A completeness property of Wilke's Tree Algebras

Saeed Salehi

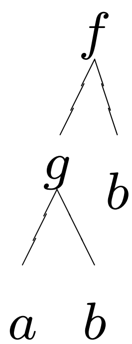
Turku Centre for Computer Science

saeed@cs.utu.fi

Trees: terms over a ranked alphabet Σ

Set of all Σ-trees: $\mathtt{T}_\Sigma$

Example $\Sigma_0 = \{a, b\}$, $\Sigma_2 = \{f, g\}$:



$$= f\Big(g(a, b), b\Big).$$

Contexts: terms over $\Sigma \cup \{\xi\}$ in which exactly one leaf is ξ.

Set of all Σ-contexts: $\mathtt{C}_\Sigma$ (Special trees)

The Σ-term algebra $\mathcal{T}_\Sigma = (\mathtt{T}_\Sigma, \Sigma)$:

– $c^{\mathcal{T}_\Sigma} = c$

– $f^{\mathcal{T}_\Sigma}(t_1, \cdots, t_m) = f(t_1, \cdots, t_m)$

Congruences of a tree language $T \subseteq \mathtt{T}_\Sigma$:

(1) for trees, $t \sim^T t'$ iff

$$p[t] \in T \leftrightarrow p[t'] \in T$$

for every context p

Syntactic algebra of $T \; = \mathtt{T}_\Sigma / \sim^T$

▶ String case: Nerode Congruence, Minimal Automata.

(2) for contexts, $p \approx^T p'$ iff

$$q[p[t]] \in T \leftrightarrow q[p'[t]] \in T$$

for all trees t, contexts q

Syntactic monoid of $T \; = \mathtt{C}_\Sigma / \approx^T$

▶ String case: Myhill/Syntactic Congruence, Syntactic Monoid/Semigroup.

Binary A-trees and A-contexts

$\Sigma_0^A = \{c_a \mid a \in A\}$ $\Sigma_2^A = \{f_a \mid a \in A\}$

$T_A =$ set of A-trees

- $c_a \in T_A$ for every $a \in A$

- $f_a(t_1, t_2) \in T_A$ if $a \in A$ and $t_1, t_2 \in T_A$

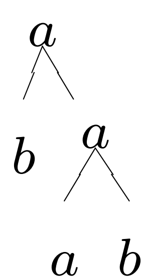
$C_A =$ set of A-contexts:

[• $\xi \in C_A$]

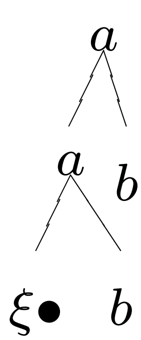
- $f_a(\xi, t), f_a(t, \xi) \in C_A$ if $a \in A$ and $t \in T_A$

- $f_a(p, t), f_a(t, p) \in C_A$ if $a \in A$, $t \in T_A$ and $p \in C_A$

Example $A = \{a, b\}$



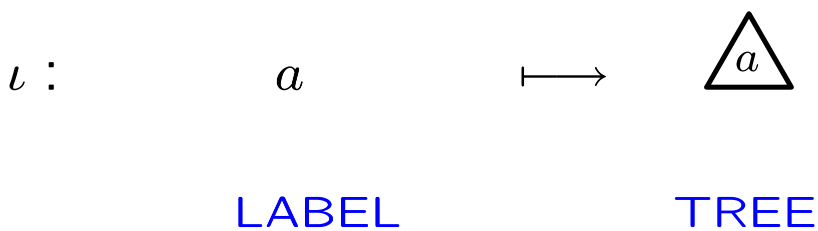
$$= f_a\Big(c_b, f_a(c_a, c_b)\Big) \in T_A,$$

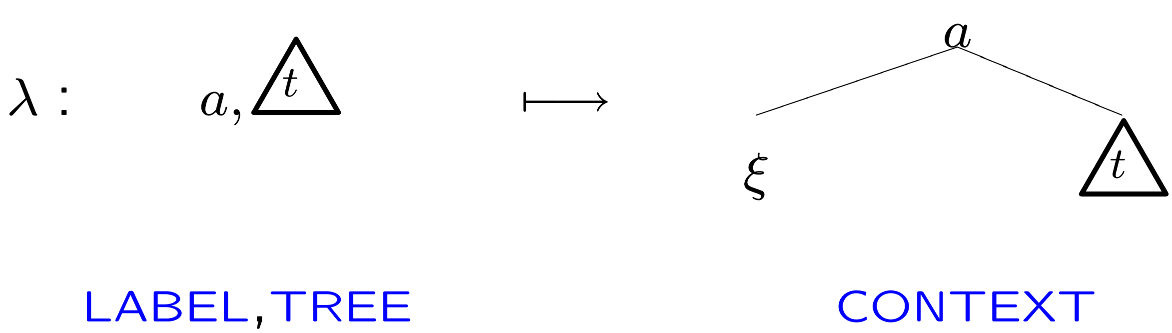


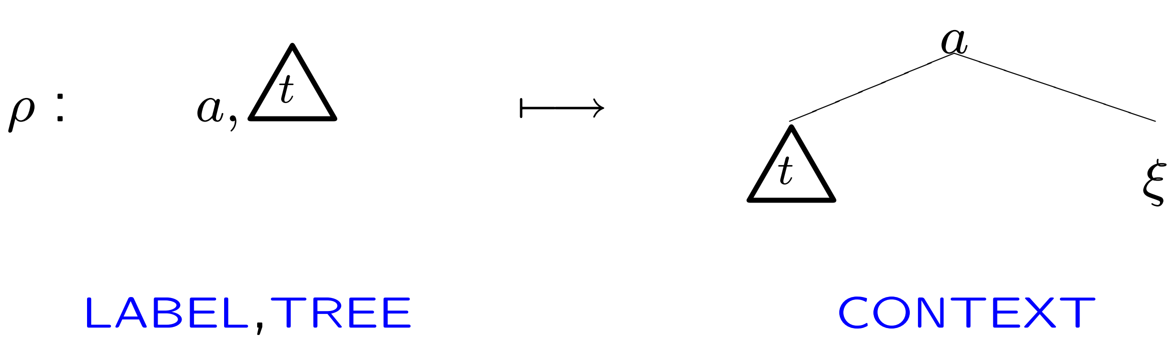
$$= f_a\Big(f_a(\xi, c_b), c_b\Big) \in C_A.$$

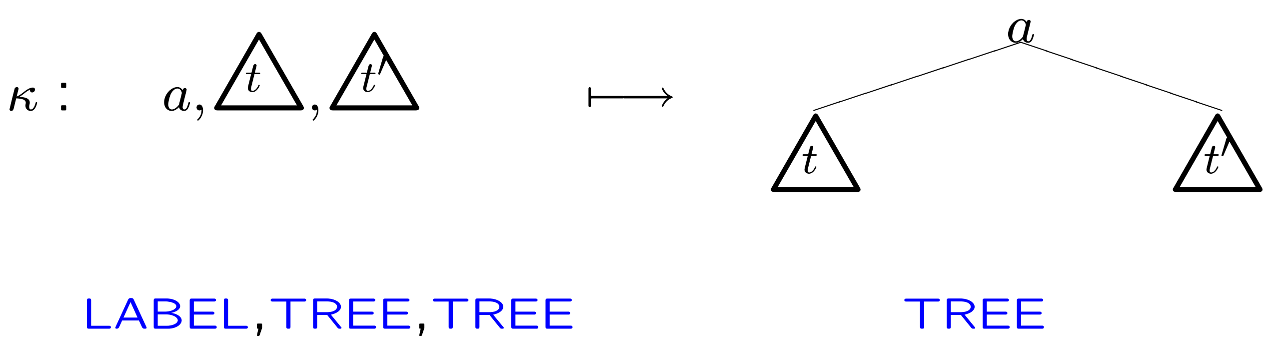
Signature of tree algebras $\Gamma = \{\iota, \kappa, \lambda, \rho, \eta, \sigma\}$

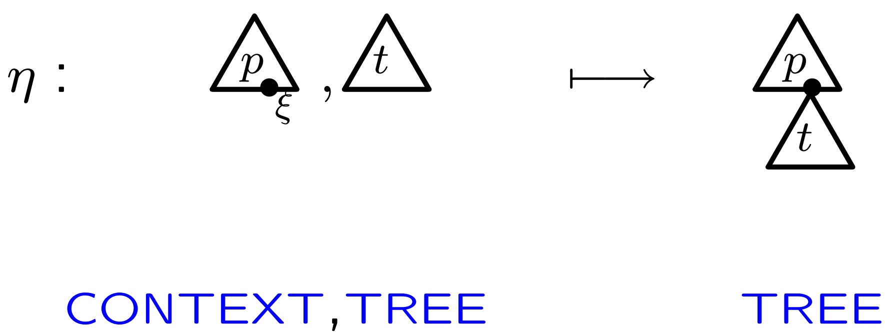
3 sorts: LABEL, TREE, CONTEXT

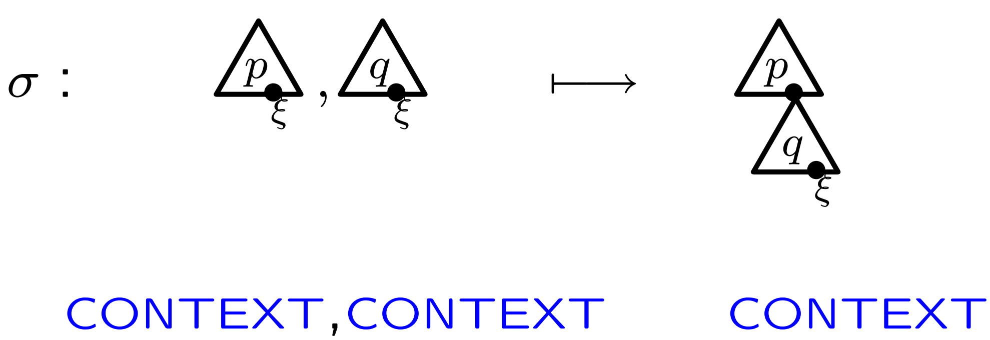







κ :
a,
t
t′
a
t
t′
LABEL,TREE,TREE
TREE


η :
p
ξ
t
p
t
CONTEXT,TREE
TREE


σ :
p
ξ
q
ξ
p
q
ξ
CONTEXT,CONTEXT
CONTEXT

Γ-Algebra of A-trees and A-contexts

$\mathcal{T}_A = \langle A, T_A, C_A, \Gamma \rangle$
labels,trees,contexts

(Wilke's functions)

- $\iota^{\mathsf{A}} : A \to T_{\mathsf{A}}$ $\qquad \iota^{\mathsf{A}}(a) = c_a$
- $\lambda^{\mathsf{A}} : A \times T_{\mathsf{A}} \to C_{\mathsf{A}}$ $\qquad \lambda^{\mathsf{A}}(a,t) = f_a(\xi, t)$
- $\rho^{\mathsf{A}} : A \times T_{\mathsf{A}} \to C_{\mathsf{A}}$ $\qquad \rho^{\mathsf{A}}(a,t) = f_a(t, \xi)$
- $\kappa^{\mathsf{A}} : A \times T_{\mathsf{A}}^2 \to T_{\mathsf{A}}$ $\qquad \kappa^{\mathsf{A}}(a, t_1, t_2) = f_a(t_1, t_2)$
- $\eta^{\mathsf{A}} : C_{\mathsf{A}} \times T_{\mathsf{A}} \to T_{\mathsf{A}}$ $\qquad \eta^{\mathsf{A}}(p,t) = p[t]$
- $\sigma^{\mathsf{A}} : C_{\mathsf{A}}^2 \to C_{\mathsf{A}}$ $\qquad \sigma^{\mathsf{A}}(p_1, p_2) = p_1[p_2]$

Tree-algebraic functions ($A^n \times T_A^m \times C_A^k \to A/T_A/C_A$):
generated by Wilke's + projection + constant functions.

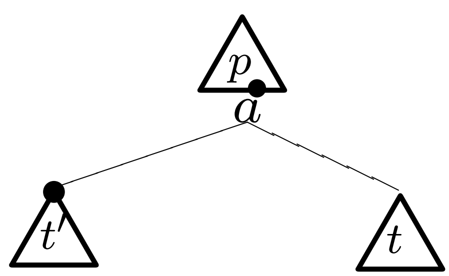
[[and their characterizations ?]]

Tree Algebra= a Γ-algebra satisfying Wilke's axioms:

- $\sigma(\sigma(p,q),r) = \sigma(p,\sigma(q,r))$ $p \circ (q \circ r) = (p \circ q) \circ r$
- $\eta(\sigma(p,q),t) = \eta(p,\eta(q,t))$ $(p \circ q)[t] = p[q[t]]$
- $\eta(\lambda(a,t),t')) = \kappa(a,t',t)$
- $\eta(\rho(a,t),t')) = \kappa(a,t,t')$

$\mathcal{T}_A$ is a tree algebra: all Wilkes' identities hold in A-trees and A-contexts.

Example: $\eta(\sigma(p,\lambda(a,t)),t') = \eta(p,\kappa(a,t',t))$



is derivable from the second and the third axioms by $q = \lambda(a,t)$.

Theorem This axiom system is sound and complete: every identity true in $\mathcal{T}_A$ is provable in the system, and vice versa.

Syntactic tree algebra congruence relations of $L \subseteq T_A$:

- $a \approx_{\mathsf{A}}^{L} a' \equiv \forall p \in C_A \forall t, t' \in T_A$
$$\Big(p[c_a] \in L \longleftrightarrow p[c_{a'}] \in L\Big) \ \&$$
$$\Big(p[f_a(t,t')] \in L \longleftrightarrow p[f_{a'}(t,t')] \in L\Big)$$

- $t \approx_{\mathsf{T}}^{L} t' \equiv \forall p \in C_A \Big(p[t] \in L \longleftrightarrow p[t'] \in L\Big)$

- $p \approx_{\mathsf{C}}^{L} p' \equiv \forall q \in C_A \forall t \in T_A$
$$\Big(q[p[t]] \in L \longleftrightarrow q[p'[t]] \in L\Big)$$

Definition $F : A^n \times T_{\mathsf{A}}^m \times C_{\mathsf{A}}^k \to A/T_{\mathsf{A}}/C_{\mathsf{A}}$

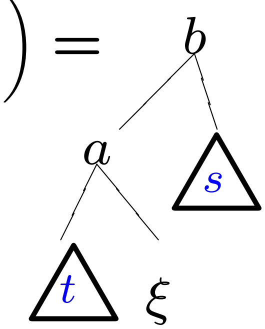
is *[syntactic] congruence preserving* if

$\forall L \subseteq T_{\mathsf{A}},\ a_j \approx_{\mathsf{A}}^{L} a'_j,\ t_j \approx_{\mathsf{T}}^{L} t'_j,\ p_j \approx_{\mathsf{C}}^{L} p'_j$, implies

$$F(a_1, \cdots, a_n, t_1, \cdots, t_m, p_1, \cdots, p_k) \approx_{\mathsf{A/T/C}}^{L}$$
$$\approx F(a'_1, \cdots, a'_n, t'_1, \cdots, t'_m, p'_1, \cdots, p'_k).$$

Example $A = \{a, b\}$.

$F : T_A^2 \to C_A$, $F(t, s) = f_b\Big(f_a(\xi, t), s\Big) =$



is congruence preserving. Indeed

$F(t, s) = \sigma^A\Big(\lambda^A(b, s), \rho^A(a, t)\Big)$, and

Lemma Tree-algebraic functions are congruence preserving.

Example $F : T_A \times A \to T_A$ defined by

"$F(t, x) =$ put x in the left-most leaf of t"

does *not* preserve the syntactic congruence of $L = \{f_a(c_b, c_b)\}$: $f_a(c_a, c_b) \approx^L f_b(c_a, c_b)$, but

$$F\Big(f_a(c_a, c_b), b\Big) = f_a(c_b, c_b) \not\approx^L f_b(c_b, c_b) = F\Big(f_b(c_a, c_b), b\Big).$$

(Main) **Theorem** For alphabet $|A| \geq 7$, every congruence-preserving function is tree-algebraic.

Not true for $|A| = 2$

Open for $|A| = 1$ and $3 \leq |A| \leq 6$ (?)

Homogeneous Version (for term algebras): For signature Σ, if $|\Sigma_0| \geq 7$, then every congruence preserving $F : (T_\Sigma)^n \to T_\Sigma$ is a term function.

Not true for $|\Sigma_0| = 1$

Open for $2 \leq |\Sigma_0| \leq 6$ (?)

Finite algebras: called hemi-primal.