●Title

A model-theoretic proof of an incompleteness theorem

Saeed Salehi

Turku Center for Computer Science
Lemminkäisenkatu 14 A
FIN-20520 Turku

saeed@cs.utu.fi

Abstract

The fragment of Peano Arithmetic with bounded induction is denoted by $I\Delta_0$. The functions ω_m, $m \geq 0$, are defined by $\omega_0(x) = x^2$, and $\omega_{m+1}(x) = 2^{\omega_m(\log x)}$, and the axiom Ω_m expresses the totality of ω_m. For a theory T, Hcon(T) denotes the Herbrand Consistency of T, see [1,2,3].

For theories $T \supseteq I\Delta_0$, the question $T \vdash^{?}$ Hcon(T) had been of interest for some time, see [1,2,3,4,5]. Here, a model-theoretic proof of $I\Delta_0 + \Omega_m \nvdash$ Hcon($I\Delta_0 + \Omega_m$) for $m \geq 2$ presented in [1] is modified for $I\Delta_0 + \Omega_1$, so we get a model-theoretic proof of $I\Delta_0 + \Omega_1 \nvdash$ Hcon($I\Delta_0 + \Omega_1$).

●<u>Earlier results</u>

The cut $\log^m$ consists of the elements x whose m-th exponential power, $\exp^m(x)$, exists.

Theorem 1.1 of [1] states that for any natural n, m, there is a bounded formula $\theta(x)$ such that

(**A**) $I\Delta_0 + \Omega_n + \exists x \in \log^m \theta(x)$ is consistent, while $I\Delta_0 + \Omega_n + \exists x \in \log^{m+1} \theta(x)$ is not.

A formula Hcon(T) is introduced in [1] that expresses the Herbrand Consistency of a theory T, and satisfies the following for $m \geq 2$:

(**B**) For *any* bounded formula $\theta(x)$, if
$I\Delta_0 + \Omega_m + \exists x \in \log^{m+1} \theta(x) + \mathrm{Hcon}(I\Delta_0 + \Omega_m)$
is consistent, then so is
$I\Delta_0 + \Omega_m + \exists x \in \log^{m+2} \theta(x)$.

• Earlier results

The proof of $I\Delta_0+\Omega_m \nvdash \mathrm{Hcon}(I\Delta_0+\Omega_m)$ for $m \geq 2$, in [1], is as follows:

By (**A**), there is a bounded $\theta(x)$ such that

$I\Delta_0 + \Omega_m + \exists x \in \log^{m+1}\theta(x)$ is consistent, while $I\Delta_0 + \Omega_m + \exists x \in \log^{m+2}\theta(x)$ is not.

Then
$I\Delta_0 + \Omega_m + \exists x \in \log^{m+1}\theta(x) + \mathrm{Hcon}(I\Delta_0 + \Omega_m)$
is inconsistent, since otherwise by (**B**),
$I\Delta_0 + \Omega_m + \exists x \in \log^{m+2}\theta(x)$
should have been consistent, contradiction.

By consistency of $I\Delta_0+\Omega_m+\exists x \in \log^{m+1}\theta(x)$, it follows that
$I\Delta_0 + \Omega_m \nvdash \mathrm{Hcon}(I\Delta_0 + \Omega_m)$. Q.E.D

●New results

A new formula $\mathsf{Hcon}(T)$ expressing the Herbrand Consistency of a theory T can be constructed in such a way that (**B**) holds for the theory $I\Delta_0 + \Omega_1$.

That is, for *any* bounded formula $\theta(x)$, if

$I\Delta_0 + \Omega_1 + \exists x \in \log^2 \theta(x) + \mathsf{Hcon}(I\Delta_0 + \Omega_1)$

is consistent, then so is

$I\Delta_0 + \Omega_1 + \exists x \in \log^3 \theta(x)$.

This supplies the necessary tools for a model-theoretic proof of $I\Delta_0+\Omega_1 \nvdash \mathsf{Hcon}(I\Delta_0+\Omega_1)$.

●Further works

Moreover, a formalized version of (**B**) can be proved (cf. [2,4]). Let the formula $\mathsf{Hcon}^{\log^2}(T)$ denote the Herbrand Consistency of a theory T relativized to the cut $\log^2$. The following implication is provable in $I\Delta_0+\Omega_1$ for every bounded $\theta(x)$:

$$\Big(\mathsf{Hcon}(I\Delta_0+\Omega_1)\wedge\exists x\in\log^2\theta(x)\Big)\longrightarrow$$
$$\mathsf{Hcon}^{\log^2}(I\Delta_0+\Omega_1+\exists x\in\log^3\theta(x)).$$

The reader is invited to consult [4] for the historical remarks and technical details.

• <u>References</u>

[1] Adamowicz Z., *Herbrand consistency and bounded arithmetic*, Fund. Math. **171** (2002), 279–292, MR1898644.

[2] Adamowicz Z. & Zbierski P., *On Herbrand consistency in weak arithmetic*, Arch. Math. Logic **40** (2001), 399–413, MR185489.

[3] Hájek P. & Pudlák P., *Metamathematics of first-order arithmetic*, Springer-Verlag, 1998.

[4] Salehi S., *Herbrand consistency in arithmetics with bounded induction*, Ph.D. thesis, Inst. Math. Polish Acad. Sci. (2002), 78 pp. `http://staff.cs.utu.fi/staff/saeed/thesis.pdf`

[5] Willard Dan E., *How to extend the semantic tableaux and cut-free versions of the second incompleteness theorem almost to Robinson's arithmetic Q*, J. Symbolic Logic **67** (2002), 465–496, MR1889562.