equations of continuity and motion for the flow system, assuming steady state, laminar, and incompressible Newtonian flow. Consider only the region $r_1 \leq r \leq r_2$ and that the flow is radially directed.
 (b) Show how the equation of continuity enables one to simplify the equation of motion to give:

$$\rho\frac{\phi^2}{r^3} = -\frac{dP}{dr} + \mu\frac{1}{r}\frac{d^2\phi}{dz^2} \qquad \text{(P24A)}$$

in which $\phi = rv_r$ is a function of z only.

 (c) It can be shown that no solution exists for equation (P24A) unless the nonlinear term containing ϕ is omitted. Omission of this term corresponds to the creeping flow assumption. Show that for creeping flow, equation (P24A) can be integrated with respect to r to give:

$$0 = (P_1 - P_2) + \left(\mu \ln\frac{r_2}{r_1}\right)\frac{d^2\phi}{dz^2} \qquad \text{(P24B)}$$

 (d) Show that further integration with respect to z gives:

$$v_r = \frac{(P_1 - P_2)b^2}{2\mu r \ln(r_2/r_1)}\left[1 - \left(\frac{z}{b}\right)^2\right] \tag{P24C}$$

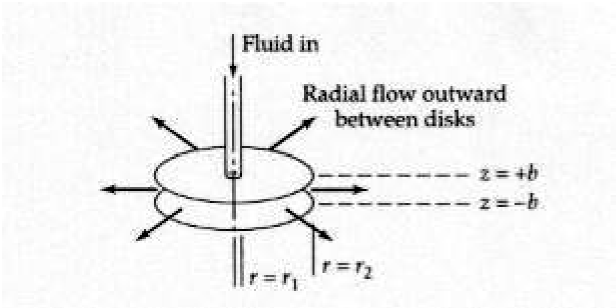

Figure P2.4. Outward radial flow in the space between two parallel circular disks

5. Consider the flow of a liquid food through a die that could be located at the exit of a screw extruder. In the die of length *D* shown in Figure P2.5, a pressure difference $p_2 - p_3$ causes a liquid food of viscosity μ to flow steadily from left to right in the annular area between two fixed concentric cylinders of radii r_1 and r_2. Find the velocity profile of the liquid food in the annular space. It is reasonable to assume here that the pressure gradient is uniform between the die inlet and exit.

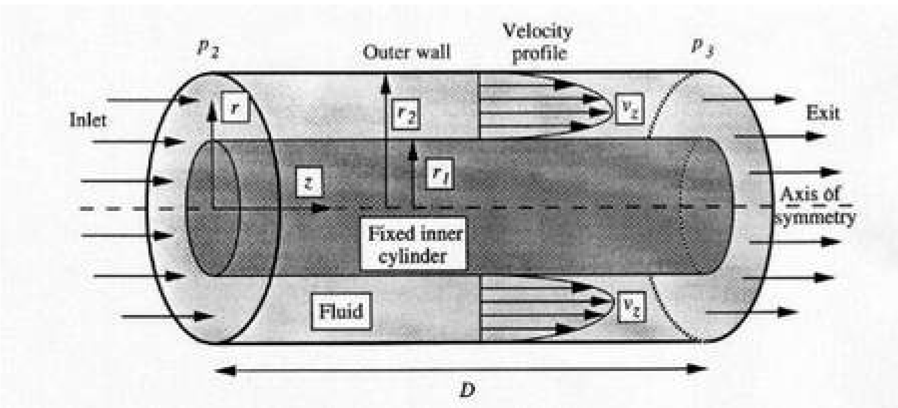

Figure P2.5. Geometry of flow through an annular die

6. Drilling mud is being pumped vertically upward in a concentric annulus. The inner cylinder with radius R_1 is the drill pipe, which is rotating at an angular speed of Ω radians per second. The outer cylinder is the well casing of radius R_2. Write the equations of motion in the appropriate coordinate system (list all simplifying assumptions made) and find the velocity components, v_θ and v_z, as functions of *r*.

7. Consider a chemically reacting rigid system depicted in Figure P2.7 within which there is a constant internal heat generation per unit volume $q^{///}$. Assuming that the system can be treated as a one-dimensional body, obtain the steady-state temperature distribution corresponding to the given boundary conditions. Also find the steady-state rate of heat transfer within the walls ($x = \pm L$).

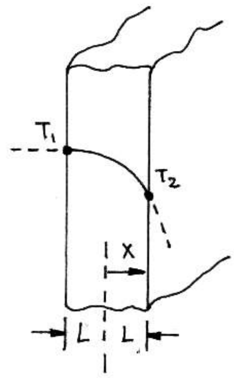

Figure P2.7. A chemically reacting system of Problem 7

8. For a fully developed laminar flow (of a constant property fluid) in a circular tube the velocity profile is given by:

$$v_z = 2\bar{v}\left[1-\left(\frac{r}{R}\right)^2\right] \tag{P28A}$$

where $\bar{v}$ is the average velocity and R is the radius of the tube. The wall of the tube is maintained at a constant temperature T_w. Far from the entrance of the tube the energy dissipated is conducted to the tube wall and the fluid temperature is not a function of axial position z.

(a) Apply the energy equation with dissipation and solve it to find the temperature profile of the flowing fluid.
(b) Define the mixing cup temperature, T_m, by:

$$T_m - T_w = \frac{\int_0^R r v_z (T - T_w) dr}{\int_0^R r v_z dr} \tag{P28B}$$

and the heat transfer coefficient, h, by:

$$q = -k\left(\frac{dT}{dr}\right)_{r=R} \equiv h(T_m - T_w) \tag{P28C}$$

Find the Nusselt number, which is defined in this case as Nu = $\frac{2hR}{k}$.

(c) Calculate the value of Brinkman number using (T_m-T_w) as the characteristic temperature difference and $\bar{v}$ as the characteristic velocity.

9. A Newtonian fluid flows between two large parallel flat plates shown in Figure P2.9. While the upper plate is stationary the lower plate moves to the left at a speed U. The spacing between the two plates is fixed at b and the flow is assumed to be laminar. Write the appropriate governing conservation equations and use them to determine the pressure gradient required to produce zero net flow (Q = 0) at any cross-section of the channel.

   Hint: In this case $Q = \int_0^b u dy$, where u is the fluid velocity in x-direction.

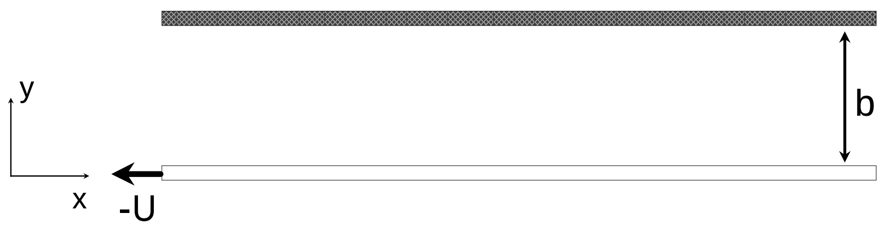

Figure P2.9. The flow system of Problem 9

10. It is desired to study the natural convection heat transfer during sterilization of a vertically placed cylindrical canned liquid food in a still retort by solving simultaneously the continuity, momentum and energy equations describing the transport processes within the can.

    A schematic sketch of the can is shown in Figure P2.10. The can outer surface temperature (bottom and side) is assumed to rise instantaneously and to be maintained at 121^{o}C throughout the heating period. The top surface of the can is insulated.

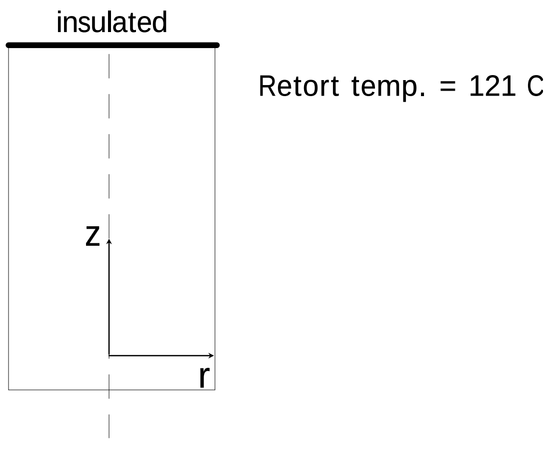

Figure P2.10. A schematic sketch of the can undergoing sterilization

For simplicity the liquid food can be assumed to behave as Newtonian fluid. This approximation is indeed valid for most liquid foods such as tomato puree, apple sauce and carrot puree, which are regularly canned and usually preserved by heating. The flow within the can may be assumed to be axisymmetry, i.e., the problem is reduced to two-dimensional (in radial and vertical directions). Viscous dissipation can be neglected due to the low velocities in the system and the Boussinesq approximation is valid. Pressure gradients in the can are also negligible.

To further simplify the analysis all physical properties but the density in the momentum equation may be assumed to be constant (this assumption is of course very crude and only serves to simplify the analysis at hand).

(a) Give the differential equations describing the velocity and temperature fields in the can.
(b) Give the initial and boundary conditions necessary to completely solve this problem.

Hint: Radial symmetry can be assumed along the centerline of the can.

11. Consider a steady flow of very viscous Newtonian oil through a fixed and a moving infinite plate as shown in Figure P2.11. The two plates are 2*z* apart and the upper plate moves at the speed *U* relative to the lower one. The upper plate is held at a temperature T_1 while the lower plate is at T_0.

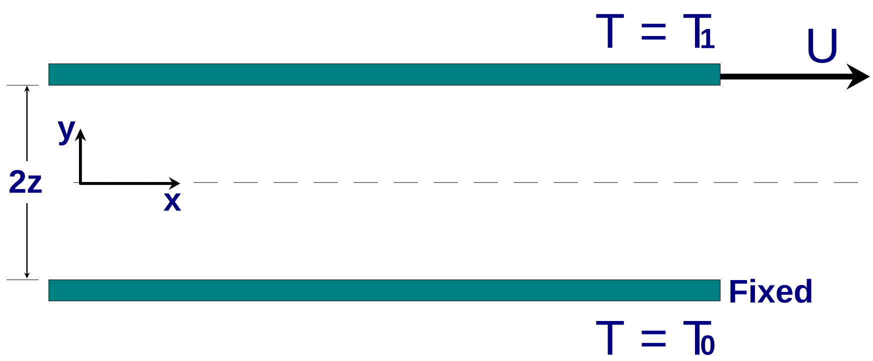

Figure P2.11. Steady flow between a fixed and a moving plate

(a) Write the differential equations that govern the above transport situation and obtain the velocity and temperature profiles of the fluid flowing between the two plates.

(b) Show that, in this case, $\mathrm{Nu} = 1 \pm \frac{\mathrm{Br}}{2}$, where Nu is the Nusselt number at each plate and Br is the Brinkman number.

Hint: The heat flux at each plate is computed as $q_w = \left| k \frac{\partial T}{\partial y} \right|_{\pm z}$. In this case, the characteristic temperature difference is $\Delta T = T_1 - T_0$ and the characteristic length is 2*z*.