Review Notes for EC750

Little Tiger

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Preface

This is note for Macro.

Introduction

Notes for Macro. It would include EC750 and EC751 which is core for comprehensive examination of Economics Ph.D in Boston College.

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Appendix I EC740

A.1 Some useful Facts

1. WE$\rightarrow$ PE, IR
2. Core $\rightarrow$ PE, IR
3. PE + IR = Contract curve
4. WE $\subseteq$ Core $\subseteq$ Contract Curve
5. First price auction $\rightarrow$ not DS, not PE too (note the seller!)
6. Second price auction $\rightarrow$ DS but not PE
7. Monotone of second price auction $\rightarrow$ not DS, not PE
8. Third price,...,last price auction $\rightarrow$ not DS, PE
9. Average price auction $\rightarrow$ not DS, PE
10. In n-person economy,
    WE $\rightarrow$ all excess demand are zero
    PE $\rightarrow$ there are no profitable bilateral trade between **each** person.
    IR $\rightarrow$ better than endowment
    Contract curve $\rightarrow$ PE + IR
    Core $\rightarrow$ there are no profitable multilateral trade between persons.
11. Short-cut Demand function:
    (a) Cobb-Doglas utility : $U = \prod_{i=1}^{n} x_i^{\alpha_i}$

$$x_i = \frac{\alpha_i I}{p_i \sum_{i=1}^{n} \alpha_i} = \frac{\alpha_i \sum_{i=1}^{n} p_i \omega_i}{p_i \sum_{i=1}^{n} \alpha_i}$$

(b) Min utility : $U = \min \{\alpha_i x_i\}_{i=1}^n$

$$x_i = \frac{\alpha_i I}{\sum_{i=1}^n p_i \alpha_i} = \frac{\alpha_i \sum_{i=1}^n p_i \omega_i}{\sum_{i=1}^n p_i \alpha_i}$$

special case: $\alpha_i = 1$ for all i,

$$x_i = \frac{I}{\sum_{i=1}^n p_i} = \frac{\sum_{i=1}^n p_i}{\sum_{i=1}^n p_i \omega_i}$$

(c) Linear utility: $U = \sum_{i=1}^n \alpha_i x_i$ or Quasi-convex utility: $U = \sum_{i=1}^n \alpha_i x_i^k$, for $k > 1$,

$$x_i = \begin{cases} I/p_i & \text{if } p_i/\alpha_i = \min \{p_i/\alpha_i\}_{i=1}^n \\ 0 & \text{if } p_i/\alpha_i > \min \{p_i/\alpha_i\}_{i=1}^n \end{cases}$$

12. In two person economy, if $u_1 = u_2$ and $\omega_1 = \omega_2$, and u_i are quasi-concave and homothetic, then set of efficient allocation is the main diagonal.

13. If initial endowment ω is efficient and preference $\succeq$ is strictly quasi-concave, then core=WE.

A.2 Frequently Asked Definition in Final Exam

1. **First price auction (99 Fall) [Description P.28]**

   Each person writes down the price for which he is willing to buy the good. The good is sold to the highest bidder for the price he offered.

2. **Second price auction (00 Fall, 01 Fall) [Description P.28]**

   Each person writes down the price for which he is willing to buy the good. The good is sold to the highest bidder for the price of second highest bid.

3. **First Theorem of Welfare Economics (00 Fall, 06 Fall) [Thm. 3]**

   If (x, p) is WE, then it is PE.

4. **Second Theorem of Welfare Economics (01 Fall) [Thm. 5]**

   Assume monotonicity, continuity and quasi concavity. If x^* is PE, then there is an initial allocation ω and a price vectoer p^* such that (p^*, x^*) is WE for ω.

5. **Walras Law (00 Fall, 06 Fall)**

   Aggregate excess demand is denoted by $z(p)$, where

   $$z(p) = \sum_i x_i (p, p \cdot \omega_i) - \sum_i \omega_i, \quad z(p) \in \mathbb{R}^k$$

   Walras Law states

   $$\forall p, p \cdot z(p) = 0$$

6. **Dominant strategy mechanism (99 Fall, 00 Fall) [Def.33 + Def. 34]**

   An allocation mechansim is a function

   $$f(u_{11}, \ldots, u_n; \omega_1, \ldots, \omega_i) \mapsto (x_1, \ldots, x_n)$$

   A mechanism is said to be dominant strategy (DS) if $\forall i$ and $\forall u$,

   $$u_i(f_i(u_1, \ldots, u_i, \ldots, u_n)) \geq u_i(f_i(u_1, \ldots, u, \ldots, u_n))$$

   is true for all possible utility profile.

7. **Equal treatment in the core theorem (99 Fall, 00 Fall) [Def. 32 + Thm. 7]**

   Two agents are of the same type if they have the same initial endowment and the same preferences. Assume strong monotonicity, strict quasi concavity and continuity. Let ε be an r-replica of ε'. If x is in the core of ε, then two agents of the same type receive the same outcome.

8. **Gross substitutes (99 Fall) [Def. 25]**

   The goods j and ℓ, $j \neq \ell$, are gross substitutes if at all p,

   $$\frac{\partial z^j(p)}{\partial p^\ell} > 0$$

9. **Core (01 Fall, 02 Fall) [Def. 29 + Def. 30]**

   A set $S \subseteq \{1, \ldots, n\}$ is said to improve upon x through x' if

   1. $\sum_{i \in S} x'_i = \sum_{i \in S} \omega_i$
   2. $\forall i \in S, x'_i \succeq_i x_i$ and $\exists i \in S, x'_i \succ_i x_i$

   An allocation x is in the core if it cannot be imporived upon by any coalition S

10. **Free Good (01 Fall) [Prop. 28 or negation of Ass. p.20]**

    Good j is free good if p^* is WE and $z^j < 0$, then $p^{j*} = 0$.

11. **Pareto Efficient (03 Fall) [Def. 16 + Def. 19]**

    An allocation x is Pareto efficient (PE) if

    1. It is feasible, that is, for $j = 1, \ldots, k$, $\sum_{i=1}^n x_i^j \leq \sum_{i=1}^n \omega_i^j$
    2. There is no other feasible allocation x' such that $\forall i, x'_i \succeq_i x_i$ and $\exists i, x'_i \succ_i x_i$

12. **Walrasian Equilibrium (02 Fall, 03 Fall) [Def. 21]**

    Walrasian equilibrium (WE) is a pair (p^*, x^*) such that

    1. x^* is feasible
    2. x_i^* solves $\max u_i(x_i)$ s.t. $p^* \cdot x_i = p^* \cdot \omega_i$

13. **Uniqueness of WE (02 Fall, 06 Fall) [Thm. 6]**

    If all goods are gross substitutes at all price vectors and there are no free goods, then there is no more than one price vector p^*.

A.3 Frequently Asked Truth/False in Final Exam

99 Fall

1. If all agents in the economy have the same utility function, then the core equals to the set of Walrasian equilibrium (WE) allocations.

2. Let S and T be two disjoint blocking coalitions for an allocation x^* (that is, $S \cap T = \varnothing$). Then $S \cup T$ is also a blocking coalition of x^*.

3. If all consumers have the same utility functions, then in (Walrasian) equilibrium, the prices of all goods are the same.

4. In an auction, giving the good to the highest bidder at the average bid offered is a dominant strategy mechanism.

00 Fall

1. In an n person economy, suppose consumers i and j have the same utility function u and suppose that $u(\omega_i) > u(\omega_j)$. Let x be a WE allocation. Then $u(x_i) > u(x_j)$.

2. In a two perons economy, $u_1 = u_2$. Then the contract curve (that is the set of efficient alloctaions) is the main diagonal of the Edgeworht box.

3. In an auction, selling the good to the second highest bidder at the third highest bid is not dominant strategy. If the statment is true (that is, if it is not dominant strategy), then what can you say about the players' bids? Will they be too high or too low?

4. Let S and T be two disjoint sets of agents (that is, $S \cap T = \varnothing$). If p^* is a WE price vector for the economy consisting of all agents of S, and if p^* is a WE price vector for the economy consisting of all agents of T, then it is also a WE price vector for the economy consisting of all agents of $S \cup T$.

01 Fall

1. The mechanism f that gives everything to person 1 is dominant strategy and efficient.

2. Let (p^*, x^*) be the WE of an n-person economy, where the initial resources are $(\omega_1, \dots, \omega_n)$. Then for every i and j, $\omega_i \succeq_i \omega_j$ implies $x_i^* \succeq_i x_j^*$.

3. If i and j have the same quasi-concave preferences and same endowments, and there is an allocation x in the core such that $x_i = x_j$. (Note: We do not assume that all types are of the same size.)

4. Assume strict quasi concavity. If the common price vector at all efficient allocations is the same, then WE is unique.

02 Fall

1. In the economy ε, there are n individuals of type 1, with the utility function $u_1\left(x^1, x^2\right) = \min\left\{x^1, x^2\right\}$ and the initial endowments $\omega_1 = (1, 0)$ and n individuals of type 2 with the utility funciton $u_2\left(x^1, x^2\right) = \min\left\{x^1, x^2\right\}$ and the initial endowment $\omega_2 = (0, 1)$. Then for every $\alpha \in [0, 1]$, the allocation that gives all individuals of type 1, (α, α) and all individuals of type 2, $(1 - \alpha, 1 - \alpha)$ is in the core of ε. (Important: You may use results proved in class.)

2. There are two individuals, and $\omega = (10, 10)$. The mechanism f is defined by

$$f(u_1, u_2) = \begin{cases} ((7,3),(3,7)) & \text{whenever } u_1(7,3) \geq u_1(5,5) \\ ((5,5),(5,5)) & \text{whenever } u_1(7,3) < u_1(5,5) \end{cases}$$

   Then f is a DS mechanism. (Remember: There are two individuals.)

3. If all individuals have the same (weakly) monotonic and continuous preferences, then the allocation that gives everyone ω/n is efficient.

4. WE $\subset$ Core (Proper subset)

03 Fall

1. We want to divide a (long...) submarine sandwich among n players. Assume all of them love this dreadful food, and would like to get as much as possible out of it. The rules of the game are these. We start moving a knife slowly along the sub. Whenever someone says "cut," we cut at that point and give him the cut slice. This person no longer participates, and we continue with the rest of the players, until another one wants us to cut, and so on.

   Assume that all ingredients are evenly spread along the sub, so the only concern each player has is how "long" is his slice. Is this a truth revealing mechanism in the sense that the person who cried "cut" first got what he believes to be 1/n of the sub, and all other players think he got no more than 1/n of it?

2. There are n consumers in the economy ε, $n > 2$. (p^*, x^*) is a WE, and it turns out that $x_1^* = \omega_1$. Is it true that $(p^*, (x_2^*, \ldots, x_n^*))$ is WE for the economy with $n - 1$ consumers $2, \ldots, n$?

3. Consider the function $G(z(p)) = \left(G^1\left(z^1(p)\right), \ldots, G^k\left(z^k(p)\right)\right)$, where $G^j(x) = x^j$ (that is, x to the power of j), $j = 1, \ldots, k$. (so $G : \mathbb{R}^k \to \mathbb{R}^k$, and $G^j : \mathbb{R} \to \mathbb{R}$, $j = 1, \ldots, k$). Does this function make senses as a price adjustment rule?

4. As a price taker with strictly quasi-concave utility function, are you better off facing a fully discriminating monopolist or a standard monopolist?

06 Fall

1. Second price auction is truth revealing but not necessarily efficient.

2. If all agents' preferences are strictly quasi concave, then a monopoly necessarily leads to an inefficient allocation.

3. Assume one price taker and one fully discriminating monopolist. Suppose the initial allocation is inefficient. Assuming strict quasi concavity and differentiability, the outcome of the interaction between them will not lead to a WE outcome, but the outcome will be in the core. (Discuss both statments.)

4. The set of PE allocations, the set of WE allocatinons and the core do not change if u_i, the utility of person i, is replaced with u_i^{2i+1} (u_i to the power of $2i+1$), $i = 1, \ldots, n$.

A.4 Solution to Hard problems in Past Paper

01 Fall Q4c

Demand function for

$$u_1\left(x^1, x^2\right) = \begin{cases} 2x^1 + x^2 & \text{if } x^1 \leq x^2 \\ x^1 + 2x^2 & \text{if } x^1 > x^2 \end{cases}$$

and $\omega_1 = (10, 0)$. Hence, we have

$$\begin{aligned} \max_{x^1, x^2} u_1\left(x^1, x^2\right) &= \begin{cases} 2x^1 + x^2 & \text{if } x^1 \leq x^2 \\ x^1 + 2x^2 & \text{if } x^1 > x^2 \end{cases} \\ s.t \quad p^1 x^1 + p^2 x^2 &\leq p^1 \omega^1 \end{aligned}$$

Normalize price vector by $p = p^1 / \left(p^1 + p^2\right)$ so that $p \in [0, 1]$ then budget constraint becomes

$$px^1 + (1 - p)\, x^2 \leq 10p$$

Now, suppose the case of $x^1 \leq x^2$, then the Lagrangian would be

$$L = 2x^1 + x^2 + \lambda\left[10p - px^1 - (1 - p)\, x^2\right] + \phi\left(x^2 - x^1\right) + \alpha x^1 + \beta x^2$$

with FOC being (note the budget constraint will be binding)

$$
\begin{aligned}
\frac{\partial L}{\partial x^1} &= 0 \Rightarrow 2 - \lambda p - \phi + \alpha = 0 \\
\frac{\partial L}{\partial x^2} &= 0 \Rightarrow 1 - \lambda\left(1 - p\right) + \phi + \beta = 0 \\
\frac{\partial L}{\partial \lambda} &= 0 \Rightarrow 10p - px^1 - \left(1 - p\right)x^2 = 0 \\
\frac{\partial L}{\partial \phi} &= 0 \Rightarrow x^2 - x^1 \geq 0 \\
\frac{\partial L}{\partial \alpha} &\geq 0 \Rightarrow x^1 \geq 0 \\
\frac{\partial L}{\partial \beta} &\geq 0 \Rightarrow x^2 \geq 0 \\
\phi &\geq 0; \quad \phi\left(x^2 - x^1\right) = 0 \\
\alpha &\geq 0; \quad \alpha x^1 = 0 \\
\beta &\geq 0; \quad \beta x^1 = 0
\end{aligned}
$$

Case 1: Suppose $x^1 \neq 0$ and $x^2 \neq 0$, so $\alpha = 0$ and $\beta = 0$.
From first two conditions, we have

$$\lambda = 3 \text{ and } \phi = 2 - 3p$$

and since $\phi \geq 0$, we have $p \leq 2/3$.
Note that when $p < 2/3$, then $\phi > 0$ and $x^2 = x^1$ and

$$x^1 = x^2 = 10p$$

and if $p = 2/3$, then

$$2x^1 + x^2 = 20$$

with constraint that $x^2 - x^1 \geq 0$, then

$$0 \leq x^1 \leq 20/3 \text{ (Rmk: } 20/3 \leq x^2 \leq 20)$$

Case 2: Suppose $x^1 = 0$, $x^2 \neq 0$, so that $\phi = 0$, $\beta = 0$ and $x^2 = 10p/\left(1 - p\right)$ and $p > 2/3$.
Case 3: Suppose $x^1 \neq 0$, $x^2 = 0$, contradicts $x^1 \leq x^2$.
Case 4: Suppose $x^1 \neq 0$, $x^2 \neq 0$, not optimal under linear utility.
Now consider the the case where the case of $x^1 > x^2$, then the Lagrangian would be

$$L = x^1 + 2x^2 + \lambda\left[10p - px^1 - \left(1 - p\right)x^2\right] + \phi\left(x^1 - x^2\right) + \alpha x^1 + \beta x^2$$

with FOC being (note the budget constraint will be binding)

$$\begin{aligned}
\frac{\partial L}{\partial x^1} &= 0 \Rightarrow 1 - \lambda p - \phi + \alpha = 0 \\
\frac{\partial L}{\partial x^2} &= 0 \Rightarrow 2 - \lambda (1-p) + \phi + \beta = 0 \\
\frac{\partial L}{\partial \lambda} &= 0 \Rightarrow 10p - px^1 - (1-p)\, x^2 = 0 \\
\frac{\partial L}{\partial \alpha} &\geq 0 \Rightarrow x^1 \geq 0 \\
\frac{\partial L}{\partial \beta} &\geq 0 \Rightarrow x^2 \geq 0 \\
\frac{\partial L}{\partial \phi} &= 0 \Rightarrow x^1 - x^2 \geq 0 \\
\phi &\geq 0; \phi \left(x^1 - x^2\right) = 0 \\
\alpha &\geq 0; \quad \alpha x^1 = 0 \\
\beta &\geq 0; \quad \beta x^1 = 0
\end{aligned}$$

Case 1: Suppose $x^1 \neq 0$ and $x^2 \neq 0$, so $\alpha = 0$ and $\beta = 0$.
From first two conditions, we have

$$\lambda = 3 \text{ and } \phi = 1 - 3p$$

and since $x^1 - x^2 \geq 0$, we have $p \leq 1/3$.
When $p < 1/3$, then $\phi > 0$ and $x^1 = x^2$. Contradicts the conditon that $x^1 > x^2$.
When $p = 1/3$, then $\phi = 0$ and

$$x^1 + 2x^2 = 10$$

with constraint that $x^1 - x^2 \geq 0$, then

$$0 \leq x^2 \leq 10/3 \text{ (Rmk: } 10/3 \leq x^1 \leq 10)$$

Case 2: Suppose $x^1 = 0$, $x^2 \neq 0$, contradicts $x^1 > x^2$.
Case 3: Suppose $x^1 \neq 0$, $x^2 = 0$, contradicts so that $\phi = 0$, $\alpha = 0$ and $x^1 = 10$, $p < 1/3$
Case 4: Suppose $x^1 \neq 0$, $x^2 \neq 0$, not optimal under linear utility.
Now, it becomes clear that the demand function would be

$$x = \begin{cases} (10, 0) & p \in [0, 1/3) \\ \left[(10,0), \left(\frac{10}{3}, \frac{10}{3}\right)\right] & p = 1/3 \\ (10p, 10p) & p \in (1/3, 2/3) \\ \left[(0,20), \left(\frac{20}{3}, \frac{20}{3}\right)\right] & p = 2/3 \\ \left(0, \frac{10p}{1-p}\right) & p \in (2/3, 1] \end{cases}$$

06 Fall Q1b

To determine it is PE or not, just try to exchange the allocation to see whether there is Pareto improvement.

1. Since person 1 and Person 2 has quasi-concave utility, they could benefit only by having more of one good. So, if there is non-zero term in both goods, it is not optimal.

2. Since person 3 and person 4 has linear utility, then it doesn't matter to manipulate their composition of goods.

3. All person have monotonic preferences, so if you take away one good, you must compensate enough.

Case 1: Yes. No profitable trade as Person 1 and Person 2 have nothing to offer for exchange and Person 3 and Person 4 could not have any profitable trade.
Case 2: No. Person 1 and Person 2 can trade, e.g. person 1 get all good 1 and person 2 get all good 2.
Case 3: Yes. No profitable trade. (Same reason as in Case 1)
Case 4: Yes. No profitable trade. (Same reason as in Case 1)
Case 5: Yes. No profitable trade. (Same reason as in Case 1)
Case 6: No. Person 1 and Person 3 can have good trade.
Case 7: Yes. Although Person 1 want to trade, person 3 and person 4 cannot trade with him.
Case 8: Yes. No profitable trade. (Same reason as in Case 1)

06 Fall Q1c

To find the core, we need to form an allocation which cannot be improved by any coalition:
Let the allocation be $((a,b),(c,d),(e,f),(g,h))$
One person coalition: $\{1\},\{2\},\{3\}$
equivalent to Individual Rationality

$$\begin{aligned}
\text{Person 1} &: \quad a^2+b^2 \geq 100 \\
\text{Person 2} &: \quad c^2+d^2 \geq 100 \\
\text{Person 3} &: \quad e+f \geq 10 \\
\text{Person 4} &: \quad g+h \geq 10
\end{aligned}$$

Four person coalition: $\{1,2,3,4\}$
Since the preference is strong monotonic, all resources should be used up.

$$\begin{aligned}
a+c+e+g &= 20 \\
b+d+f+h &= 20
\end{aligned}$$

This also requires Pareto Efficiency.

Two person coalition:
Since 3 and 4 have linear utility. There is no need to worry about them.
Since 1 and 2 have quai-convex utility, $\{1,2\}$ can block coalition when at the same time have non-zero in both goods.
$\{1,3\}$, $\{1,4\}$, $\{2,3\}$, $\{2,4\}$ could block the core only when 1 or 2 have non-zero in both goods but cannot have all quantity of either good.
However, by individual rationality, 1 or 2 cannot have all quantity in either goods. Therefore, we have

$$((a,b),(c,d)) = ((\alpha,0),(0,\beta)) \text{ or } ((0,\alpha),(\beta,0))$$

Three person coalition:
Since 3,4 cannot make good trade among themselves, so three person coalition is same as two person coalition.
Sum up:
Condition 1:

$$\begin{array}{ll} \text{Person 1:} & a^2+b^2 \geq 100 \\ \text{Person 2:} & c^2+d^2 \geq 100 \\ \text{Person 3:} & e+f \geq 10 \\ \text{Person 4:} & g+h \geq 10 \end{array}$$

Condition 2:

$$\begin{aligned} a+c+e+g &= 20 \\ b+d+f+h &= 20 \end{aligned}$$

Condition 3:

$$((a,b),(c,d)) = ((\alpha,0),(0,\beta)) \text{ or } ((0,\alpha),(\beta,0))$$

Combine conditions 1 and 3, we know that $\alpha = \beta = 10$. Then condition 2 becomes

$$\begin{aligned} e+g &= 10 \\ f+h &= 10 \end{aligned}$$

Hence the core would be

$$\begin{array}{cccc} (10,0) & (0,10) & (\alpha,10-\alpha) & (10-\alpha,\alpha) \\ (0,10) & (10,0) & (\alpha,10-\alpha) & (10-\alpha,\alpha) \end{array}$$

A.5 Solution to Written problems in Past Paper

02 Fall

1. **Why is the core?**

   Allocation is rearrangement of distribution of endowment. An allocation would be in core if there cannot be a single individual or any group of individual can have better outcome by rearranging their own endomwents.

2. **Why is it interesting?**

   The concept is interesting is that it does not require any force or contractual arrangement to make it stable. No institutional setup is needed in a core to ensure no one would have any incentive to break away from that. This stability concept embodies two basic desirable concept: Pareto efficiency and extended version individual rationality. Pareto efficiency means there is no other allocation can make some better off without making other worse off, and individual rationality means means each individual must be better off than being left alone. The generalized version is just extends to the case that any group of people should be better off in the core rather than themselves alone.

3. **How does it relate to WE?**

   With such simple structure, it is amazing that there does have certain kind of arrangement involving all individuals exists. WE is the core if we allow the assumptions for the existence of WE.

4. **What are the differences between the core and WE?**

   Since core has less restriction, it is reasonable to expect WE only of the cores. However, it is amazing that in large economies, core would be smaller and smaller and only core is WE. Core concept is based on coalition formation, with huge information requirement but not institutional restriction while WE is based on market mechanism, which has institution restriction but less information requirement as all individual would take price as given and allow to trade freely.

03 Fall

1. **What is definition of PE?**

   Pareto efficienct allocation means no other allocation can make some better off without making other worse off.

2. **Why is this concept interesting?**

   It is intuitively attractive because it is unambiguously better to have PE than not. For a social planner, it is reason not to make someone better off without making other worse off. Compare to other different measure, it is indisputable that PE should be an desirable criteria to judge the efficiency of an allocation criteria.

3. **How does it relate to prices?**

   PE has no direct relation to prices and PE can even exists without the existence of any explicit mention of price. However, relative price of two different goods, which represent the generally accept rate of exchange of two goods should equal to the internal rate of exchange of every single individuals. Otherwise, there is better way to allocate the resource within the concept of PE.

4. **How does it related to WE? Core?**

   For any WE or Core, it must be PE. It is because WE use price mechanism to establish allocation, and, as said above, the price should reflect preferences of individuals so it must be PE. For core, since it requires no individual or group individual can do better by themselves, it must be PE.

5. **What are the differences between PE, the core and WE?**

   PE is just efficiency concept without consideration of any initial endowment. Core is combination of generalized individual rationality and PE under concept of coaltion formation. WE is an allocation under price mechanism. To sum up, WE must be core and PE; Core must be PE and include WE; PE is basic requirement for core and WE.

Appendix B

Appendix II EC751

Afterword

Acknowledgements

Bibliography