Varieties of Tree Languages

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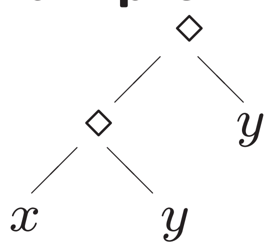
Trees

Ranked alphabet Σ, Leaf alphabet X

$T(\Sigma, X)$ = set of trees with

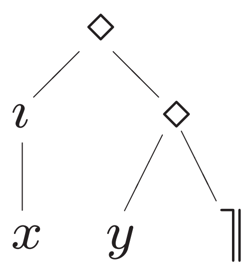
node labels from Σ and leaf labels from X.

Example $\Sigma = \{\diamond/2\}$, $X = \{x, y\}$



$$= \diamond\Big(\diamond (x, y), y\Big) \in T(\Sigma, X)$$

$\Omega = \{\rrbracket/0,\ \imath/1,\ \diamond/2\}$



$$= \diamond\Big(\imath(x), \diamond(y, \rrbracket)\Big) \in T(\Omega, X)$$

Words $\Lambda = \{a/1, b/1, \ldots\}$, $Y = \{\epsilon\}$

$$= baa = a(a(b(\epsilon))) \in T(\Lambda, Y)$$

Tree Languages

Any $T \subseteq T(\Sigma, X)$ is a ΣX-tree language.

Family of Tree Languages

For a fixed Σ,
$\mathscr{V} = \{\mathscr{V}(X)\}$, where $\mathscr{V}(X)$ is a set of ΣX-tree languages for each X.

Generalized family of tree languages

$\mathscr{V} = \{\mathscr{V}(\Sigma, X)\}$, where $\mathscr{V}(\Sigma, X)$ is a set of ΣX-tree languages for each pair Σ,X.

Variety of Tree Languages

A family of tree languages $\mathscr{V} = \{\mathscr{V}(X)\}$
closed under

- finite intersections, finite unions,
and complements, i.e., if $T, T' \in \mathscr{V}(X)$ then
$T \cap T', T \cup T', T(X) \setminus T \in \mathscr{V}(X)$

- inverse translations, and

- inverse morphisms.

Generalized variety of tree languages

A generalized family $\mathscr{V} = \{\mathscr{V}(\Sigma, X)\}$
closed under

- finite intersections, finite unions,
and complements,

- inverse translations, and

- inverse generalized morphisms.

Algebras

Σ-algebra $\mathcal{A} = (A, \Sigma)$:

- $A \neq \emptyset$ set of elements
- $c \in \Sigma_0$ (nullary) defines $c^{\mathcal{A}} \in A$
- $f \in \Sigma_m$ $(m > 0)$ defines $f^{\mathcal{A}} : A^m \to A$
- algebra $\mathcal{A}$ is finite, if A is finite

Example

$\Sigma = \{\diamond/2\}$

$$\text{Semigroup} : \diamond(a, \diamond(b, c)) = \diamond(\diamond(a, b), c).$$

$\Omega = \{\rrbracket/0, \imath/1 \diamond/2\}$

$$\begin{aligned}\text{Group} : \ &\diamond(a, \diamond(b, c)) = \diamond(\diamond(a, b), c),\\ &\diamond(\rrbracket, a) = \diamond(a, \rrbracket) = a,\\ &\diamond(\imath(a), a) = \diamond(a, \imath(a)) = \rrbracket.\end{aligned}$$

Term Algebra

$\mathcal{T}(\Sigma, X) = (T(\Sigma, X), \Sigma)$

- $c^{\mathcal{T}(\Sigma,X)} = c$ for $c \in \Sigma_0$
- $f^{\mathcal{T}(\Sigma,X)}(t_1, \ldots, t_m) = f(t_1, \ldots, t_m)$
for $f \in \Sigma_m$ $(m > 0)$
and $t_1, \ldots, t_m \in T(\Sigma, X)$.

Tree Recognizers

ΣX-tree recognizer $(\mathcal{A}, \alpha, F)$:

- $\mathcal{A} = (A, \Sigma)$ finite Σ-algebra
- $\alpha : X \to A$ initial assignment
- $F \subseteq A$ final states

Extend α to a homomorphism

$$\varphi : \mathcal{T}(\Sigma, X) \to \mathcal{A}$$

tree language recognized by $(\mathcal{A}, \alpha, F)$:

$$F\varphi^{-1} = \{t \in T(\Sigma, X) \mid t\varphi \in F\}.$$

Minimal algebra recognizing a tree language, is the *syntactic algebra* of the language.

Variety of finite algebras

A variety of finite algebras, is a class of Σ-algebras closed under

- subalgebras,
- homomorphic images, and
- finite direct products.

The choice of Σ here is essential. Subsemigroups of groups are not necessarily groups.

Variety Theorem

Variety of tree languages $\mathscr{V} = \{\mathscr{V}(X)\}$

$\updownarrow$

Variety of finite algebras $\mathbf{K}$

(M. Steinby 1979, 1992; J. Almeida 1990)

- Any T in $\mathscr{V}$ can be recognized by an algebra in $\mathbf{K}$.

- All tree languages recognized by an algebra in $\mathbf{K}$ belong to $\mathscr{V}$.

Other Syntactic Structures

associated with a tree language:

- Syntactic Monoid (the translation monoid of the syntactic algebra)

(W. Thomas 1982,4; K. Salomaa 1983)

- Syntactic Ordered Algebra and
  Syntactic Ordered Monoid

(T. Petković and S. Salehi 2005)

- Syntactic Tree Algebra (3-sorted algebra; for binary trees)

(T. Wilke, 1996)

- Syntactic Theory (Clone) and Preclone

(Z. Ésik 1999; Z. Ésik and P. Weil 2005)

New Results (Variety Theorems)

- Varieties of Many-Sorted Subsets of Free Algebras
  $\updownarrow$
  Varieties of Many-Sorted (Syntactic) Algebras

(S. Salehi and M. Steinby; Chapter 2)

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- Positive Varieties of Tree Languages
  $\updownarrow$
  Varieties of Ordered Algebras

(T. Petković and S. Salehi; Chapter 3)

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- Certain Varieties of Tree Languages
  $\updownarrow$
  Varieties of Monoids

(S. Salehi; Chapter 4)

- Certain Positive Varieties of Tree Languages
  $\updownarrow$
  Varieties of Ordered Monoids

(T. Petković and S. Salehi; Chapter 5)

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- Certain Families of Binary Tree Languages
  $\updownarrow$
  Varieties of Tree Algebras

(S. Salehi and M. Steinby; Chapter 6)

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- A case study (instance) of variety theorems

(T. Petković and S. Salehi; Chapter 5)

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- Some algebraic properties of tree algebras

(S. Salehi; Chapter 6)