Outline of PhD Thesis

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1 Notation and Basic Definitions

The language of arithmetical theories considered here is $\mathcal{L} = \langle +, \times, \leq, 0, 1\rangle$ in which the symbols are interpreted as usual in elementary mathematics. Robinson's arithmetic is denoted by Q; it is a finitely axiomatized basic theory of the function and predicate symbols in $\mathcal{L}$. Peano's arithmetic PA is the first-order theory that extends Q by the induction schema for any $\mathcal{L}$-formula $\varphi(x)$: $\varphi(0) \,\&\, \forall x(\varphi(x) \rightarrow \varphi(x+1)) \rightarrow \forall x \varphi(x)$. Fragments of PA are extensions of Q with the induction schema restricted to a class of formulas. The most studied hierarchy of formulas is defined as follows: let Δ_0 be the class of bounded formulas. A formula is called bounded if its every quantifier is bounded, i.e., is either of the form $\forall x \leq t(\ldots)$ or $\exists\, x \leq t(\ldots)$ where t is a term; they are read as $\forall x(x \leq t \rightarrow \ldots)$ and $\exists x(x \leq t \wedge \ldots)$ respectively. It is easy to see that bounded formulas are decidable. The theory $\mathsf{I\Delta_0}$, also called bounded arithmetic, is axiomatized by Q plus the induction schema for bounded formulas.

The next level in the hierarchy are the classes of Σ_1 and Π_1 formulas which constitute bounded formulas prefixed with, respectively, a block of existential, and universal quantifiers. So, for example the formula $\exists\, x \forall y \leq x(y \neq x \wedge \exists\, z \leq x[y \times z = x] \rightarrow y = 2)$ is a Σ_1-formula, and its negation $\forall x \exists y \leq x(y \neq x \wedge \exists\, z \leq x[y \times z = x] \wedge y \neq 2)$ is a Π_1-formula. We note that Σ_1-definable properties are exactly the computationally verifiable ones, and Π_1-definable properties are exactly the computationally refutable ones. The classes Σ_m and Π_m are defined inductively: Σ_{n+1}-formulas are obtained from Π_n-formulas by putting a block of existential quantifiers behind them, and Π_{n+1}-formulas are Σ_n-formulas prefixed with a block of universal quantifiers. The theory $\mathsf{I\Sigma_n}$ is the extension of Q by the induction schema for Σ_n-formulas. Note that $\mathsf{PA} = \bigcup_{n \geq 0} \mathsf{I\Sigma_n}$.

The exponentiation function exp is defined by $\exp(x) = 2^x$; the formula Exp expresses its totality $(\forall x \exists y[y = \exp(x)])$. The inverse of exp is log. Let us recall that Exp is not provable in $\mathsf{I\Delta_0}$; and sub-theories of $\mathsf{I\Delta_0} + \mathsf{Exp}$ are called weak arithmetics. Between $\mathsf{I\Delta_0}$ and $\mathsf{I\Delta_0} + \mathsf{Exp}$ another hierarchy of theories is considered in the literature, which has close connections with computational complexity. Let $\omega_1(x) = x^{\log x}$; note that it dominates all the polynomials, and in turn all the $\mathsf{I\Delta_0}$-provably total functions are dominated by polynomials. Let $\omega_{n+1} = \exp(\omega_n(\log x))$ be defined inductively, and let Ω_m express the totality of ω_m. We have $\mathsf{I\Delta_0} + \mathsf{\Omega_n} \subseteq \mathsf{I\Delta_0} + \mathsf{\Omega_{n+1}}$ for every $n \geq 1$. Finally, we recall that the super-exponential function is defined by $\text{Superexp}(x) = 2^x_x$, applying the exp function x times on x; $2^x_0 = x$ and $2^x_{n+1} = \exp(2^x_n)$.

2 Abstract of the Thesis

By Gödel's celebrated incompleteness theorems, **truth** is not conservative over **provability** in theories that contain sufficiently strong fragments of arithmetic. In other words, for any give reasonable arithmetical theory T, there exists a true arithmetical sentence G_T which is not provable in T. Moreover, this formula G_T can be chosen to be a Π_1-formula; thus **truth** is not even Π_1-conservative over **provability** in general arithmetics. Needless to say, this G_T may be provable in a stronger theory (than T). Thus, Π_1-conservativity of a theory over its sufficiently strong sub-theories is an interesting, and often difficult, question in mathematical logic. As a technical example, we can mention that the hierarchy $\{\mathsf{I\Sigma_n}\}_n$ of PA is Π_1-separable; that is to say there are Π_1-sentences A_n such that $\mathsf{I\Sigma_{n+1}} \vdash \mathrm{A}_n$ but $\mathsf{I\Sigma_n} \nvdash \mathrm{A}_n$. Another example is the important open problem of the Π_1-conservativity of the fragments of bounded arithmetic: is $\mathsf{I\Delta_0} + \Omega_{\mathsf{n+1}}$ conservative over $\mathsf{I\Delta_0} + \Omega_{\mathsf{n}}$ for Π_1-formulas?

A natural candidate for showing the Π_1-unconservativity of T over its sub-theory $S \subset T$ is the consistency statement of S, $\mathrm{Con}(S)$; i.e., one would wish to show that $T \vdash \mathrm{Con}(S)$, and then use Gödel's Second Incompleteness Theorem to infer that $S \nvdash \mathrm{Con}(S)$. Let us recall that for Zermelo-Frankel set theory ZFC we have $\mathsf{ZFC} \vdash \mathrm{Con}(\mathsf{PA})$, though $\mathsf{PA} \nvdash \mathrm{Con}(\mathsf{PA})$. Also, $\mathsf{I\Sigma_{n+1}} \vdash \mathrm{Con}(\mathsf{I\Sigma_n})$ and $\mathsf{I\Sigma_n} \nvdash \mathrm{Con}(\mathsf{I\Sigma_n})$ for all $n \geq 0$. For weak arithmetics this candidate does not work for Π_1-separating $\mathsf{I\Delta_0} + \mathsf{Exp}$ over $\mathsf{I\Delta_0}$: we have $\mathsf{I\Delta_0} + \mathsf{Exp} \nvdash \mathrm{Con}(\mathsf{I\Delta_0})$ (and also $\mathsf{I\Delta_0} \nvdash \mathrm{Con}(\mathsf{I\Delta_0})$). In 1981, J. Paris and A. Wilkie [8] proposed cut-free consistency statement for this purpose; though at that time it was not yet proved that $\mathsf{I\Delta_0} \nvdash \mathrm{CFCon}(\mathsf{I\Delta_0})$, where CFCon stands for cut-free consistency. However, it was known that $\mathsf{I\Delta_0} + \mathsf{Exp} \vdash \mathrm{CFCon}(\mathsf{I\Delta_0})$. We note that the cost of cut-elimination in proof theory is super-exponential, so in weak arithmetics cut-free provability is not equivalent to the usual (Hilbert style) provability. Indeed, in those theories CFCon is a stronger predicate than Con.

From another point of view, unprovability of cut-free consistency of weak arithmetics in themselves is an interesting generalization of Gödel's Second Incompleteness Theorem. The original proof of this theorem was presented for the usual (Hilbert) consistency predicate of theories that contain primitive recursive arithmetic (or contain $\mathsf{I\Sigma_1}$ if the language is $\mathcal{L}$). However, later on, the theorem was proved for all r.e. extensions of Q. So, one direction of generalizing the theorem was investigating the boundary cases: finding the weakest possible theories whose r.e. extensions cannot prove their own consistency. Another direction could be weakening the consistency predicate in addition to weakening the underlying theory. By 1985, another $(\mathsf{I\Delta_0} + \mathsf{Exp})$-provable Π_1-sentence that is unprovable in $\mathsf{I\Delta_0}$ had been found; however the question of the unprovability of cut-free consistency in theories weaker than $\mathsf{I\Delta_0} + \mathsf{Exp}$ remained open (see Pudlák's paper [9] where he mentions the problem explicitly for Herbrand consistency in 1985). Let us recall that Herbrand consistency of a theory is the propositional satisfiability of every (finite) set of its Skolem instances. Herbrand consistency of a theory T is denoted by $\mathrm{HCon}(T)$.

The first demonstration of the unprovability of cut-free consistency in weak arithmetics was made by Z. Adamowicz who proved in an unpublished manuscript in 1999 (later appeared as a technical report [1]) that the Tableau-consistency of $\mathsf{I\Delta_0} + \Omega_1$ is not provable in itself. Later on with P. Zbierski [2] she proved the theorem (Gödel's Second Incompleteness Theorem) for Herbrand consistency of $\mathsf{I\Delta_0} + \Omega_2$, and in [3] she gave a model theoretic proof of it. Extending these results for $\mathsf{I\Delta_0}$ was proposed to me as a topic for my PhD thesis by her.

By modifying the definition of Herbrand consistency, the model-theoretic proof of [3] was generalized to $\mathsf{I\Delta_0} + \Omega_1$ in Chapter 5 of the thesis (the result is not published anywhere else). Much later, in [6] L.A. Kołodziejczyk extended her proof to show the unprovability of $\mathrm{HCon}(\mathsf{I\Delta_0} + \Omega_2)$ in $\mathsf{I\Delta_0} + \bigcup_{\mathsf{n}} \Omega_{\mathsf{n}}$. He could generalize this result for $\mathrm{HCon}(\mathsf{I\Delta_0} + \Omega_1)$ with the condition that a function symbol for ω_1 is added to $\mathcal{L}$. The result in Chapter 5 is more general in a sense, as it does not require expanding the langauge.

Also, it was shown in Chapter 3 that $\mathsf{I}\Delta_0 \nvdash \mathrm{HCon}(\overline{\mathsf{I}\Delta_0})$ where the theory $\overline{\mathsf{I}\Delta_0}$ is axiomatized by a conventional axiomatization of $\mathsf{I}\Delta_0$ augmented with two $\mathsf{I}\Delta_0$-provable sentences. Chapter 4 proves $\mathsf{I}\Delta_0 + \Omega \nvdash \mathrm{HCon}(\mathsf{I}\Delta_0 + \Omega)$ where Ω expresses the totality of $\omega(x) = x^{\log\log x}$; here the conventional axiomatization of $\mathsf{I}\Delta_0$ is taken in the proof. The theory $\mathsf{I}\Delta_0 + \Omega$ lies between $\mathsf{I}\Delta_0$ and $\mathsf{I}\Delta_0 + \Omega_1$. In the end of [2] three questions were asked. In Chapter 5 the second question is answered negatively, by elaborating a concrete counter-example (introduced in Chapter 2).

Independently, D. Willard [12] introduced an $\mathsf{I}\Delta_0$-provable Π_1-formula V and showed that any theory whose axioms contains $\mathsf{Q} + V$ cannot prove its own Tableaux consistency. He also showed there that Tableaux consistency of $\mathsf{I}\Delta_0$ is not provable in itself; this proved the conjecture of J. Paris & A. Wilkie mentioned above.

The main result of Chapter 3 is published in [10], and a talk on these results was presented in the Logic Colloquium 2001 [11]. The thesis is referred to in e.g. [7],[4] (2003), [5] (2004), [13] (2005), and [14] (2006).

Extensions and New Results

Other engagements have prevented me from polishing and publishing the results in their full generality. Very recently (as of summer 2006) some new results which connect the results of the dissertation to (weak - subnormal) modal logics have been obtained. As an application, our old result $\mathsf{I}\Delta_0 \nvdash \mathrm{HCon}(\overline{\mathsf{I}\Delta_0})$ can be used to derive the more natural unprovability statement $\mathsf{I}\Delta_0 \nvdash \mathrm{HCon}(\mathsf{I}\Delta_0)$ where the conventional axiomatization of $\mathsf{I}\Delta_0$ has been considered.

Hopefully, the old results with these recently found extensions and generalizations will be submitted for publication in near future. Some talks based on these results, however, have been delivered in several seminars.

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