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Unprovability of Herbrand Consistency in Weak Arithmetics

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Godel's Second Incompleteness Theorem

$T \nvdash Con(T)$ for strong enough T's

- Diagonalization $T \vdash G(\bar{\theta}) \leftrightarrow \theta$

$\underline{1}$ $T \vdash \varphi \Longrightarrow T \vdash Pr_T(\bar{\varphi})$

$\underline{2}$ $T \vdash Pr_T(\bar{\varphi}) \longrightarrow Pr_T(\overline{Pr_T(\varphi)})$

$\underline{3}$ $T \vdash Pr_T(\bar{\varphi}) \wedge Pr_T(\overline{\varphi \to \psi}) \to Pr_T(\bar{\psi})$

(Consistent)

<u>Problem</u> $T \overset{?}{\vdash}$ Cut-Free $Con(T)$

for $T \subsetneq I\Delta_0 + Exp$

Formalized Σ_1-completeness Theorem

$$T \vdash \varphi \to Pr_T(\overline{\varphi}) \quad \text{for } \varphi \in \Sigma_1$$

A weak formalized Σ_1-completeness

- $T \vdash Con(T) \wedge \varphi \to Con_T(\overline{\varphi}) \quad \text{for } \varphi \in \Sigma_1$
- Diagonalization $(T \supseteq PA^-)$
- Σ_1-complete $(T \supseteq PA^-)$ + Consistent

$\therefore\ T \nvdash Con(T)$

<u>Proof</u> Let $Con_T(\overline{\neg\psi}) \equiv \psi \quad (\in \Pi_1)$

if $T \vdash Con(T)$ then $T \vdash \neg\psi \to Con_T(\overline{\neg\psi})$

$\to \psi$

So $T \vdash \psi$. Hence $\mathbb{N} \models \neg Con_T(\overline{\neg\psi})$

So $T \vdash \neg Con_T(\overline{\neg\psi})$

$T \vdash \neg\psi$

$\times$

Cut rule : $$\frac{\varphi \to \psi \quad \psi \to \eta}{\varphi \to \eta}$$

$$\text{Proof} \xrightarrow[\text{Sup Exp}]{\rightsquigarrow} \text{Cut Free Proof}$$

$$I\Delta_0 + Exp \nvdash Pr \equiv CFPr$$

<u>Pudlak</u> $I\Delta_0 + Exp \nvdash CFCon(I\Delta_0 + Exp)$

<u>Adamowicz</u> $I\Delta_0 + \Omega_1 \nvdash HCon(I\Delta_0 + \Omega_1)$

$I\Delta_0 + (\Omega_m)$ $(+\Omega_m)$

<u>Willard</u> $Q + V \nvdash$ Tableaux $Con(Q + V)$

V Π_1-sentence

Here: for $I\Delta_0$

Herbrad Consistency

PNF $\forall x_1 \exists y_1 \forall x_2 \exists y_2 \cdots \forall x_m \exists y_m \, A(x_1, y_1, \ldots, x_m, y_m)$

Skolemization $\forall x_1 \cdots \forall x_m \, A(x_1, f_1(x_1), \ldots, x_m, f_m(x_1, \ldots, x_m))$

Skolem instance $A(t_1, f_1(t_1), \ldots, t_m, f_m(t_1, \ldots, t_m))$

Herbrand's Theorem

A theory is consistent iff every finite set of its Skolem instances is propositionally satisfiable.

Evaluation p: a set of atomic formulae $\to \{0,1\}$

- $p[a=b]=1 \Rightarrow p[\varphi(a)]=p[\varphi(b)]$ atomic φ
- $p[a=a]=1$

T-evaluation satisfies all the available Skolem instances of T.

$\rightsquigarrow$ Evaluations on a set of terms

$\Lambda =$ a set of terms

evaluation: $\begin{array}{c}\text{atomic formulae with} \\ \text{terms from } \Lambda\end{array} \longrightarrow \{0,1\}$

Herbrand Consistency (of T):

for any set of terms, there is an T-evaluation on it.

Language of Arithmetic

$\langle 0, \underbrace{+, \cdot, \leq, S}_{\text{predicate}} \rangle$

$x_1 + x_2 = x_3$

$x_1 \cdot x_2 = x_3$

$x_1 \leq x_2$

$x_2 = S(x_1)$

$x_1 = x_2$

There are $2|\Lambda|^3 + 3|\Lambda|^2$ atomic formulae constructed from the elements of Λ ($|\Lambda| = \mathrm{Card}(\Lambda)$)

So, there are $2^{2|\Lambda|^3+3|\Lambda|^2}$ evaluations on Λ.

<u>admissible</u> set of terms (Λ):

all (the intuitionally $2^{2|\Lambda|^3+3|\Lambda|^2}$ possible) evaluations are available

(modified) <u>Herbrand Consistency</u> of T:

for any admissible set of terms, there is an T-evaluation on it.

$$\mathrm{HCon}(T) \equiv \forall x \underbrace{\chi(x)}_{\Delta_0}$$

$$\mathrm{HCon}^*(T) \equiv \forall x \in \log^2 \chi(x)$$

$$x \in \log^2 \equiv 2^{2^x} \text{ exists}$$

$$I\Delta_0 \vdash HCon(\overline{\overline{I\Delta_0}}) \wedge \overline{\exists x \in \log^2 \theta(x)} \rightarrow$$
$$HCon^*(\overline{\overline{I\Delta_0}} + \overline{\exists x \in \log^2 \theta(x)})$$

for $\theta \in \Delta_0$.

$\overline{\overline{I\Delta_0}}$ = a certain (unusual) axiomatization of $I\Delta_0$

$\rightsquigarrow$ Diagonalization & Σ_1-completeness in $I\Delta_0$:

$I\Delta_0 \nvdash HCon(\overline{\overline{I\Delta_0}})$:

__Proof__ Take $\psi \equiv HCon^*(\overline{\overline{I\Delta_0}} + \overline{\neg\psi})$, $(\in \Pi_1^*)$

if $I\Delta_0 \vdash HCon(\overline{\overline{I\Delta_0}})$ then

$$I\Delta_0 \vdash \neg\psi \rightarrow HCon^*(\overline{\overline{I\Delta_0}} + \overline{\neg\psi})$$
$$\rightarrow \psi$$

so $I\Delta_0 \vdash \psi$ then $\mathbb{N} \models \neg HCon^*(\overline{\overline{I\Delta_0}} + \overline{\neg\psi})$

so $I\Delta_0 \vdash \neg HCon^*(\overline{\overline{I\Delta_0}} + \overline{\neg\psi})$

or $I\Delta_0 \vdash \neg\psi$ $\because$

$T \overset{?}{\vdash} HCon(T)$ if $T \subsetneq I\Delta_0$ (and $T \supseteq PA^-$)