Review Notes for EC750

Little Tiger

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Contents

Preface

This is note for Macro.

Introduction

Notes for Macro. It would include EC750 and EC751 which is core for comprehensive examination of Economics Ph.D in Boston College.

Part I

EC750 Macro

Chapter 1

Economic Growth

1.1 Introduction to Solow Model

`Lecture Note: #1,#2,#3`
`Textbook: Romer ch.1,3; Acemoglu, ch.1,2,3,4`

Neoclassical Production Function:

Assumptions:

1. Constant Return to Scale (CRS): $F(\alpha K, \alpha L) = \alpha F(K, L)$
    (a) By Euler's Theorem, this implies $F_K K + F_L L = F$
    (b) Under competitive market, factor payment exhausts all revenue as $F_K = R$ and $F_L = W$, $RK + WL = F(K, L) = Y$.
    (c) Intensive form: $y \equiv Y/L = F(K/L, 1) = f(k)$ where $k \equiv K/L$

    Rmk: Technical convenience (macro rationale), Easy Aggregation (micro rationale), increasing return or decreasing return globally is unattractive.

2. Positive diminishing Marginal Product

    $MP_K \equiv F_K > 0$ and $MP_L \equiv F_L > 0$; $\partial^2 F(K, L)/\partial K^2 \equiv F_{KK} < 0$ and $\partial^2 F(K, L)/\partial L^2 \equiv F_{LL} < 0$

3. Inada Conditions:

    $F_K(0, L) = \infty$, $F_L(K, 0) = \infty$, $F_K(\infty, L) = 0$, $F_L(K, \infty) = 0$, $F(0, 0) = 0$ and $F(\infty, \infty) = \infty$.

Cobb-Douglas production function is usually as standard equation form. because

1. $\alpha + \beta \gtreqless 1$ would represent the scale of return

2. linear in logs
3. shares of income is represented by α and β under CRS, which fits the long-run data

Cambridge-Cambridge Debate over the problem of aggregation of single capital stock K.

Solow Growth Model with constant population and technology

Central assumption: constant saving rate
Capital formation equation: capital adjustment is equal to investment reduced by depreciation

$$\begin{aligned}\dot{K}(t) &= \frac{dK(t)}{dt} = I(t) - \delta K(t) \\ &= sY(t) - \delta K(t) \\ &= sF[K(t), L(t)] - \delta K(t)\end{aligned}$$

Given stable technology and population,

$$\begin{aligned}\dot{k}(t) &= \frac{d}{dt}\frac{K(t)}{L(t)} = \frac{\dot{K}(t)}{L(t)} \\ &= \frac{1}{L(t)}[sF[K(t), L(t)] - \delta k(t)] \\ &= sf[k(t)] - \delta k(t)\end{aligned}$$

so that at steady state,

$$\dot{k} = 0 \Rightarrow s[f(k^*)] = \delta k^*$$

Such steady state is stable because if $k_0 > k^*$, then $\dot{k} < 0$ and if $k_0 > k^*$, then $\dot{k} < 0$. The existence of the steady state is confirmed by the Inada condition $\lim_{k\to\infty} f'(k) = 0$ and $\lim_{k\to 0} f'(k) = \infty$.
For Cobb-Douglas example,$y = f(k) = k^\alpha$, so that

$$sk^{*\alpha} = \delta k^* \Rightarrow k^* = \left(\frac{s}{\delta}\right)^{1/(1-\alpha)}$$

and hence

$$y^* = \left(\frac{s}{\delta}\right)^{\alpha/(1-\alpha)}$$

This is not a proper growth model in steady state, at most growth during transition.

Solow Growth Model with constant population growth but constant techonlogy

New assumption on population:

$$\frac{\dot{L}(t)}{L(t)} = n$$

The equation of motion changes:

$$\begin{aligned}\dot{k} &= \frac{dK/L}{dt} = \frac{\dot{K}L - \dot{L}K}{L^2} = \frac{\dot{K}}{L} - \frac{\dot{L}}{L}k \\ &= \frac{sF(K,L) - \delta K}{L} - nk \\ &= sf(k) - (n+\delta)k\end{aligned}$$

which implies golden rule to be

$$\max_{k^*} c^* = f(k^*) - (n+\delta)k^*$$

so that

$$\frac{\partial c^*}{\partial k^*} = 0 \Rightarrow f'(k^*) = (n+\delta)$$

This is still an improper growth model as aggregate Y, C and K grows but per capita y, c and k remains constant in steady state which contradicts our recent 200 years data, showing growth in standard of living.

Solow Growth Model with constant population and techonlogy growth

New assumption on production function by incorporating labor efficiency E:

$$Y(t) = F[K(t), E(t)L(t)]$$

and efficiency grows at exogenous rate g:

$$\frac{\dot{E}}{E} = g$$

The equation of motion changes as production function now is $Y = F(K, EL)$

$$\begin{aligned}\dot{k} &= \frac{dK/EL}{dt} = \frac{\dot{K}EL - \left(\dot{L}EK + L\dot{E}K\right)}{(EL)^2} \\ &= \frac{\dot{K}}{EL} - \frac{\dot{L}}{L}\frac{K}{EL} - \frac{\dot{E}}{E}\frac{K}{EL} \\ &= \frac{sF(K,EL) - \delta K}{EL} - nk - gk \\ &= sf(k) - (n+\delta+g)k\end{aligned}$$

which implies golden rule to be

$$\max_{k^*} c^* = f(k^*) - (n+\delta+g)\, k^*$$

so that

$$\frac{\partial c^*}{\partial k^*} = 0 \Rightarrow f'(k^*) = (n+\delta+g)$$

This model can explain the growth in capita term but in an exogenous manner. Though it is not complete but it does say that there is no multipler effect.
Different forms of technical change:

1. Harrod-neutral: $Y = F(K, AL) \Rightarrow \left.\frac{KF_K}{LF_L}\right|_{K/Y=\text{constant}}$ =constant
2. Hicks-neutral: $Y = AF(K, L) \Rightarrow \left.\frac{F_K}{F_L}\right|_{K/L=\text{constant}}$ =constant
3. Solow-neutral: $Y = AF(K, L) \Rightarrow \left.\frac{LF_L}{KF_K}\right|_{L/Y=\text{constant}}$ =constant

Existence of steady state would require Harrod-neutral if production function is not Cobb-Douglas.

Transitional Dynamics

Trying to explain cross-country income difference by transitional dynamics as all countries grows at the same rate g if they are in the steady state even if they have in different steady states.
There are two convergence concepts:

1. Absolute convergence: tendency of a poorer country to grow faster under all circumstances.
2. Conditional convergence: If two countries has the same steady state k^*, then poorer countries grows faster.

Solow Model only has conditional convergence. From data, transitional dynamics does explain convergence in OECD countries, which have similar economic fundamentals.
By Cobb-Douglas example,

$$\frac{\dot{k}}{k} \equiv \gamma_k = sk^{\alpha-1} - (n+g+\delta)$$

and

$$\frac{\dot{y}}{y} = \alpha\frac{\dot{k}}{k} = \alpha\gamma_k$$

since growth rate of k is decreasing in transition, it implies if 2 countries have same k^*, poor country grows faster as it has greater γ_k as $\dot{k}/k$ is decreasing function of $|k-k^*|$.

A more rigorous quantitative analysis can be done by log-linearlization around steady state:

$$\begin{aligned}
\frac{\dot{y}}{y} &= \alpha\frac{\dot{k}}{k} = \alpha\left[sk^{\alpha-1} - (n+g+\delta)\right] \\
&= \alpha s \exp\left[(\alpha-1)\ln k\right] - \alpha(n+g+\delta) \\
&\approx \alpha(\alpha-1)sk^{*\alpha-1}\left[\ln k - \ln k^*\right] \\
&= (\alpha-1)(n+g+\delta)\left[\ln y - \ln y^*\right] \\
&\equiv \gamma\left[\ln y^* - \ln y\right]
\end{aligned}$$

where $\gamma \equiv (1-\alpha)(n+g+\delta)$ to be the speed of convergence. Now,

$$\frac{d\ln y(t)}{dt} = \gamma\left[\ln y^* - \ln y\right]$$

or, taking integrating factor $e^{\gamma t}$

$$\begin{aligned}
&\frac{de^{\gamma t}\ln y(t)}{dt} = \gamma e^{\gamma t}\ln y^* \\
\Rightarrow\ &\int_0^t \frac{de^{\gamma v}\ln y(v)}{dv}dv = \int_0^t \gamma e^{\gamma v}\ln y^* dv \\
\Rightarrow\ &e^{\gamma t}\ln y(t) - \ln y(0) = \left(e^{\gamma t}-1\right)\ln y^* \\
\Rightarrow\ &\ln y(t) = e^{-\gamma t}\ln y(0) + \left(1-e^{-\gamma t}\right)\ln y^*
\end{aligned}$$

which implies half life $t_{1/2}$ is

$$\exp\left(-\gamma t_{1/2}\right) = 1/2$$

or

$$t_{1/2} = \ln 2/\gamma$$

Under reaonable estimates of $\alpha = 1/3$, $n = 0.01/year$ and $g + \delta = 0.05/year$, $t_{1/2}$ is 17 years, which is too fast to fit into actual-world data.

Adding Human captial

To explain to quick convergence, the production function becomes $Y = F(K, EH)$ where $H(t) = L(t)G(e)$ is human capital, which is a function of year of eduation e and raw labor L. The alegbra is the same but there is extra source of cross-country variation in per-capita income.

$$\frac{Y}{L} = y(s,n,\delta)EG(e)$$

The quantitative analysis would be to fit a regression equation:

$$Y = K^\alpha (EH)^{1-\alpha} \Rightarrow y = k^\alpha$$

In steady state, for country i,

$$s_i y_i^* = (n_i + g + \delta)\, k_i^*$$

or

$$\begin{aligned} \frac{k_i^*}{y_i^*} &= \frac{s_i}{(n_i + g + \delta)} \\ \Rightarrow \quad y_i^* &= \left(\frac{s_i}{n_i + g + \delta}\right)^{\alpha/(1-\alpha)} \\ \Rightarrow \quad \left(\frac{Y_i}{L_i}\right)^* &= \left(\frac{s_i}{n_i + g + \delta}\right)^{\alpha/(1-\alpha)} G(e_i)\, E(t) \end{aligned}$$

For unknown A, one can do the regression based on ratio

$$\frac{Y_i/L_i}{Y_{US}/L_{US}}$$

so that there is no need to find out A as $A(t)$ is same all over the world. Empirical result shows that the regression is not really good fit.

1.2 Cross-Country Income Difference

`Lecture Note: #1,#2,#3`
`Textbook: Romer 3; Acemoglu ch.3,4,7;BSM ch10`

Development Accounting

Assuming Cobb-Douglas production function, for each country i,

$$Y_i = K_i^{\alpha} (E_i H_i)^{1-\alpha}$$

or

$$\ln Y_i = \alpha \ln K_i + (1-\alpha) \ln E_i + (1-\alpha) \ln H_i$$

Using US as comparsion basis, we have

$$\ln \frac{E_i}{E_{US}} = \frac{1}{1-\alpha}\left[\ln\left(\frac{Y_i}{Y_{US}}\right) - \alpha \ln\left(\frac{K_i}{K_{US}}\right) - (1-\alpha)\ln\left(\frac{H_i}{H_{US}}\right)\right]$$

Empirical regression shows $R^2 = 0.8$ which means productivity explains most of variation. This also explains major puzzles:

1. high-human-capital people leave poor countries
2. relatively little physical capital flows to truly poor countries

Interpretation of variation in productivity

1. Technology

   Makiw argument: technology is knowledge and low capital/labor ratio does not represent knowledge

   Basu-Weil (1998), Acemoglu-Zilibotti (2001): most technologies developed by rich countires and profit motive determine the type of technologies to be produced. However, poor countries would eventually use those technologies.

2. Health

   Weil (2005): Direct effect affect wage and indirect effect affect human capital accumulation

3. Geography

   Landes (1998), Saches, et al. (2000-2003): lattitude and development has negative correlation, diseases and pests in tropical is difficult to kill

   Diamond (1997): transmission of ideas easier in long East-West than North-South.

4. Institutions

   Acemogula-Johnson-Robinson: mortality rate of early colonialists as IV

Growth Accounting

Solow (1957) took Cobb-Douglas production function:

$$\ln(Y) = \ln\left[F(K, EL)\right]$$

and log-linearized

$$\begin{aligned}
\frac{\dot{Y}}{Y} &\approx \frac{1}{F}F_1\dot{K} + \frac{1}{F}F_2\left[\dot{E}L + E\dot{L}\right] \\
&= \frac{F_1K}{F}\frac{\dot{K}}{K} + \frac{F_2(EL)}{F}\left[\frac{\dot{E}}{E} + \frac{\dot{L}}{L}\right] \\
&= s_K\frac{\dot{K}}{K} + s_L\left[\frac{\dot{E}}{E} + \frac{\dot{L}}{L}\right] \\
&= (1-s_L)\frac{\dot{K}}{K} + s_L\left[\frac{\dot{E}}{E} + \frac{\dot{L}}{L}\right]
\end{aligned}$$

Hence, contribution to per-capita growth is capital deepening and efficiency improvement:

$$\frac{\dot{Y}}{Y} - \frac{\dot{L}}{L} \approx (1-s_L)\left(\frac{\dot{K}}{K} - \frac{\dot{L}}{L}\right) + s_L\frac{\dot{E}}{E}$$

and contribution to technology is

$$s_L \frac{\dot{E}}{E} \approx \frac{\dot{Y}}{Y} - (1 - s_L) \frac{\dot{K}}{K} - s_L \frac{\dot{L}}{L}$$

In either case, there is no regression is needed.
Rmk: Development is cross-country, cross-section exercise but growth accounting is within-country, time series regression.

1.3 Ramsey-Cass-Koopman Model

`Lecture Note:  #6,#7,#8,#9`
`Textbook:  Romer ch.2; Acemoglu ch.8;BSM ch.2;Intriligator ch.16`

Social Planner Ramsey Model without technological change

Relax exogenous constant saving assumption from Solow model to endogenous optimal saving model.
Setup:

1. Neoclassical production function
2. Population growth at rate n and normalize $L(0) = 1$
3. Inada Type of utility: $u' > 0, u'' < 0, u'(0) = \infty$ and $u'(\infty) = 0$
4. social planner would discount future utility at rate ρ where $\rho > n$ to prevent unbounded utility
5. No technological change

Social planner is to maximizes sum of total discounted utility:

$$\begin{aligned} \max_{c(t)} \quad & \int_0^\infty e^{-pt} \left[u(c(t)) L(t) \right] dt \\ s.t \quad & \dot{k}(t) = f(k(t)) - c(t) - (n + \delta) k(t) \\ & k(0) = k_0 \end{aligned}$$

Rewrite objective function

$$\begin{aligned} \int_0^\infty e^{-pt} \left[u(c(t)) L(t) \right] dt &= \int_0^\infty e^{-pt} \left[u(c(t)) L(0) e^{nt} \right] dt \\ &= \int_0^\infty e^{-(p-n)t} u[c(t)] \, dt \end{aligned}$$

The Hamiltonian would be

$$H = e^{-(\rho - n)t} \left\{ u[c(t)] + \lambda \left[f(k(t)) - c(t) - (n + \delta) k(t) \right] \right\}$$

with F.O.C.s being

$$\begin{aligned} \frac{\partial H}{\partial c(t)} = 0 \quad &\Rightarrow \quad u'[c(t)] = \lambda \\ \frac{de^{-(\rho-n)t}\lambda}{dt} = -\frac{\partial H}{\partial k(t)} \quad &\Rightarrow \quad e^{-(\rho-n)t}\dot{\lambda} - (\rho-n)e^{-(\rho-n)t}\lambda = -e^{-(\rho-n)t}\lambda[f'[k(t)] - (n+\delta)] \\ &\Rightarrow \quad \frac{\dot{\lambda}}{\lambda} = -f'[k(t)] + (\rho+\delta) \end{aligned}$$

and TVC being

$$\lim_{t\to\infty} e^{-(\rho-n)t}\lambda(t)k(t) = 0$$

Combining two F.O.C.s, we have

$$\frac{u''}{u'}\dot{c} = \frac{\dot{\lambda}}{\lambda} = -f'(k) + (\rho+\delta)$$

or

$$\frac{\dot{c}}{c} = \frac{1}{\sigma(c)}[f'(k) - (\rho+\delta)]$$

where $\sigma(c) = -u'(c)/[cu''(c)]$ is called coefficient of relative risk aversion or intertemporal elasticity of substitution.

At steady state,

$$\begin{aligned} \dot{c} &= 0 \Rightarrow f'(k) = \rho+\delta \\ \dot{k} &= 0 \Rightarrow c(t) = f(k) - (n+\delta)k \end{aligned}$$

so that the modified golden rule is

$$\begin{aligned} f'\left(k^{MGR}\right) &= \rho+\delta \\ &> n+\delta = f'\left(k^{GR}\right) \end{aligned}$$

which implies

$$k^{MGR} < k^{GR}$$

This is due to discounting effect so that saving is more expensive than before.

From $\dot{c}$ equation, we can derive Ramsey-Keynes condition,

$$\sigma(c)\frac{\dot{c}}{c} + \rho = f'(k) - \delta$$

which means that if social planner is to invest current output rather than consuming it, the extra output $f'(k) - \delta$ should be enough to cover patience ρ and curvature correction $\sigma(c)\dot{c}/c$.

TVC implies there is nothing-left over condition in present value. Given perfect foresight, TVC excludes all but one transition path, which is saddle-point-stable path(SSP) equilibrium.

Elasticity of intertemporal tradeoff $\sigma(c)$

$$\begin{aligned}\sigma(c) &= -\frac{u'(c)}{cu''(c)} \\ &= \left[-\frac{c(t_1)/c(t_2)}{u'[c(t_1)]/u'[c(t_2)]}\frac{du'[c(t_1)]/u'[c(t_2)]}{d[c(t_1)/c(t_2)]}\right]\end{aligned}$$

Itwould affect the path of transition as it represents the elasticity of intertemporal tradeoff. If σ is bigger, the elasticity goes down, which implies people are willing to sacrifice present consumption for that of the future.

Comparative Dynamics:

1. Anticipated permanent shock
2. Anticipated temporary shock
3. Unanticipated permanent shock
4. Unanticipated temporary shock

Social Planner Ramsey Model without technological change

Requires utility in form constant elasticity of substituion (CES):

$$u(c) = \begin{cases} \dfrac{c^{1-\sigma}-1}{1-\sigma} & \text{for } \sigma \neq 1 \\ \ln(c) & \text{for } \sigma = 1 \end{cases}$$

The revised planner problem is

$$\begin{aligned}\max_{c(t)} \quad & \int_0^\infty e^{-pt}\left[u(\tilde{c}(t))L(t)\right]dt \\ s.t \quad & \dot{k}(t) = f(k(t)) - c(t) - (n+\delta)k(t) \\ & k(0) = k_0\end{aligned}$$

where $\tilde{c} = C/L$. Rewrite the objective function,

$$\begin{aligned}\int_0^\infty e^{-pt}\left[u(\tilde{c}(t))L(t)\right]dt &= \int_0^\infty e^{-pt}L(0)e^{nt}\frac{[E(t)c(t)]^{1-\sigma}}{1-\sigma}dt \\ &= \int_0^\infty e^{-pt}L(0)e^{nt}E(0)^{1-\sigma}e^{gt(1-\sigma)}\frac{c(t)^{1-\sigma}}{1-\sigma}dt \\ &= L(0)E(0)^{1-\sigma}\int_0^\infty e^{-[p-n-g(1-\sigma)]t}\frac{c(t)^{1-\sigma}}{1-\sigma}dt\end{aligned}$$

Normalize $L(0)E(0)^{1-\sigma} = 1$ and letting $\beta = p - n - g(1-\sigma)$, we have

$$\begin{aligned}\max_{c(t)} \quad & \int_0^\infty e^{-\beta t}\frac{c(t)^{1-\sigma}}{1-\sigma}dt \\ s.t \quad & \dot{k}(t) = f(k(t)) - c(t) - (n+\delta)k(t) \\ & k(0) = k_0\end{aligned}$$

so that the equation of motion becomes

$$\begin{aligned}\frac{\dot{c}}{c} &= \frac{1}{\sigma}\left[f'(k) - (\rho+\delta+\sigma g)\right] \\ \dot{k} &= f(k) - c - (\rho+\delta+\sigma g)\, k\end{aligned}$$

so the new golden rule would implies

$$f'\left(k^{MGR}\right) = \rho + \delta + \sigma g$$

and per worker result would be

$$\begin{aligned}\frac{\dot{\tilde{c}}}{\tilde{c}} &= \frac{\dot{A}}{A} + \frac{\dot{c}}{c} = g + \frac{\dot{c}}{c} \\ &= \frac{1}{\sigma}\left[f'(k) - (\rho+\delta+\sigma g)\right] + g \\ &= \frac{f'(k) - \rho - \delta}{\sigma}\end{aligned}$$

Hence, the steady state is just like Solow model as steady state $\dot{c} = 0$ and $\dot{k} = 0$ so that $\tilde{c}$ and $\tilde{k}$ grows at rate g. Therefore, saving rate cannot affect long-run growth rate.

Competitive Equilibrium Ramsey Model

To justify infinite horizon planning for finite life consumer, caring parents could justify this. Barro (1974) supposed everyone lives just one period/generation:

$$\begin{aligned}V_t &= u(c_{t+1}) + \gamma V_{t+1} \\ &= \sum_{t=0}^{\infty} \gamma^t u(c_t)\end{aligned}$$

Setup:

1. Continuum of dynasties growing at rate of n
2. Members of dynamsties would inherit his own share in the family
3. No new household
4. Identical household implies representative agent
5. Production would be equipped with neoclassical production function

Household: For each household i,

$$\begin{aligned}\max_{c_i(t)} \quad & \int_0^{\infty} e^{-\beta t} \frac{c_i(t)^{1-\sigma}}{1-\sigma} dt \\ s.t. \quad & A_i(t) = B_i(t) + K_i(t) \\ & \dot{A}_i(t) = r(t) B_i(t) + W(t) L_i(t) + R(t) K_i(t) - C_i(t) - \delta K_i(t) \\ & A_i(0) = A_0\end{aligned}$$

Given identical household, and taking $R - \delta = r$, we have only to track on A but not separate B and K,

$$\dot{A}(t) = r(t)A(t) + W(t)L(t) - C(t)$$

and no-Ponzi-game condition (NPG):

$$\lim_{T\to\infty} \exp\left[-\int_0^T r(s)\,ds\right] A(T) \geq 0$$

Rewrite the constraint into per-household term:

$$\begin{aligned}
\dot{a}(t) &= \frac{d}{dt}\frac{A_i(t)}{E(t)L(t)} = \frac{E(t)L(t)\dot{A}(t) - A(t)\left[\dot{E}(t)L(t) + E(t)\dot{L}(t)\right]}{\left[E(t)L(t)\right]^2} \\
&= \frac{\left[r(t)A(t) + W(t)L(t) - C(t)\right]}{E(t)L(t)} - \frac{A(t)}{E(t)L(t)}\left[\frac{\dot{E}(t)}{E(t)} + \frac{\dot{L}(t)}{L(t)}\right] \\
&= r(t)a(t) + w(t) - c(t) - a(t)(g+n)
\end{aligned}$$

where $w(t) \equiv W(t)/E(t)$ is the efficiency wage.
The Hamiltonian is

$$H = e^{-\beta t}\left\{\frac{c^{1-\sigma}}{1-\sigma} + \lambda\left[r(t)a(t) + w(t) - c(t) - a(t)(g+n)\right]\right\}$$

with F.O.C.s being

$$\begin{aligned}
\frac{\partial H}{\partial c} = 0 \quad &\Rightarrow \quad c^{-\sigma} = \lambda \\
\frac{de^{-\beta t}\lambda}{dt} = -\frac{\partial H}{\partial k} \quad &\Rightarrow \quad \frac{\dot{\lambda}}{\lambda} = (n+g-\beta) - r
\end{aligned}$$

Hence, the equations of motion are

$$\begin{aligned}
\frac{\dot{c}}{c} &= \frac{r-n-g-\beta}{\sigma} \\
\dot{a} &= ra + w - c - (n+g)a
\end{aligned}$$

Firm: Since each time is a static problem, maximization of profit in each point of time is same as maxizimation of sume of discounted profit over time:

$$\max_{K,L} F(K, EL) - RK - WL$$

with F.O.C.s being

$$\begin{aligned}
F_K &= R \\
F_L &= W
\end{aligned}$$

Now, write the firm F.O.C. in per capita term:

$$\begin{aligned} F_K &= \frac{dF(K,EL)}{dK} = \frac{dELf(K/EL)}{dK} = ELf'\left(\frac{K}{EL}\right)\frac{1}{EL} = f'(k) \\ F_L &= \frac{dF(K,EL)}{dL} = \frac{dELf(K/EL)}{dL} = Ef\left(\frac{K}{EL}\right) - EL\frac{K}{EL^2}f'\left(\frac{K}{EL}\right) = E\left[f(k) - kf'(k)\right] \end{aligned}$$

so that

$$\begin{aligned} r &= f'(k) - \delta \\ w &= \frac{W}{E} = \frac{E\left[f(k) - kf'(k)\right]}{E} = f(k) - kf'(k) \end{aligned}$$

Equilibrium:

$$\begin{aligned} \frac{\dot{c}}{c} &= \frac{r - n - g - \beta}{\sigma} \\ &= \frac{f'(k) - n - g - \left[\rho - n - (1-\sigma)g\right]}{\sigma} \\ &= \frac{f'(k) - \delta - \rho - \sigma g}{\sigma} \end{aligned}$$

and since all households are identical, $B = b = 0$,

$$\begin{aligned} \dot{k} &= \dot{a} = ra + w - c - (n+g)a \\ &= rk + w - c - (n+g)k \\ &= \left[f'(k) - \delta\right]k + \left[f(k) - kf'(k)\right] - c - (n+g)k \\ &= f(k) - c - (n+g+k) \end{aligned}$$

so the result is same as the social planner's outcome. The modified golden rule would be

$$r^* = \rho + \sigma g$$

but in practice, Fed uses

$$r = \sigma\frac{\dot{c}}{c} + \rho + \sigma g$$

Consumption Function (Ramsey Type)

Assumption:

1. No population and technology growth: $n = 0$ and $g = 0$ so that $L = E = 1$.
2. Allow variation in interest rate

From budget constraint:

$$\dot{A}(t) = r(t)A(t) + W(t)L(t) - C(t)$$

or

$$\dot{A}(t) = r(t)A(t) + W(t) - C(t)$$

and taking integrating factor $R(t,s) = \exp\left[-\int_t^s r(v)\,dv\right]$, we have

$$\frac{dR(t,s)A(s)}{ds} = R(t,s)\left[W(s) - C(s)\right]$$

and taking $t = 0$ and integrating s from 0 to ∞,

$$\int_0^\infty \frac{dR(0,s)A(s)}{ds}ds = \int_0^\infty R(0,s)\left[W(s) - C(s)\right]ds$$

by TVC, $\lim_{T\to\infty} R(0,T)A(T) = 0$, hence,

$$\int_0^\infty R(0,s)C(s)\,ds = A(0) + \int_0^\infty R(0,s)W(s)\,ds$$

Taking $r = \rho = \bar{r}$, so that

$$\frac{\dot{C}}{C} = \frac{1}{\sigma}(r - \rho) = 0$$

implies $C(t) = \bar{C}$ and

$$\int_0^\infty e^{-\bar{r}t}\bar{C}dt = A(0) + \int_0^\infty e^{-\bar{r}t}W(t)\,dt$$

or

$$\bar{C}\int_0^\infty e^{-\bar{r}t}dt = A(0) + \int_0^\infty e^{-\bar{r}t}W(t)\,dt$$

so that

$$\bar{C} = \bar{r}\left[A(0) + \int_0^\infty e^{-\bar{r}t}W(t)\,dt\right] \equiv Y^P$$

where C is the annuity value of lifetime wealth and is called permanent income. Therefore, only permanent shock would have larege effect on consumption, and temporary shcok would only have little effect over time as consumer would smooth consumption overtime.

1.4 Fiscal Policy, Ricardian Equivalence

`Lecture Note: #10,#11`

`Textbook: Romer ch.11; Acemoglu ch.8;BSM ch.2.1,3.1;`

`Paper: Elmendorf & Mankiw, NBER wp 6470`

Ricardian Equivalence

Assumption:

1. No population growth $n = 0$
2. No technology change $g = 0$

Household: When government levies lump-sum tax, household budget constraint becomes

$$\begin{aligned} \dot{A}(t) &= r(t)A(t) + Y(t) - C(t) - T(t) \\ A(0) &= A_0 \\ \lim_{T\to\infty} R(t,T)A(T) &= 0 \end{aligned}$$

Using integrating factor $R(t,T)$, we have

$$\int_t^\infty \frac{dA(v)R(t,v)}{dv} = \int_t^\infty R(t,v)\left[Y(v) - C(v) - T(v)\right]dv$$

which implies

$$\lim_{T\to\infty} R(t,T)A(T) - A(t) = \int_t^\infty R(t,v)\left[Y(v) - C(v) - T(v)\right]dv$$

and so that we have

$$\int_t^\infty R(t,v)C(v)\,dv + \int_t^\infty R(t,v)T(v)\,dv = A(t) + \int_t^\infty R(t,v)Y(v)\,dv$$

Government: budget depends on real expenditure $G(t)$, tax $T(t)$ and debt $D(t)$,

$$\begin{aligned} \dot{D}(t) &= r(t)D(t) + G(t) - T(t) \\ D(0) &= D_0 \\ \lim_{T\to\infty} R(t,T)D(T) &= 0 \end{aligned}$$

Similarly, using integrating factor $R(t,T)$, we have

$$\int_t^\infty \frac{dD(v)R(t,v)}{dv} = \int_t^\infty R(t,v)\left[G(v) - T(v)\right]dv$$

which implies

$$\lim_{T\to\infty} R(t,T)D(T) - D(t) = \int_t^\infty R(t,v)\left[G(v) - T(v)\right]dv$$

and so that we have

$$D(t) = \int_t^\infty R(t,v)\left[T(v) - G(v)\right]dv$$

This implies debt is present value of future surpluses. Note that Primary Surplus ($T > G$) is consistent with consolidated deficit ($T < G + rD$).

Unified Budget constraint:

Now the household assets would include governmetn debt $D(t)$ and capital $A(t)$:

$$\int_t^{\infty} R(t,v)\,C(v)\,dv + \int_t^{\infty} R(t,v)\,T(v)\,dv = A(t) + D(t) + \int_t^{\infty} R(t,v)\,Y(v)\,dv$$

which implies

$$\int_t^{\infty} R(t,v)\,C(v)\,dv = A(t) + \int_t^{\infty} R(t,v)\,[Y(v) - G(v)]\,dv$$

which is precise statement of Ricardian equivalence that a change in the timing of taexs holding constant the time path of real government expenditure has no real effects.

The arugement does not rely on

1. form of preference as the arguement works through budget constraint, and
2. infinite horizon as it only requires no economic disconnection between debt buying and tax paying

However, the argument relies on

1. lump-sum taxes,
2. people being forward-looking in consumption and obey permanent income hypothesis (PIH), and
3. those who hold government debt are also those people who have to pay later taxes.

Optimal Debt Management

Barro (1979) proposed that since in Ricardian equivalence requires lump-sum tax but actual most tax are distortionary, there may be optimal tax policy to minimize the distortion, which is the problem he created in 1974!

Assumption: welfare cost is convex in tax rate. $C = f(T_t/Y_t)$, $f(0) = 0$, $f' > 0$ and $f'' < 0$.

Government objective:

$$\begin{aligned} &\min_{T_t} \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t Y_t f\left(\frac{T_t}{Y_t}\right) \\ s.t. \quad &\sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t T_t = D_0 + \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t G_t \end{aligned}$$

The Lagrangian is

$$L = \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t Y_t f\left(\frac{T_t}{Y_t}\right) + \lambda \left[\sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t T_t - D_0 - \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t G_t\right]$$

with F.O.C. being

$$\frac{\partial L}{\partial T_t} = 0 \Rightarrow \left(\frac{1}{1+r}\right)^t Y_t f'\left(\frac{T_t}{Y_t}\right) \frac{1}{Y_t} = \lambda \left(\frac{1}{1+r}\right)^t$$

so that for all t,

$$f'\left(\frac{T_t}{Y_t}\right) = \lambda$$

or

$$\frac{T_t}{Y_t} = \frac{T_{t+1}}{Y_{t+1}} = \cdots = \frac{T_{t+k}}{Y_{t+k}}$$

From empirical work, large expenditure of government occurs usually during war time and usually most war are funded by debt rather than tax.

1.5 Overlapping Generations Model

`Lecture Note: #12,#13,#14,#15,#16`
`Textbook: Romer ch.2; BSM ch.3;`

2-period OLG Model (Decentralized Version)

Setup:

1. Compared to Ramsey Model, it includes
   (a) finite horizon/no altruism
   (b) retirement.
2. OLG consumer has
   (a) no initial endowment of wealth
   (b) lives two period
   (c) die without bequest
3. Population grows at rate $n > 0$
4. No technology progress $g = 0$

Household:
Life cycle:

1. Born at time t without endowment.
2. Supply 1 unit of labor inelasticity and work in competitive labor market, and earns wage w_t and then retires.
3. Saving/consumption decision to split wages into c_t consumption of period t and saving s_t.
4. receive principle plus interest $s_t\left(1+r_{t+1}\right)$ and consumes all of the saving and die without any bequest.

Hecne, the budget constraint is

$$\begin{aligned}
\max_{c_{1t},c_{2t+1}} \qquad & U_t=\frac{c_{1t}^{1-\theta}}{1-\theta}+\frac{1}{1+\rho}\frac{c_{2t+1}^{1-\theta}}{1-\theta} \\
s.t. \qquad & c_{1t}+s_t=w_t \\
& c_{2t+1}=\left(1+r_{t+1}\right)s_t
\end{aligned}$$

with Lagrangian being

$$L=\frac{c_{1t}^{1-\theta}}{1-\theta}+\frac{1}{1+\rho}\frac{c_{2t+1}^{1-\theta}}{1-\theta}+\lambda\left[w_t-c_{1t}-\frac{c_{2t+1}}{1+r_{t+1}}\right]$$

and F.O.C.s being

$$\begin{aligned}
\frac{\partial L}{\partial c_{1t}} &= 0\Rightarrow c_{1t}^{-\theta}=\lambda \\
\frac{\partial L}{\partial c_{2t+1}} &= 0\Rightarrow \frac{1}{1+\theta}c_{2t+1}^{-\theta}=\lambda\frac{1}{1+r_{t+1}}
\end{aligned}$$

so that

$$\frac{c_{2t+1}^{-\theta}}{c_{1t}^{-\theta}}=\frac{1+\rho}{1+r_{t+1}}$$

Rmk:

$$\ln\left(c_{2t+1}\right)-\ln\left(c_{1t}\right)=-\frac{1}{\theta}\left[\ln\left(1+\rho\right)-\ln\left(1+r_{t+1}\right)\right]$$

Under approximation, $\ln\left(1+x\right)\approx x$ when x is small, so that

$$\ln\left(c_{2t+1}\right)-\ln\left(c_{1t}\right)\approx-\frac{1}{\theta}\left(\rho-r_{t+1}\right)=\frac{1}{\theta}\left(r_{t+1}-\rho\right)$$

which is the F.O.C. in Ramsey model.
From the budget constraint, we have

$$w_t-c_{1t}-\frac{1}{1+r_{t+1}}\left[\frac{1+\rho}{1+r_{t+1}}\right]^{-1/\theta}c_{1t}=0$$

or

$$\frac{\left(1+\rho\right)^{1/\theta}+\left(1+r_{t+1}\right)^{(1-\theta)/\theta}}{\left(1+\rho\right)^{1/\theta}}c_{1t}=w_t$$

so that

$$c_{1t}=\frac{(1+\rho)^{1/\theta}w_t}{(1+\rho)^{1/\theta}+(1+r_{t+1})^{(1-\theta)/\theta}}$$

Define saving rate be $s(r_{t+1})=1-c_{1t}/w_t$. Then

$$\begin{aligned}s(r_{t+1}) &= 1-\frac{(1+\rho)^{1/\theta}}{(1+\rho)^{1/\theta}+(1+r_{t+1})^{(1-\theta)/\theta}}\\ &= \frac{(1+r_{t+1})^{(1-\theta)/\theta}}{(1+\rho)^{1/\theta}+(1+r_{t+1})^{(1-\theta)/\theta}}\end{aligned}$$

Saving rate and interest rate depends on parameter θ:

$$\begin{aligned}\frac{\partial s(r_{t+1})}{\partial r_{t+1}} &= \frac{(1-\theta)/\theta(1+r_{t+1})^{(1-2\theta)/\theta}}{\left[(1+\rho)^{1/\theta}+(1+r_{t+1})^{(1-\theta)/\theta}\right]}-\frac{(1-\theta)/\theta(1+r_{t+1})^{(1-\theta)/\theta}(1+r_{t+1})^{(1-2\theta)/\theta}}{\left[(1+\rho)^{1/\theta}+(1+r_{t+1})^{(1-\theta)/\theta}\right]^2}\\ &= \frac{(1-\theta)/\theta(1+r_{t+1})^{(1-2\theta)/\theta}\left[(1+\rho)^{1/\theta}+(1+r_{t+1})^{(1-\theta)/\theta}-(1+r_{t+1})^{(1-\theta)/\theta}\right]}{\left[(1+\rho)^{1/\theta}+(1+r_{t+1})^{(1-\theta)/\theta}\right]^2}\\ &= \frac{(1-\theta)/\theta(1+r_{t+1})^{(1-2\theta)/\theta}(1+\rho)^{1/\theta}}{\left[(1+\rho)^{1/\theta}+(1+r_{t+1})^{(1-\theta)/\theta}\right]^2}\end{aligned}$$

so that

$$\frac{\partial s(r_{t+1})}{\partial r_{t+1}}\gtreqless 0 \text{ iff } \theta\lesseqgtr 1$$

The reason is when interest rate rises, relative price future consumption drops in terms of present value, which increases saving(substitution effect) and effective income has goes up, which reduces saving(income effect). $\theta=1$ implies inocme and substitution effects cancel each other and $\theta>1$ implies income effect is less than substitution effect, and $\theta<1$ implies income effect is greater than substitution effect.

Note that interest rate does have unambiguous effect on grwoth rate of c but only ambiguous effect on the level of c.

Firm: Being standard, neoclassical, the result is standard that

$$\begin{aligned}F_k &= f'(k)\Rightarrow r=f'(k)-\delta\\ F_L &= w=f(k)-kf'(k)\end{aligned}$$

Equilibrium:

Further assumes that:

1. Saving at t accounts for entire capital stocks at $t+1$:

$$L_{t-1}s_{t-1}=K_t$$

   or by $c_t=(1+r_t)s_{t-1}$,

$$L_{t-1}c_t=(1+r_t)K_t$$

2. Saving rate affects y and w through k_t and affect r by k_{t+1} through MPK

From National income identity,

$$\begin{aligned} & Y_t = C_t + I_t \\ \Rightarrow \quad & F(K_t, L_t) = C_t + (K_{t+1} - K_t + \delta K_t) \\ \Rightarrow \quad & w_t L_t + R_t K_t = L_t c_{1t} + L_{t-1} c_{2t} + (K_{t+1} - K_t + \delta K_t) \\ \Rightarrow \quad & K_{t+1} - K_t = w_t L_t + r_t K_t - L_t c_{1t} - L_{t-1} c_{2t} \end{aligned}$$

Recall

$$\begin{aligned} c_{1t} &= w_t - s_t \\ &= [1 - s(r_{t+1})] w_t \end{aligned}$$

and

$$L_{t-1} c_{2t} = (1 + r_t) K_t$$

so we have

$$K_{t+1} - K_t = w_t L_t + r_t K_t - L_t [1 - s(r_{t+1})] w_t - (1 + r_t) K_t$$

or

$$\begin{aligned} K_{t+1} &= s(r_{t+1}) w_t L_t \\ &= \frac{1}{1+n} s(r_{t+1}) w_t L_{t+1} \end{aligned}$$

and hence

$$k_{t+1} = \frac{1}{1+n} s[f'(k_{t+1}) - \delta][f(k_t) - kf'(k_t)]$$

However, this equation has different kinds of strange dynamics, so we will analyze under two canonical assumptions:

1. Log utility (implies $\theta = 1$). Since

$$s(r_{t+1}) = \frac{(1 + r_{t+1})^{(1-\theta)/\theta}}{(1+\rho)^{1/\theta} + (1 + r_{t+1})^{(1-\theta)/\theta}}$$

   when $\theta = 1$, we have

$$s(r_{t+1}) = \frac{1}{2+\rho}$$

2. Cobb-Douglas Utility (implies $f(k_t) = k_t^\alpha$)

Therefore, the equation would simplify into

$$\begin{aligned} k_{t+1} &= \frac{1}{1+n} \frac{1}{2+\rho} \left[k_t^\alpha - k_t \alpha k_t^{\alpha-1}\right] \\ &= \frac{1}{1+n} \frac{1}{2+\rho} (1-\alpha) k_t^\alpha \end{aligned}$$

which has steady state as

$$k_{t+n}-k_{t+n-1}=\left[\frac{1}{1+n}\frac{1}{2+\rho}\left(1-\alpha\right)\right]^{\alpha(n-1)}k_t^{\alpha n}\left[k_{t+1}-1\right]$$

which goes to zero as n goes to infinity.
For steady state, we have

$$k^*=\frac{1}{1+n}\frac{1}{2+\rho}\left(1-\alpha\right)k^{*\alpha}$$

or

$$k^*=\left[\frac{1-\alpha}{\left(1+n\right)\left(2+\rho\right)}\right]^{1/(1-\alpha)}$$

which implies no growth by only accumulating capital, same as neoclassical growth model.

2-period OLG Model (Social Planner Version)

Social planner is to maximize weighted sum of utility across generation: (as in Acemogolu P351)

$$\begin{array}{lll}\max\limits_{\{c_{1t}\},\{c_{2t}\}} & \sum\limits_{t=1}^{\infty}\left(\frac{1}{1+\rho}\right) & {}^tU\left(c_t\right)=\sum\limits_{t=1}^{\infty}\left(\frac{1}{1+\rho}\right)^t\left[\ln c_{1t}+\frac{1}{1+\rho}\ln c_{2t+1}\right]\\ & s.t. & F\left(K_t,L_t\right)=I_t+C_t=K_{t+1}-K_t+\delta K_t+C_t\end{array}$$

where the capital accumulation can be rewritten as

$$f\left(k_t\right)L_t=L_{t+1}k_{t+1}-L_tk_t+\delta L_tk_t+\left(L_tc_{1t}+L_{t-1}c_{2t}\right)$$

or

$$f\left(k_t\right)=\left(1+n\right)k_{t+1}-k_t+\delta k_t+c_{1t}+\frac{c_{2t}}{1+n}$$

so that F.O.C.s are

$$\begin{aligned}\frac{\partial L}{\partial c_{1t}}&=0\Rightarrow\left(\frac{1}{1+\rho}\right)^t\frac{1}{c_{1t}}-\lambda_t=0\\ \frac{\partial L}{\partial c_{2t}}&=0\Rightarrow\left(\frac{1}{1+\rho}\right)^t\frac{1}{c_{2t}}-\lambda_t\frac{1}{1+n}=0\\ \frac{\partial L}{\partial k_{t+1}}&=0\Rightarrow\lambda_{t+1}\left[f'\left(k_t\right)+\left(1-\delta\right)\right]-\lambda_t\left(1+n\right)=0\end{aligned}$$

so that we have

$$\frac{\left[f'\left(k_t\right)+\left(1-\delta\right)\right]}{1+n}=\frac{\lambda_t}{\lambda_{t+1}}=\frac{\left(\frac{1}{1+\rho}\right)^t\frac{1}{c_{1t}}}{\left(1+n\right)\left(\frac{1}{1+\rho}\right)^{t+1}\frac{1}{c_{2t+1}}}=\frac{1+\rho}{1+n}\frac{c_{2t+1}}{c_{1t}}$$

or

$$\frac{1}{c_{1t}} = \frac{1}{1+\rho}\left[f'\left(k_t\right) + (1-\delta)\right]\frac{1}{c_{2t+1}}$$

so that in steady state, we have

$$k_{t+1} = k_t$$

or

$$c_{1t} = c_{1t+1}; c_{2t} = c_{2t+1}$$

and hence

$$\frac{c_{2t+1}}{c_{1t}} = \frac{c_{2t}}{c_{1t}} = 1+n = \frac{1}{1+\rho}\left[f'\left(k_t\right) + (1-\delta)\right]$$

so that

$$1+n = \frac{1}{1+\rho}\left[f'\left(k_t\right) + (1-\delta)\right]$$

or under Cobb-Douglas production function

$$k^* = \left[\frac{\rho + n(1+\rho) + \delta}{\alpha(1+\rho)(1+n)}\right]^{1/(1-\alpha)}$$

which is different from decentralized version of the model. Also, we have

$$f\left(k^*\right) = (1+n)k^* - k^* + c_{1t} + \frac{c_{2t}}{1+n} + \delta k^*$$

or

$$c^* \equiv c_1^* + \frac{c_2^*}{1+n} = f\left(k^*\right) - nk^* - \delta k^*$$

Hence

$$\frac{\partial c^*}{\partial k^*} = f'\left(k^*\right) - n - \delta$$

implies golden rule would be

$$f'\left(k_{GR}\right) = \delta + n$$

ForCobb-Douglas function, we have

$$k_{GR} = \left(\frac{\alpha}{\delta + n}\right)^{1/(1-\alpha)}$$

Hence, to gether with log-utility, we have $k^* > k_{GR}$ if

$$\frac{1-\alpha}{(1+n)(2+\rho)} > \frac{\alpha}{\delta + n}$$

so that competitive equilibrium is not necessarily Pareto Efficient. The source of problem is due to the saving for retirement.
Inefficiency lead to the justification of government action: unfunded social security or rolling over her debt at interest rate r.

Since golden rule implies

$$f'(k_{GR}) = \delta + n$$

the test for inefficiency would be

$$\begin{aligned} k^* \gtreqless k_{GR} &\Leftrightarrow f'(k^*) \lesseqgtr f'(k_{GR}) \\ &\Leftrightarrow n + \delta \lesseqgtr r + \delta \\ &\Leftrightarrow n \lesseqgtr r \end{aligned}$$

Empirical studies show dynamic inefficiency requires $\alpha < 0.32$ for standard parameter. Abel-Mankiw-Summers-Zeckhauser (1989) shows that under uncertainty the economy is dynamically efficient if capital income exceeds investment expenditure, which is not satisfied by data. Therefore, modern economies is unlikely to be dynamically inefficient. Hence, there should not be any rational bubble for which government would not repay her debt.

Blanchard Finite-Horizon Model without Government

Another finite horizon version of Ramsey but more tractable than OLG. Compared to OLG, it has no retirement, or more precisely, its wage remains non-zero level throughout the lifetime of worker.
Setup:

1. Continuum of agents

2. Continuous time

3. Constant probability of death p (capture no literal death, but economic disconnection from the future; e.g. a family with no children or parents are no longer altruistic)

4. Population size is always 1 (BSM has treatment with $n > 0$)

   This implies same fraction of p of new population is born

5. Expected remaining life always $1/p$.

6. Perfect insurance market with zero profit competition environment:

   Everyone signs annuity contract with insurance company: Insurance company pays $p\cdot$(expected wealth) if alive and gets wealth if dead so the company would expect to pay $1/p \cdot p \cdot w = w$ but is expected to get w so it is actuarially fair. In real life, this is similar to reverse mortgage (refer to appendix for detailed explanation)

7. No bequest so that anyone would start with zero wealth (key different from Ramsey; same as OLG)

Individual: For each cohort s, they maximize lifetime utility: (refer to appendix for detailed explanation for objective function)

$$\begin{aligned}
\max_{\{c(s,v)\}_{v=t}^{\infty}} \quad & E_t\left[\int_t^{\infty} \ln\left[c\left(s,v\right)\right] e^{-\theta(v-t)} dv\right] = \int_t^{\infty} \ln\left[c\left(s,v\right)\right] e^{-(\theta+p)(v-t)} dv \\
s.t. \quad & \frac{dw\left(s,v\right)}{dv} = \left[r\left(v\right)+p\right] w\left(s,v\right) + y\left(s,v\right) - c\left(s,v\right) \\
& \lim_{v\to\infty} w\left(s,v\right) \exp\left\{-\int_t^{v}\left[r\left(u\right)+p\right] du\right\} = 0
\end{aligned}$$

The Hamiltonian would be

$$H = e^{-(\theta+p)(v-t)}\left\{\ln c\left(s,v\right) + \lambda\left[\left(r\left(v\right)+p\right) w\left(s,v\right) + y\left(s,v\right) - c\left(s,v\right)\right]\right\}$$

and the corresponding F.O.C.s are

$$\begin{aligned}
\frac{\partial H}{\partial c\left(s,v\right)} &= 0 \quad \Rightarrow \quad \frac{1}{c\left(s,v\right)} = \lambda \\
\frac{de^{-(\theta+p)(v-t)}\lambda}{dv} &= -\frac{\partial H}{\partial w\left(s,v\right)} \quad \Rightarrow \quad \frac{d\lambda}{dv} - \left(\theta+p\right)\lambda = -\left(r\left(v\right)+p\right)\lambda
\end{aligned}$$

so that

$$\frac{dc\left(s,v\right)}{dv} = \left[r\left(v\right)-\theta\right] c\left(s,v\right)$$

Taking integrating factor, $\exp\left[-\int_t^{v}\left[r\left(u\right)-\theta\right] du\right]$, we have

$$\frac{dc\left(s,v\right)}{dv}\exp\left[-\int_t^{v}\left[r\left(u\right)-\theta\right] du\right] - \left[r\left(v\right)-\theta\right]\exp\left[-\int_t^{v}\left[r\left(u\right)-\theta\right] du\right] c\left(s,v\right) = 0$$

or, by Leibniz rule

$$\frac{d}{dv}\left\{c\left(s,v\right)\exp\left[-\int_t^{v}\left[r\left(u\right)-\theta\right] du\right]\right\} = 0$$

so that by integrating v from t to z, we have

$$c\left(s,z\right)\exp\left[-\int_t^{z}\left[r\left(u\right)-\theta\right] du\right] - c\left(s,t\right) = 0$$

or

$$c\left(s,z\right) = c\left(s,t\right)\exp\left[\int_t^{z}\left[r\left(u\right)-\theta\right] du\right]$$

From the budget constraint, taking integrating factor $\exp\left[-\int_t^{v}\left[r\left(u\right)+p\right] du\right]$,

$$\left[\frac{dw\left(s,v\right)}{dv} - \left[r\left(v\right)+p\right] w\left(s,v\right)\right]\exp\left[-\int_t^{v}\left[r\left(v\right)+p\right] du\right] = \left[y\left(s,v\right)-c\left(s,v\right)\right]\exp\left[-\int_t^{v}\left[r\left(u\right)+p\right] du\right]$$

so that

$$\begin{aligned}\int_t^\infty c(s,v)\exp\left[-\int_t^v [r(vu)+p]\,du\right]dv &= \int_t^\infty \frac{d}{dv}\left\{w(s,v)\exp\left[-\int_t^v [r(u)+p]\,du\right]\right\}dt + \\ &\quad \int_t^\infty y(s,v)\exp\left[-\int_t^v [r(u)+p]\,du\right]dv\end{aligned}$$

then, by TVC,

$$\int_t^\infty c(s,v)\exp\left[-\int_t^v [r(u)+p]\,du\right]dv = w(s,t)+h(s,t)$$

where

$$h(s,t)=\int_t^\infty y(s,v)\exp\left[-\int_t^v [r(u)+p]\,du\right]dv$$

is defined to be present value of labor income. Note that

$$\begin{aligned}\int_t^\infty c(s,v)\exp\left[-\int_t^v [r(u)+p]\,du\right]dv &= \int_t^\infty c(s,t)\exp\left[\int_t^v [r(u)-\theta]\,du\right]\exp\left[-\int_t^v [r(u)+p]\,du\right]dv \\ &= c(s,t)\int_t^\infty \exp\left[\int_t^v -(\theta+p)\,du\right]dv \\ &= c(s,t)\int_t^\infty \exp\left[-(\theta+p)(v-t)\right]dv \\ &= \frac{c(s,t)}{\theta+p}\end{aligned}$$

and hence

$$c(s,t)=(\theta+p)\left[w(s,t)+h(s,t)\right]$$

The reason for the indepedent of r is due to log utility and MPC is indepednet of age is due to constant mortality.

Aggregation:

$$\begin{aligned}W(t) &= \int_{-\infty}^t w(s,t)\,pe^{-p(t-s)}ds \\ H(t) &= \int_{-\infty}^t h(s,t)\,pe^{-p(t-s)}ds\end{aligned}$$

so that under equilibrium,

$$\begin{aligned}C(t) &= \int_{-\infty}^t c(s,t)\,pe^{-p(t-s)}ds \\ &= \int_{-\infty}^t (\theta+p)\left[w(s,t)+h(s,t)\right]pe^{-p(t-s)}ds \\ &= (\theta+p)\left[W(t)+H(t)\right]\end{aligned}$$

and

$$\begin{aligned}
H(t) &= \int_{-\infty}^{t} h(s,t)\, p e^{-p(t-s)} ds \\
&= \int_{-\infty}^{t} \left\{ \int_{t}^{\infty} y(s,v) \exp\left[-\int_{t}^{v} [r(u)+p]\, du \right] dv \right\} p e^{-p(t-s)} ds \\
&= \int_{-\infty}^{t} \int_{t}^{\infty} y(s,v) \exp\left[-\int_{t}^{v} [r(u)+p]\, du \right] p e^{-p(v-s)} dv ds \\
&= \int_{t}^{\infty} \int_{-\infty}^{v} y(s,v)\, p e^{-p(v-s)} \exp\left[-\int_{t}^{v} [r(u)+p]\, du \right] ds dv \\
&= \int_{t}^{\infty} \int_{-\infty}^{v} y(s,v)\, p e^{-p(v-s)} ds \exp\left[-\int_{t}^{v} [r(u)+p]\, du \right] dv \\
&= \int_{t}^{\infty} Y(v) \exp\left[-\int_{t}^{v} [r(u)+p]\, du \right] dv
\end{aligned}$$

so that, by Leibniz rule, assuming $d\infty/dt = 0$,

$$\begin{aligned}
\dot{H}(t) &= -Y(t) + \int_{t}^{\infty} Y(v) \exp\left[-\int_{t}^{v} [r(u)+p]\, du \right] [r(t)+p]\, dv \\
&= -Y(t) + H(t)\,[r(t)+p]
\end{aligned}$$

Non-human wealth would be

$$W(t) = \int_{-\infty}^{t} w(s,t)\, p e^{-p(t-s)} ds$$

so that, by Lebniz rule, assuming $d\infty/dt = 0$,

$$\begin{aligned}
\dot{W}(t) &= pw(t,t) + \int_{-\infty}^{t} \left[\frac{dw(s,t)}{dt} p e^{-p(t-s)} - pw(s,t)\, p e^{-p(t-s)} \right] ds \\
&= pw(t,t) - pW(t) + \int_{-\infty}^{t} \frac{dw(s,t)}{dt} p e^{-p(t-s)} ds
\end{aligned}$$

as $w(t,t) = 0$, and substitute the budget constraint,

$$\begin{aligned}
\dot{W}(t) &= -pW(t) + \int_{-\infty}^{t} \{[r(t)+p]\, w(s,t) + y(s,t) - c(s,t)\}\, p e^{-p(t-s)} ds \\
&= -pW(t) + [r(t)+p]\, W(t) + Y(t) - C(t) \\
&= r(t)\, W(t) + Y(t) - C(t)
\end{aligned}$$

Now, we have the following system of differential equations:

$$\begin{aligned}
C(t) &= (\theta + p)\,[W(t) + H(t)] \\
\dot{H}(t) &= -Y(t) + H(t)\,[r(t)+p] \\
\dot{W}(t) &= r(t)\, W(t) + Y(t) - C(t)
\end{aligned}$$

Differentiate $C(t)$ with respect to time, taking away $H(t)$,

$$\begin{aligned}
\dot{C}(t) &= (\theta+p)\left[\dot{W}(t)+\dot{H}(t)\right] \\
&= (\theta+p)\{r(t)W(t)+Y(t)-C(t)+-Y(t)+H(t)[r(t)+p]\} \\
&= (\theta+p)\left\{r(t)W(t)-C(t)+\left[\frac{C(t)}{\theta+p}-W(t)\right][r(t)+p]\right\} \\
&= [r(t)-\theta]C(t)-p(\theta+p)W(t)
\end{aligned}$$

For steady state, we have $W=K$ and neoclassical production function implies

$$\begin{aligned}
F(K(t),L(t)) &= F(K(t),1) \\
&= F_K K(t)+F_L \\
&= R(t)K(t)+Y(t) \\
&= [r(t)-\delta]K(t)+Y(t)
\end{aligned}$$

so that

$$\begin{aligned}
\dot{C}(t) &= [r(t)-\theta]C(t)-p(\theta+p)W(t) \\
&= [F'(K(t),1)-\delta-\theta]C(t)-p(p+\theta)K
\end{aligned}$$

and

$$\begin{aligned}
\dot{K}(t) &= r(t)K(t)+Y(t)-C(t) \\
&= [R(t)-\delta]K(t)+Y(t)-C \\
&= F[K(t)]-\delta K(t)-C
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\dot{C}(t) &= 0 \Rightarrow C^* = \frac{p(p+\theta)K^*}{[F'(K^*)-\delta-\theta]} \\
\dot{K}(t) &= 0 \Rightarrow C^* = F[K^*]-\delta K^*
\end{aligned}$$

Non-negativity implies $F'(K^*)-\delta-\theta>0$ so that

$$F'(K^*)=f'(k^*)>\delta+\theta=f'\left(k^{MGR}\right)=F'\left(K^{MGR}\right)$$

so that

$$K^*<K^{MGR}$$

Moreover, when ρ goes up, C^* goes up as consumers are more patient so that more capitals are accumulated for more production.

Blanchard Finite-Horizon Model With Government

Similar to Ricardian equivalence model, government Budget constraints would be

$$\begin{aligned}
\dot{D}(t) &= r(t)D(t)+G(t)-T(t) \\
D(0) &= D_0 \\
\lim_{T\to\infty} R(0,T)D(T) &= 0
\end{aligned}$$

or more compactly,

$$\begin{aligned} D(t) &= \int_t^{\infty} R(t,s)\,T(s)\,ds - \int_t^{\infty} R(t,s)\,G(s)\,ds \\ &= \int_t^{\infty} R(t,s)\left[T(s) - G(s)\right] ds \end{aligned}$$

The only change in condition is that $W(t) = K(t)$ is no longer true as debt $D(t)$ would be one part of wealth and tax $T(t)$ would reduce wealth accumulation $\dot{W}(t)$, so

$$\begin{aligned} C(t) &= (\theta + p)\left[W(t) + H(t)\right] \\ W(t) &= D(t) + K(t) \\ \dot{W}(t) &= r(t)\,W(t) + Y(t) - C(t) - T(t) \end{aligned}$$

and

$$H(t) = \int_t^{\infty} Y(v)\exp\left[-\int_t^{v}\left[r(u)+p\right]du\right]dv - \int_t^{\infty} T(v)\exp\left[-\int_t^{v}\left[r(u)+p\right]du\right]dv$$

or

$$\dot{H}(t) = -Y(t) + T(t) + H(t)\left[r(t) + p\right]$$

Also,

$$\begin{aligned} W(t) &= D(t) + K(t) \\ &= \int_t^{\infty} R(t,s)\left[C(s) + T(s) - Y(s)\right] ds \end{aligned}$$

so that

$$K(t) = \int_t^{\infty} R(t,s)\left[C(s) + G(s) - Y(s)\right] ds$$

which is standard Ricardian result tha timing of taxes would not affect the financial wealth.

Taxing timing without changing the real government spending, (see appendix)

$$dT_t = -R(t, t+\tau)\,dT_{t+\tau}$$

so that

$$dH = -dT_t - dT_{t+\tau}\exp\left[-\int_t^{t+\tau}\left[r(s)+p\right]ds\right]$$

or

$$dH_t = -dT_t\left(1 - e^{-p\tau}\right)$$

and this implies when $p = 0$ (Ramsey model), has Ricardian equivalence. The reason why non-ricardian result is that people discount Y and T at rate $r + p$ but government only discounts at r.

Suppose $\bar{T} = G + r\bar{D}$, the new system becomes (see appendix)

$$\begin{aligned}\dot{C} &= \left[F'(K) - \delta - \theta\right]C - p(p+\theta)(D+K) \\ \dot{K} &= F(K) - C - \delta K - \bar{G} \\ r\bar{D} &= T - G\end{aligned}$$

Therefore, this implies the effect of debt on C would depends on when the later tax is levied. The steady state effect is negative impact on capital accumulation as income would need to devote to new tax for those new born.

1.6 Endogenous Growth Model

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Lecture Note:  #17,#18,#19,#20,#21
Textbook:  Romer ch.3; BSM ch.1,4;Acemoglu ch.11,
Paper:  Jones, “Growth and Ideas” (2004)
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Three main problems of exogenous growth model

1. No explanation of growth (which means the theory is far from complete)
2. Too large implied interest rate differential
3. cross-country non-convergence

Solow Ak model

1. Assuming raw labor L is not necessary in production
2. Capital K includes physical capital and human capital

Then Production function becomes $Y = AK$ or in per captia term $y = Ak$. Now, capital accumulation equation:

$$\begin{aligned}\dot{k} &= sf(k) - (n+\delta)k \\ &= sAk - (n+\delta)k\end{aligned}$$

or

$$\frac{\dot{k}}{k} = \gamma_A = sA - (n+\delta)$$

When $sA > n + \delta$, we have

$$\frac{\dot{k}}{k} = \gamma_A > 0$$

Compare with the exogenous-growth model, the three problems are addressed:

1. This capital accumulation would promote growth, which also implies saving rate s would change growth rate!
2. Every country has same interest rate as $r = f'(k) - \delta = A - \delta$.
3. convergence problem can be explain by no convergence as different in n and s as difference in n and s would imply different long-term growth rate

Sobelo Model

Include transitional dynamics so that convergence between OECD countries could be included.
Rebelo (1991) suggests the production function could be $Y = AK + BK^{\alpha}L^{1-\alpha}$ or in per capita terms, $y = Ak + Bk^{\alpha}$.
The positive growth rate will fall with k but asymptotically same as the Ak model yet it remains non-zero to represent a source of growth.

Ramsey Ak model

Consumption:

$$\begin{array}{rl} \max\limits_{c(t)} & \displaystyle\int_0^{\infty} e^{-(\rho-n)t} \frac{c(t)^{1-\sigma}}{1-\sigma} dt \\ s.t & \dot{a}(t) = r(t)a(t) + w(t) - c(t) - na(t) \\ & A(0) = A_0 \end{array}$$

Since it is unrelated to production function, the optimal conditions remains unchanged

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\sigma}(r(t) - \rho)$$
$$\dot{a}(t) = r(t)a(t) + w(t) - c(t) - na(t)$$
$$\lim_{T\to\infty} \exp\left\{-\int_0^T [r(s) - n]\,ds\right\} a(T) = 0$$

Production:

$$\max_{K,L} \quad F(K,L) - WL - RK$$

so that we have

$$\begin{aligned} R &= F_K(K,L) = A \\ W &= F_L = 0 \end{aligned}$$

By setting $r = R - \delta = A - \delta$, $w = 0$ and $a = k$, we have

$$\begin{aligned} \frac{\dot{c}(t)}{c(t)} &= \frac{1}{\sigma}(A - \delta - \rho) \\ \dot{k}(t) &= (A-\delta)k(t) - c(t) - nk(t) \\ &= Ak(t) - c(t) - (n+\delta)k(t) \end{aligned}$$

Assuming $A > \rho + \delta > (A - \delta)(1-\sigma) + \sigma n + \delta$ (1st inequalty is for growth and 2nd is for finite utility), we have

$$\begin{aligned} \frac{\dot{c}(t)}{c(t)} &= \gamma_c = \frac{1}{\sigma}(A - \delta - \rho) > 0 \\ \frac{\dot{k}(t)}{k(t)} &= \gamma_k = A - \frac{c(t)}{k(t)} - (n+\delta) \end{aligned}$$

We would like to show $\gamma_k = \gamma_c$ using the fact that $c(t) = c(0)\exp\left[\frac{1}{\sigma}(A-\delta-\rho)t\right]$

$$\begin{aligned}
& \dot{k}(t) - (A-n-\delta)k(t) = -c(t) \\
\Rightarrow\quad & \frac{dk(t)\exp\left[-(A-n-\delta)t\right]}{dt} = -c(t)\exp\left[-(A-n-\delta)t\right] \\
\Rightarrow\quad & \int_0^t \frac{dk(v)\exp\left[-(A-n-\delta)v\right]}{dv}dv = \int_0^t -c(v)\exp\left[-(A-n-\delta)v\right]dv \\
\Rightarrow\quad & k(t)\exp\left[-(A-n-\delta)t\right] - k(0) = \int_0^t -c(0)\exp\left[\frac{1}{\sigma}(A-\delta-\rho)v\right]\exp\left[-(A-n-\delta)v\right]dv \\
\Rightarrow\quad & k(t)\exp\left[-(A-n-\delta)t\right] - k(0) = -c(0)\int_0^t \exp\left[-\left(\frac{(A-\delta)(\sigma-1)}{\sigma}+\frac{\rho}{\sigma}-n\right)v\right]vdv \\
\Rightarrow\quad & k(t)\exp\left[-(A-n-\delta)t\right] - k(0) = \frac{-c(0)}{-\left(\frac{(A-\delta)(\sigma-1)}{\sigma}+\frac{\rho}{\sigma}-n\right)}\left[\exp\left[\left(\frac{(A-\delta)(1-\sigma)}{\sigma}-\frac{\rho}{\sigma}+n\right)t\right]-1\right]
\end{aligned}$$

so that

$$\begin{aligned}
k(t) &= k(0)\exp(A-n-\delta)t \\
&\quad + \frac{c(0)}{\frac{(A-\delta)(\sigma-1)}{\sigma}+\frac{\rho}{\sigma}-n}\exp\left[(A-n-\delta)t\right]\left\{\exp\left[\left(\frac{(A-\delta)(1-\sigma)}{\sigma}-\frac{\rho}{\sigma}+n\right)t\right]-1\right\} \\
&= k(0)\exp(A-n-\delta)t + \frac{c(0)}{\frac{(A-\delta)(\sigma-1)}{\sigma}+\frac{\rho}{\sigma}-n}\left\{\exp\left[\frac{1}{\sigma}(A-\delta-\rho)t\right]-\exp\left[(A-n-\delta)t\right]\right\} \\
&= \left\{k(0)+\left[\frac{(A-\delta)(\sigma-1)}{\sigma}+\frac{\rho}{\sigma}-n\right]^{-1}c(0)\right\}\exp(A-n-\delta)t \\
&\quad + c(0)\left[\frac{(A-\delta)(\sigma-1)}{\sigma}+\frac{\rho}{\sigma}-n\right]^{-1}\exp\left[\frac{1}{\sigma}(A-\delta-\rho)t\right]
\end{aligned}$$

By TVC,

$$\lim_{T\to\infty}\exp\left[-(A-\delta-n)T\right]k(T) = 0$$

so we have

$$k(0) + \left[\exp\left(\frac{(A-\delta)(1-\sigma)}{\sigma}-\frac{\rho}{\sigma}+n\right)\right]^{-1}c(0) = 0$$

and hence

$$\begin{aligned}
k(t) &= c(0)\left[\frac{(A-\delta)(\sigma-1)}{\sigma}+\frac{\rho}{\sigma}-n\right]^{-1}\exp\left[\frac{1}{\sigma}(A-\delta-\rho)t\right] \\
&= k(0)\exp\left[\frac{1}{\sigma}(A-\delta-\rho)t\right]
\end{aligned}$$

which also implies the optimal initial consumption

$$c(0) = k(0)\left[\frac{(A-\delta)(\sigma-1)}{\sigma}+\frac{\rho}{\sigma}-n\right]$$

Implication is the similar as Solow Ak that change in patience ρ would change the long-run growth rate. This key property is due Ak techonology of non-diminishing return to capital and violation inada condition.

Beyond Neoclassical: Increasing return models

Increasing return implies $AC > MC$ as for a cost-minimizing firm, return to scale $\gamma = AC/MC$. See proof in appendix.
Hence, for increasing return, if we set $P = MC$, then firm would have perpetual losses as $AC > MC = P$. To have zero profit, we need to have $P \geq AC > MC$ but we need imperfect competition models if we are to model internal IRS. To retain perfect competition, we are to augment increasing return by externality.

Learning-by-doing models

Each producer i, production function $Y_i = F(K_i, EL_i)$.
By small firm assumption, E is taken as given. However, E is endogenous for whole economy.
Learning-by-doing implies productivity A would be proportional to discounted sum of past output:

$$E = \sum_{r=0}^{\infty} \beta^{t-r+1} Y_{t-r} = Y_t + \beta Y_{t-1} + \beta^2 Y_{t-2} + \cdots$$

However, by assuming fixed saving rate, we have

$$\begin{aligned}
K_t &= I_t + (1-\delta) K_{t-1} \\
&= sY_t + (1-\delta)(I_{t-1} + K_{t-2}) \\
&= sY_t + (1-\delta) sY_{t-1} + (1-\delta) s^2 Y_{t-2} + \cdots
\end{aligned}$$

Therefore, the production function could be rewritten as $Y_i = F(K_i, KL_i)$ so that F is HD 1 in K_i and K.
Key assumption is that new knowledge is costlessly and immediately available to all firms.
Assumptions:

1. Standard Ramsey Consumer
2. No exogenous technical change $g = 0$
3. No population growth $n = 0$

Firms' problem:
Every firm i would maximize his own profit:

$$\begin{aligned}
\max_{K_i, L_i} \pi_i &= F(K_i, KL_i) - WL_i - RK_i \\
&= L_i \left[F(k_i, K) - W - Rk_i \right]
\end{aligned}$$

so that the first order condition would be

$$\begin{aligned}
\frac{\partial \pi_i}{\partial K_i} = 0 \quad &\Rightarrow \quad L_i \left[F_{k_i}(k_i, K) \frac{1}{L_i} - R \frac{1}{L_i} \right] = 0 \\
&\Rightarrow \quad F_{k_i}(k_i, K) = R = r + \delta \\
\frac{\partial \pi_i}{\partial L_i} = 0 \quad &\Rightarrow \quad [F(k_i, K) - W - Rk_i] + L_i \left[F_{k_i}(k_i, K) \left(-\frac{K_i}{L_i^2} \right) - R \left(-\frac{K_i}{L_i^2} \right) \right] = 0 \\
&\Rightarrow \quad F(k_i, K) - Rk_i = W
\end{aligned}$$

Since all firms are identical, so that $k_i = k = K/L$ and that we can define

$$f(L) = F(1, L) = F\left(1, \frac{K}{K/L}\right) = \frac{1}{k} F(k, K) = \frac{1}{k_i} F(k_i, K)$$

and that $f'(L) > 0$ and $f''(L)$ since

$$f'(L) = F_L(1, L)$$

which, under neoclassical production function assumption would have decreasing positive marginal return of labor. Note that average product capital APK is invariant to K and increasing in L as

$$APK_i = \frac{F(K_i, KL_i)}{K_i} = F\left(1, \frac{K}{K_i/L_i}\right) = F\left(1, \frac{K}{K/L}\right) = f(L) = APK$$

so that K has no role in the average product.
Now rewrite the first order conditions by $f(L)$:

$$\begin{aligned}
F_{k_i}(k_i, K) &= \frac{\partial}{\partial k_i} F(k_i, K) = \frac{\partial}{\partial k_i} \left[k_i F\left(1, \frac{K}{K_i/L_i}\right) \right] = \frac{\partial}{\partial k_i} \left[k_i F\left(1, \frac{K}{k_i}\right) \right] \\
&= F(1, k_i K) - F_2(1, k_i K) K/k_i = F(1, L) - F_2(1, L) L \\
&= f(L) - Lf'(L)
\end{aligned}$$

In equilibrium:

$$\begin{aligned}
\frac{\dot{c}}{c} &= \frac{1}{\sigma}(r - \rho) \\
&= \frac{1}{\sigma}[f(L) - Lf'(L) - \delta - \rho]
\end{aligned}$$

and

$$\begin{aligned}
\dot{K} &= \textstyle\sum_i F(K_i, KL_i) - C - \delta \sum_i K_i \\
&= \textstyle\sum_i L_i F(k_i, K) - C - \delta K \\
&= F(k, K) \textstyle\sum_i L_i - C - \delta K \\
&= F(k, K) L - cL - \delta Lk \\
&= L[F(k, K) - c - \delta k] \\
&= L[kf(L) - c - \delta k]
\end{aligned}$$

so that

$$\dot{k} = kf(L) - c - \delta k$$

Hence, consumption grows at rate $[f(L) - Lf'(L) - \delta - \rho]/\sigma$. It can be shown (detailed in appendix) that in the case of social planner, the consumption would grow at $[f(L) - \delta - \rho]/\sigma$ because the social marginal product of labor is $f(L)$ as

$$F_k(k, K) = \frac{\partial}{\partial k} F(k, K) = \frac{\partial}{\partial k}[kf(L)] = f(L)$$

instead of the private one $f(L) - Lf'(L)$. This is due to the fact that the private firms would ignore the positive externality by accumulation of capital. The inefficiency could be reduced by government subsidy through lump sum tax. *(There are various way to do this — work on this later)*
There is **scale effect** on the growth rate as consumption would grow with the population. This originates from the fact that more people can have more idea and idea is non-rival so that every one would be more efficient.

Simple Idea-Based Growth Model

Basic principle is that idea is non-rival.
Setup:

1. Solow-style exogenous saving without explicit optimization
2. 2 sectors: final good sector and knowledge sector
3. No captial is needed in production (To keep economies in one good world)
4. Production function for output: $Y_t = A_t^\sigma L_{Y_t}$ where $\sigma > 0$. A is knowledge and idea and L_{Y_t} is labor force working on final goods sector. Y_t is CRS in L_{Y_t} but IRS in A_t and L_{Y_t}.
5. Production function for ideas: $\dot{A}_t = L_{A_t}\nu(A_t) = \nu L_{A_t} A_t^\phi$ where $\phi > 0$. L_{A_t} is labor force allocated to generating new ideas and A is number of idea existence. $\dot{A}_t$ (change in A_t, which in fact the quantitative measure of output of the research) is CRS in L but IRS in L and A. ϕ catches effect of "standing on shoulder effect" ($\phi > 1$) and "fishing out effect" ($\phi < 1$).
6. Labor force growth: $L_t = L_{A_t} + L_{Y_t} = L_0 e^{nt}$

Steady State:
By having Solow-style of saving, $L_{A_t} = sL_t$ and $L_{Y_t} = (1-s)L_t$, we have

$$Y_t = A_t^\sigma L_{Y_t} = (1-s) A_t^\sigma L_t$$

or

$$y_t \equiv \frac{Y_t}{L_t} = (1-s) A_t^\sigma$$

so that

$$\frac{\dot{y}}{y} = \sigma\frac{\dot{A}_t}{A_t} = \sigma\frac{\nu L_{A_t} A_t^{\phi}}{A_t} = \sigma\frac{\nu s L_t}{A_t^{1-\phi}}$$

Define growth rate of A by g_A :

$$g_A = \frac{\dot{A}_t}{A_t} = \frac{\nu s L_t}{A_t^{1-\phi}}$$

so that

$$\dot{g}_A = \frac{\nu s \dot{L}_t}{A_t^{1-\phi}} - (1-\phi)\frac{\nu s L_t}{A_t^{1-\phi}}\frac{\dot{A}_t}{A_t}$$

In steady state, $\dot{g}_A = 0$, implies

$$g_A = \frac{n}{1-\phi}$$

and

$$g_Y = \frac{\sigma n}{1-\phi}$$

As the growth rate is not related to L but n, it only has weak scale effect.

Richer idea-based growth model

Setup:

1. 3-sector economy
   (a) competitive firms produce final output using differentiated capital goods and labor
   (b) R&D sector produces new ideas, blueprints for new types of capital goods and sells patents
   (c) Monoplist buy patent and retn differentiated capital goods to final output producing firms
2. Production function for output:

$$Y_t = \left[\left(\int_0^{A_t} x_{it}^{\theta} di\right)^{1/\theta}\right]^{\alpha} L_{Y_t}^{1-\alpha}$$

   which is generalized version of Cobb-Douglas and known as "Love for variety" as raising A postpone diminishing returns.
3. Production function for ideas:

$$\dot{A}_t = \nu L_{A_t}^{\lambda} A_t^{\phi}$$

   and λ captures the effect of "Stepping on Toes": duplication of research at point of time

4. Capital Accumulation: $\dot{K}_t = Y_t - C_t - \delta K_t$
5. Resources Constraints:
    (a) Sum of different capital goods would be equal to existing K

$$\int_0^{A_t} x_{it} di = K_t$$

    (b) Labor force constraint: $L_{A_t} + L_{Y_t} = L_t = L_0 e^{nt}$

Equilibrium:
Rule of thumb method: Fix ratio at exogenous level (as true in steady state if it exists), and solved for implied growth rate. The process is correct if growth rate is inedpendent of assumed number.

1. Constant saving rate for investment of capital: $\bar{s}_K = 1 - C_t/Y_t$
2. Constant saving rate for investment of ideas: $\bar{s}_A = L_{A_t}/L_t$
3. Full uniform use of all varities of capital: $x_{it} = \bar{x} = K_t/A_t$

With the above rules, we write the production function final goods as:

$$\begin{aligned} Y_t &= \left[\left(\int_0^{A_t} x_{it}^{\theta} di\right)^{1/\theta}\right]^{\alpha} L_{Y_t}^{1-\alpha} \\ &= \left(\int_0^{A_t} \left(\frac{K_t}{A_t}\right)^{\theta} di\right)^{\alpha/\theta} L_{Y_t}^{1-\alpha} \\ &= \left(\frac{K_t}{A_t}\right)^{\alpha} A_t^{\alpha/\theta} L_{Y_t}^{1-\alpha} \\ &= A_t^{\sigma} K_t^{\alpha} L_{Y_t}^{1-\alpha} \end{aligned}$$

where $\sigma \equiv \alpha\left(\frac{1}{\theta} - 1\right)$. Simplify it to our usual Solow format,

$$\begin{aligned} Y_t &= K_t^{\alpha}\left[A_t^{\sigma/(1-\alpha)} L_{Y_t}\right]^{1-\alpha} \\ &= K_t^{\alpha}\left[E_t L_{Y_t}\right]^{1-\alpha} \end{aligned}$$

where $E_t \equiv A_t^{\sigma/(1-\alpha)}$, so that

$$y_t \equiv \frac{Y_t}{E_t L_{Y_t}} = \left(\frac{K_t}{E_t L_{Y_t}}\right)^{\alpha}$$

Note that for $y_t = k_t^{\alpha}$ the steady state would be

$$y^* = \left(\frac{s_K}{n + g_E + \delta}\right)^{\alpha/(1-\alpha)}$$

and now that

$$\frac{Y^*}{L_{Y^*}} = E^* y^* = A^{*\sigma/(1-\alpha)} \left(\frac{s_K}{n + g_E + \delta} \right)^{\alpha/(1-\alpha)}$$

or

$$\begin{aligned} \hat{y}^* &\equiv \frac{Y^*}{L^*} = \frac{Y^*}{L_{Y^*}} (1 - \bar{s}_A) \\ &= (1 - \bar{s}_A) A^{*\sigma/(1-\alpha)} \left(\frac{s_K}{n + g_E + \delta} \right)^{\alpha/(1-\alpha)} \end{aligned}$$

Second, consider the idea production,

$$\dot{A}_t = \nu L_{A_t}^{\lambda} A_t^{\phi}$$

and denote $g_A = \dot{A}/A$ would result in

$$g_A = (A_t^*)^{\phi-1} \nu L_{A^*}^{\lambda}$$

or

$$A^* = \left[\frac{\nu}{g_A} \right]^{1/(1-\phi)} (\bar{s}_A L_t)^{\lambda/(1-\phi)}$$

by settting steady state $\dot{g}_A = 0$, we have

$$\dot{g}_A = \nu \lambda L_{A_t}^{\lambda-1} A_t^{\phi-1} \dot{A}_t + \nu (\phi - 1) L_{A_t}^{\lambda-1} A_t^{\phi-2} \dot{A}_t = 0$$

or

$$\lambda \frac{\dot{L}_{A_t}}{L_{A_t}} + (\phi - 1) \frac{\dot{A}_t}{A_t} = 0$$

Now, given constant saving rate rule, $\dot{L}_{A_t}/L_{A_t} = \dot{L}_t = L_t = n$,

$$g_A = \frac{\lambda n}{1 - \phi}$$

As in solow technology augmented model,

$$g_{\hat{y}} = g_{\hat{k}} = g_E = \frac{\sigma}{1-\alpha} g_A = \frac{\lambda n \sigma}{(1-\phi)(1-\alpha)}$$

Now since g_A is independent of s_A and $g_{\hat{y}}$, $g_{\hat{k}}$ are indpedent of s_K so that our procedure is correct.
This model has only weak scale effect as

$$g_A = \frac{\lambda n}{1 - \phi}$$

is independent of population but proportional to growth rate of labor. Taking $\phi = 1$, we have a strong skill effect as in the simple model:

$$\dot{A} = \nu L_A^{\lambda} A \Rightarrow g_A = \frac{\dot{A}}{A} = \nu L_A^{\lambda}$$

Full Derivation: (See appendix)

Conclusion on Endogenous Growth Model

1. First-generation endogenous growth models (learning-by-doing, simple idea-based model) are inconsistent with R&D/TFP evidence
2. Second generation endogenous growth models (rich idea-based model) show that TFP growth can arise from economic decisions taken by indiviudals and firms
3. Policy only has level effect but not growth effect; long-run growth rate depends on parameters

Kremer's Test on Population on growth

Basic model:

1. Production function with fixed supply of land T: $Y_t = T^{\alpha}\left(A_t L_t\right)^{1-\alpha}$
2. Idead production function: $\dot{A}_t = B L_t A_t^{\phi}$
3. Malthus assumption:
$$\frac{Y_t}{L_t} = \bar{y}$$

Equilibrium:

$$\bar{y} = \frac{T^{\alpha}\left(A_t L_t\right)^{1-\alpha}}{L_t}$$

or

$$L_t = \left[\frac{1}{\bar{y}}\right]^{\alpha} A_t^{(1-\alpha)/\alpha}$$

so that when we assume $\phi = 1$,

$$\frac{\dot{L}_t}{L_t} = \frac{1-\alpha}{\alpha}\frac{\dot{A}_t}{A_t} = \frac{1-\alpha}{\alpha} B L_t$$

Kremer's (1993) finding strongly supports the scale effect but not fully decides the issue of scale effect.

Chapter 2

Elements of Aggregate Demand

2.1 Fixed Investment

`Lecture Note: #22,#23,#24,#25`
`Textbook: Romer ch.3,8;BSM ch.3;B-F,ch.2.4`

Tobin's q

Tobin (1969) proposed $I = I(q)$ against Keyesnian assumption of $I = I(r)$

$$q = \frac{\text{value of the firm}}{\text{replacement cost of capital}}$$

Reasoning:

$$\begin{aligned} q &> 1 \Rightarrow \text{issue stock to buy capital} \\ q &= 1 \Rightarrow \text{no investment} \\ q &< 1 \Rightarrow \text{selling capital} \end{aligned}$$

Investment model with adjustment costs:

Assumption:

1. Adjustment cost $C(I_t, K_t)$ is high when investment is high $\partial C_t / \partial I_t > 0$
2. Symmetric cost: cost incurred for installation or removal
3. Cost form: convex (usually assumed to be function of I_t / K_t)
4. Interest rate r is constant for simplicity
5. Neoclassical firm: CRS, competitive in output and factor markets

6. Firm acts to maximize shareholder value (no agency problem)

Firm's Problem:

$$\max_{L_t, I_t} V_0 = \int_0^\infty e^{-rt} \left\{ F\left(K_t, L_t\right) - W_t L_t - I_t \left[1 + \phi\left(\frac{I_t}{K_t}\right)\right] \right\} dt$$

$$\begin{aligned} s.t. \qquad & \dot{K}_t = I_t - \delta K_t \\ & K_0 \text{ is given} \end{aligned}$$

with assmption that $\phi' > 0, \phi'' > 0$ and $\phi(0) = 0$. (Note this is the convex assumption)
The currrent-value Hamiltonian would be

$$H = e^{-rt} \left\{ F\left(K_t, L_t\right) - W_t L_t - I_t \left[1 + \phi\left(\frac{I_t}{K_t}\right)\right] + q_t \left(I_t - \delta K_t\right) \right\}$$

where the F.O.C.s are

$$\begin{aligned} \frac{\partial H}{\partial L_t} = 0 \quad & \Rightarrow \quad F_L\left(K_t, L_t\right) = W \\ \frac{\partial H}{\partial I_t} = 0 \quad & \Rightarrow \quad 1 + \phi\left(\frac{I_t}{K_t}\right) + \frac{I_t}{K_t}\phi'\left(\frac{I_t}{K_t}\right) = q_t \end{aligned}$$

and

$$\begin{aligned} \frac{de^{-rt} q_t}{dt} = -\frac{\partial H}{\partial K_t} \quad & \Rightarrow \quad -re^{-rt} q_t + e^{-rt}\dot{q}_t = -e^{-rt}\left[F_K\left(K_t, L_t\right) + \left(\frac{I_t}{K_t}\right)^2 \phi'\left(\frac{I_t}{K_t}\right) - \delta q_t\right] \\ & \Rightarrow \quad \dot{q}_t = (r + \delta) q_t - F_K\left(K_t, L_t\right) - \left(\frac{I_t}{K_t}\right)^2 \phi'\left(\frac{I_t}{K_t}\right) \end{aligned}$$

with TVC being

$$\lim_{t \to \infty} e^{-rt} q_t K_t = 0$$

Interpreatation:
By FOC of I_t, we can write

$$\frac{I_t}{K_t} = h\left(q_t\right) \quad \text{where} \quad h' > 0$$

and in steady state so that $\dot{K} = 0$ os that $I_t = \delta K_t$, we have

$$q^* = 1 + \phi(\delta) + \delta\phi'(\delta) > 1$$

From the $\dot{q}_t$ equation we have

$$F_K\left(K_t, L_t\right) + \left(\frac{I_t}{K_t}\right)^2 \phi'\left(\frac{I_t}{K_t}\right) = (r + \delta) q_t - \dot{q}_t$$

or

$$\frac{F_K\left(K_t, L_t\right)+\left(\frac{I_t}{K_t}\right)^2 \phi'\left(\frac{I_t}{K_t}\right)}{q_t}+\frac{\dot{q}_t}{q_t}-\delta=r$$

so that when there is no adjustemet cost ($q_t=1, \dot{q}_t=0$ and $\phi=0, \phi'=0$), we get the usual rental cost equation $r=F_K-\delta$.
It means that the real interest rate would be equal to the rate of marginal return of captial $F_K\left(K_t, L_t\right)+\left(\frac{I_t}{K_t}\right)^2 \phi'\left(\frac{I_t}{K_t}\right)$ plus change in shadow value of captial $\dot{q}_t/q_t$ minus depreciation δ so that the firm is indifferent between investing in K and buying bonds. The reason for $\left(\frac{I_t}{K_t}\right)^2 \phi'\left(\frac{I_t}{K_t}\right)$ is due to the saving in adjustment cost due to having more capital.
Moreover, from the $\dot{q}_t$ equation,

$$-\left[\dot{q}_t-(r+\delta) q_t\right]=F_K\left(K_t, L_t\right)+\left(\frac{I_t}{K_t}\right)^2 \phi'\left(\frac{I_t}{K_t}\right)$$

or

$$-\int_t^{\infty} \frac{d e^{-(r+\delta)(v-t)} q_v}{d v} d v=\int_t^{\infty} e^{-(r+\delta)(v-t)}\left[F_K\left(K_v, L_v\right)+\left(\frac{I_v}{K_v}\right)^2 \phi'\left(\frac{I_v}{K_v}\right)\right] d v$$

so that

$$q_t=-\left[\lim_{v \rightarrow \infty} e^{-(r+\delta)(v-t)} q_t-q_t\right]=\int_t^{\infty} e^{-(r+\delta)(v-t)}\left[F_K\left(K_v, L_v\right)+\left(\frac{I_v}{K_v}\right)^2 \phi'\left(\frac{I_v}{K_v}\right)\right] d v$$

where TVC implies $\lim_{v \rightarrow \infty} e^{-(r+\delta)(v-t)} q_t=0$. Therefore, this implies that shadow value of capital would be marginal product of capital discounnt for interest r and depreciation δ.

Transitional dynamics and steady state:
The system is characterized by

$$\begin{aligned}
\dot{K}_t &= I_t-\delta K_t=h\left(q_t\right) K_t-\delta K_t \\
\dot{q}_t &= (r+\delta) q_t-F_K\left(K_t, L_t\right)-\left(\frac{I_t}{K_t}\right)^2 \phi'\left(\frac{I_t}{K_t}\right)
\end{aligned}$$

In steady state, we have $\dot{K}_t=0$ and $\dot{q}_t=0$:

$$\begin{aligned}
\dot{K}_t &= 0 \Rightarrow h\left(q_t\right)=\delta \Rightarrow q=q^*>1 \\
\dot{q}_t &= 0 \Rightarrow (r+\delta) q_t=F_K\left(K_t, L_t\right)+\left(\frac{I_t}{K_t}\right)^2 \phi'\left(\frac{I_t}{K_t}\right)
\end{aligned}$$

Note that

$$\frac{\partial q_t}{\partial K_t}=\left[F_{KK}\left(K_t, L_t\right)-2\left(\frac{I_t^2}{K_t^3}\right) \phi'\left(\frac{I_t}{K_t}\right)-\frac{I_t^3}{K_t^4} \phi''\left(\frac{I_t}{K_t}\right)\right]<0$$

Neoclassical Interpretation of Tobin's q

Tobin's average concept $q_t = V_t/K_t$
However, neoclassical implies $q_t = \partial V_t/\partial K_t$ which is marginal concept.
Only when V is linear then $V_t = \alpha K_t$ so that $V_t/K_t = \alpha = \partial V_t/\partial K_t$.

Hayashi Theorem

Marginal q equal to average q. (Neoclassical q is Tobin's q.)
Assumptions:

1. Need $\pi(K) = \max_L F(K,L) - WL$ be linear in K (implies zero profit)
2. Constant return in investment: $C(I,K)$ is HD1 in I and K
3. Capital goods needs to be homogenous
4. Stock market must be efficient with symmetric information between market and manager

Proof:

$$\begin{aligned}
\frac{d}{dt}V_t &= \frac{d}{dt}(q_t K_t) \\
&= \dot{q}_t K_t + \dot{K}_t q_t \\
&= K_t\left[(r+\delta)q_t - F_K(K_t,L_t) - \left(\frac{I_t}{K_t}\right)^2 \phi'\left(\frac{I_t}{K_t}\right)\right] + (I_t - \delta K_t) q_t \\
&= K_t\left[rq_t - F_K(K_t,L_t) - \left(\frac{I_t}{K_t}\right)^2 \phi'\left(\frac{I_t}{K_t}\right)\right] + I_t q_t \\
&= K_t\left[rq_t - F_K(K_t,L_t) - \left(\frac{I_t}{K_t}\right)^2 \phi'\left(\frac{I_t}{K_t}\right)\right] + I_t\left[1 + \phi\left(\frac{I_t}{K_t}\right) + \left(\frac{I_t}{K_t}\right)\phi'\left(\frac{I_t}{K_t}\right)\right] \\
&= rq_t K_t - K_t F_K(K_t,L_t) + I_t + I_t\phi\left(\frac{I_t}{K_t}\right)
\end{aligned}$$

By Euler's Thm, we have $F_K(K_t,L_t)K_t + F_L(K_t,L_t)L_t = F(K_t,L_t)$, so

$$\begin{aligned}
& \frac{d}{dt}(q_t K_t) = rq_t K_t + F_L(K_t,L_t)L_t - F(K_t,L_t) + I_t\left[1 + \phi\left(\frac{I_t}{K_t}\right)\right] \\
\Rightarrow\ & \frac{d}{dt}\left(q_t K_t e^{-rt}\right) = e^{-rt}\left\{WL_t - F(K_t,L_t) + I_t\left[1 + \phi\left(\frac{I_t}{K_t}\right)\right]\right\} \\
\Rightarrow\ & \int_0^\infty \frac{d}{dt}\left(q_t K_t e^{-rt}\right) dt = -\int_0^\infty e^{-rt}\left\{F(K_t,L_t) - WL_t - I_t\left[1 + \phi\left(\frac{I_t}{K_t}\right)\right]\right\} dt \\
\Rightarrow\ & \lim_{t\to\infty} q_t K_t e^{-rt} - q_0 K_0 = V_0 \\
\Rightarrow\ & q_0 = V_0/K_0
\end{aligned}$$

Empirical Result

Method I: Taking

$$C\left(I,K\right)=I+\phi\left(\frac{I}{K}\right)I\quad\text{and}\quad\phi\left(\frac{I}{K}\right)=a\left(\frac{I}{K}\right)^{2}$$

we have optimal condition: (the differentiation is made w.r.t to I_t)

$$\begin{aligned}
& 1+\phi\left(\frac{I_t}{K_t}\right)+\frac{I_t}{K_t}\phi'\left(\frac{I_t}{K_t}\right)=q_t\\
\Rightarrow\quad & 1+a\frac{I_t^2}{K_t^2}+\frac{I_t}{K_t}2a\frac{I_t}{K_t}=q_t\\
\Rightarrow\quad & \frac{I_t}{K_t}=\left(\frac{1}{3a}\right)^{1/2}\left(q_t-1\right)^{1/2}
\end{aligned}$$

Method II:Taking

$$C\left(I,K\right)=I+\phi\left(\frac{I}{K}\right)K\quad\text{and}\quad\phi\left(\frac{I}{K}\right)=a\left(\frac{I}{K}\right)^{2}$$

so that the optimal condition:

$$\begin{aligned}
& 1+\phi'\left(\frac{I_t}{K_t}\right)=q_t\\
\Rightarrow\quad & 1+2a\left(\frac{I_t}{K_t}\right)=q_t\\
\Rightarrow\quad & \frac{I_t}{K_t}=\frac{1}{2a}\left(q_t-1\right)
\end{aligned}$$

Hence, q is sufficient statisitcs.
In empirical studies, firm cash flow is found to be significiant in the regression I_t/K_t on V_t/K_t. One explanation is that credit market imperfection makes external borrowing more expensive than interal funds and other interpretation could be cash flow is proxying some omitted fundamental factor.

Interpretation

If there is no adjustment cost, investment would adjust immediately when there is shock in productivity. However, in real world, we have such cost so that the adjustment would be made overtime. The reason for such huge magnitude is that $K/Y\approx 3$ but $I/Y\approx 0.2$ so that little adjustment in K would result in huge adjustment in I.
Tobin's q is found from valuation of firm, which usually comes from stock market. Is pricing from stock market useful for manager to make decision? It would depend on whether the valuation is based on fundamentals or bubbles. (See Blanchard,Rhee & Summers 1993)

Extension to fixed cost

Simple fixed adjustment cost model:

$$C\left(I_t, K_t\right)=\left\{\begin{array}{cl} F K_t+I_t\left[1+\phi\left(I_t / K_t\right)\right] & \text{if } I \neq 0 \\ 0 & \text{if } I=0 \end{array}\right.$$

The result is we have region of inaction which fits the empirical data. Moreover, region of inaction implies policy would be state dependent. (See Adda-Copper, "Balladurette & Juppette", JPE 2002)
To model irreversible investment, we might model it as resale price < purchase price.
Aggregating over heterogeneous firms smooths out sharp implications of fixed-cost model. This implies convex adjustment costs may be a good approximation for industry data. (Cooper-Willis, 2003)

Bubbles

No arbitrage equation:

$$\frac{E_t\left(p_{t+1}+d_{t+1}\right)}{p_t}=1+r$$

implies

$$\begin{aligned} p_t &= \frac{E_t\left(p_{t+1}+d_{t+1}\right)}{1+r} \\ &= \frac{E_t d_{t+1}}{1+r}+\frac{1}{1+r} E_t\left[\frac{E_{t+1}\left(p_{t+2}+d_{t+2}\right)}{1+r}\right] \\ &= \frac{E_t d_{t+1}}{1+r}+\frac{E_t E_{t+1} d_{t+2}}{(1+r)^2}+\frac{E_t E_{t+1} p_{t+2}}{(1+r)^2} \\ &= \frac{E_t d_{t+1}}{1+r}+\frac{E_t d_{t+2}}{(1+r)^2}+\frac{E_t p_{t+2}}{(1+r)^2} \\ &= E_t \sum_{s=0}^{\infty} \frac{d_{t+s}}{(1+r)^s}+\lim_{T \rightarrow \infty} \frac{E_t p_{t+T}}{(1+r)^T} \end{aligned}$$

Assumption:

1. dividend do not grows faster than interest rate
2. present value of terminal price is zero

These two implies no bubbles.If we drops, the second assumption, we can define price to be sum of fundamentals and bubble term:

$$p_t=p_t^*+b_t$$

where $E_t\left(b_{t+1} \mid I_t\right)=(1+r) b_t$. Examples of bubble includes:

1. non-stochastic bubble:

$$b_{t+1} = (1+r)\, b_t$$

2. bursting bubble:

$$b_{t+1} = \begin{cases} \dfrac{1+r}{q} b_t + e_{t+1} & \text{with probability } q \\ e_{t+1} & \text{with probability } 1-q \end{cases}$$

Tirole (1982, 1985) proved that:

1. There cannot be bubble for a economy with finite number of infinitely living agents.

2. There can be bubble for a economy with infinite number of infinitely living agents only if the economy is dynamically inefficient.

Shiller's Volatility Test

If markets are fully efficient, then (1) non-bubble and (2) rational expectation. Define ex-post optimal price p^*:

$$p_t^* = \sum_{s=1}^{\infty} (1+r)^{-s}\, d_{t+s}$$

with no bubble assumption,

$$p_t^* = p_t + u_t$$

and rational expectation

$$E\left(u_t \mid p_t\right) = 0 \Rightarrow cov\left(p_t, u_t\right) = 0$$

so we have

$$Var\left(p_t^*\right) = Var\left(p_t\right) + Var\left(u_t\right) \geq Var\left(p_t\right)$$

Shiller (1981) asserted that measurement error of estimated p^* would be biased towards null hypothesis as it would only increase the variance of p_t^* so it does not affect the result. Marsh-Merton (1983) denied the test by the claim that p and d has unit roots. Campbell-Shhiller (1987) allowed time-varying r, still rejecting null but less dramatic. Kleidon (1986) quoted "peso problems", which means stock price factoring improbable events in the stock price, but Shiller replied that implied crash would have to be greater than Great Depression. Overall, this is a strong blow to to Efficient Market Hypothesis (EMH).

Small Open Economy

This is de facto a partial equilibrium analysis.
Setup:

1. Open implies free movement of capital (consumer can lend and borrow without bounded by local investment opportunity)
2. small implies interest rate is exogenous (consumer can lend and borrow at fixed rate)
3. Consumer: Ramsey
4. Producer: q-theory

Representaive Firm problem:

$$\begin{aligned}
\max_{K_t, L_t} V_0 &= \int_0^\infty e^{-rt} \Pi_t dt \\
s.t. \quad \Pi_t &= F(K_t, L_t) - W_t L_t - I_t \left[1 + \phi\left(\frac{I_t}{K_t}\right)\right] \\
\dot{K}_t &= I_t - \delta K_t \\
K(0) &= K_0
\end{aligned}$$

with F.O.C.s being

$$\begin{aligned}
\frac{I_t}{K_t} &= h(q_t), h' > 0 \Rightarrow \dot{K}_t = h(q_t) K_t - \delta K_t \\
\dot{q}_t &= (r+\delta) q_t - F_{Kt} - \left[\frac{I_t}{K_t}\right]^2 \phi'\left(\frac{I_t}{K_t}\right) \\
\lim_{t\to\infty} e^{-rt} q_t K_t &= 0
\end{aligned}$$

Household problem: (with $n = 0$, $g = 0$)

$$\begin{aligned}
\max_{c(t)} \quad & \int_0^\infty e^{-\rho t} \frac{c(t)^{1-\sigma}}{1-\sigma} dt \\
s.t \quad & \dot{D}(t) = rD(t) + C(t) - W(t) - \Pi(t) \\
& D(0) = D_0 \\
& \lim_{T\to\infty} e^{-rT} D(T) \leq 0
\end{aligned}$$

with F.O.C.s being

$$\begin{aligned}
\frac{\dot{C}}{C} &= \frac{1}{\sigma}(r-\rho) \\
\dot{D}(t) &= rD(t) + C(t) - W(t) - \Pi(t) \\
\lim_{T\to\infty} e^{-rT} \lambda(T) D(T) &= 0
\end{aligned}$$

Now assume $r = \rho$ so that $C(t) = \bar{C} > 0$ for all t. This implies $\lambda(T) < \infty$, so we can simplify TVC into

$$\lim_{T\to\infty} e^{-rT} D(T) = 0$$

Equilibrium:
Define trade surplus be $NX = X - M$

$$\begin{aligned} NX &= Y - C - I\left[1 + \phi(I/K)\right] \\ &= S - I(1+\phi) \end{aligned}$$

and then current account deficit is change in D:

$$\begin{aligned} CA &= rD - NX \\ &= rD - \left[S - I(1+\phi)\right] \\ &= rD - Y + C + I(1+\phi) \\ &= rD - \Pi - WL - I(1+\phi) + C + I(1+\phi) \\ &= rD + C - WL - \Pi \\ &= \dot{D} \end{aligned}$$

Household consumption could be found by integrate budget constraint from 0 to ∞ by integrating facotr e^{-rt}:

$$\int_0^\infty \frac{de^{-rt}D(t)}{dt}dt = \int_0^\infty e^{-rt}\left[C(t) - W(t) - \Pi(t)\right]dt$$

or

$$-D(0) = \bar{C}\int_0^\infty e^{-rt}dt - \int_0^\infty e^{-rt}\left[F(K_t, L_t) - I_t\left(1 + \phi\left(\frac{I_t}{K_t}\right)\right)\right]dt$$

so that

$$\bar{C} = r\left\{\int_0^\infty e^{-rt}\left[F(K_t, L_t) - I_t\left(1 + \phi\left(\frac{I_t}{K_t}\right)\right)\right]dt - D(0)\right\}$$

Steady State: Open economy separates consumption and investment. By borrowing and lending throught international perfect credit market, consumption can immediately reach the steady state but capital adjustment needs time as there is cost associated with that. Therefore, the reactions to shocks are completely different from other models.
Shocks:
Suppose producitn function is $Y = AF(K, L) + B$

1. Additive permanent shock

   Suppose B increases permanently, it implies the production Y goes up permenantly. From $\bar{C}$ equation, C would also follows. As $\dot{q}$ equation has no change, there is no change in K, I and q. As there is no change in I, so Δnet output $= \Delta F(K, L) - \Delta I(1+\phi) = \Delta Y$ implies $\Delta C = \Delta B = \Delta Y$ and $\Delta D = 0$. Hence, $NX = Y - C - I(1+\phi)$ should have no change and $CA = rD - NX$ should also have no change.

2. Additive temporary shock

   The shock would be in the form of

   $$B_t = \alpha B_{t-1} + e_t,\ 0 < \alpha < 1$$

   Suppose B_t increase temporarily, it implies that production Y would goes up for a while and return to original level eventually. Since $\bar{C}$ would adjust to smooth consumption, the change of C would be less than change of Y, that is $\Delta C < \Delta Y$ initally but $\Delta C > \Delta Y$ after some time because consumer is to save transitory boost of income and spread it over time, so $\Delta D < 0$ initially and eventually $\Delta D > 0$. Again, note that $\dot{q}$ equation remains unchange so that K, I and q stay the same. Now, that $NX = Y - C - I\left(1 + \phi\right)$, ΔNX should be positive at the beginning but negative later. Moreover, $CA = rD - NX$, so that ΔCA should be negative at the beginning but positive later.

3. Multiplicative permanent shock

   Increase in A would change MP_K so that $\dot{q}$ equation is changed so that steady state K^* increased. In transition, q would first goes up and then go back to original state while I is increasing. Since MP_K goes up, with increasing captial stock, Y is increasing. Since Δnet output $= \Delta F\left(K, L\right) - \Delta I\left(1 + \phi\right)$ would depends on the relative size of two changes during the transition but would be strictly positive as I would be zero in steady state, ΔC would be increasing as net output should have overall increase in presevent value term. Since $NX = Y - C - I\left(1 + \phi\right)$, the initial change in NX is ambigous so that the final result is also ambiguous which reverses the beginning movement. CA also have the same ambiguity as rD and NX are uncertain.

Extension:
Borrowing country has strong incentive to default when NX turns. If lenders know this, they won't lend. Institutions can easily force a private firm to repay but not the government. By Barro, Mankiw & Sala-i-Martin (1995), it may be easier to enforce repayment for physical capital than human capital.
Kehoe & Perri (2002) adopted endogenous borrowing constraint by assuming no exernal agency enforces repayment. Lenders can threaten not to lend agian if borrower defaults. Threat would make sense if ther is uncertainty or business cycle.

Part II

Pre-EC751

Chapter 3

Real-Business Cycle Theory (Romer Ch.4)

3.1 Facts about fluctuations

1. Fluctuations do not exhibit any simple regular or cyclical pattern
2. Fluctuations are distributed very unevenly over the components of output: Investment is the major factor!
3. Asymmetry in output movement
   (a) there are no large asymmetries between rises and falls in output; that is, output growth is distributed roughly symmetrically around its mean
   (b) output seems to be characterized by relatively long periods when it is slightly above its usual path, interrupted by brief periods when it is relatively far below
4. Okun's law: shortfalll in GDP of 3 percent relative to normal growth produces 1 percentage-point rise in unemployment rate

3.2 Theoris of Fluctuation

Traditionally, two major schools of thought: Classical v.s. Keynesian.

1. Classical school: Walarisan equilibrium, micro-foundation, flexible price, optimizing behaviour, fluctuation are result of shocks from technology, real business cycle model(RBC model)
2. Keyneisan school: behavior are assumed, rigidity in real variable adjustment (sticky price, wage)

3.3 Baseline RBC Model

Setup:

1. large number of identical price-taking firms
2. large number of identical price-taking households
3. Infinitely lived household
4. Each household has to decide houeshold consumption c and lesiure $1-\ell$ with maximization on lifetime utility:

$$\max U = \sum\nolimits_{t=0}^{\infty} e^{-\rho t} u_t\left(c_t, \ell_t\right) \frac{N_t}{H}$$

   where $u(\cdot)$ is assumed to be

$$u\left(c_t, \ell_t\right) = \ln c_t + b \ln\left(1-\ell_t\right)$$

5. Production function:

$$Y_t = K_t^{\alpha}\left(A_t L_t\right)^{1-\alpha}, 0<\alpha<1$$

6. Capital accumulation:

$$\begin{aligned} K_{t+1} &= K_t + I_t - \delta K_t \\ &= K_t + Y_t - C_t - G_t - \delta K_t \end{aligned}$$

7. Population Growth: $\ln N_t = \bar{N} + nt$ which implies $N_t = \exp\left\{\bar{N} + nt\right\}$.
8. Technology growth:

$$\ln A_t = \bar{A} + gt + \tilde{A}_t$$

   where in the absence of shock $\ln A_t = \bar{A} + gt$ and the shock $\tilde{A}_t$ takes the AR(1) process:

$$\tilde{A}_t = \rho_A \tilde{A}_{t-1} + \varepsilon_{A,t}, \quad -1 < \rho_A < 1$$

9. Government's purchases are financed by lump-sum taxes that are assumed to equal to purchases each period. Hence, G_t should grow as the same rate as A_t in steady state:

$$\ln G_t = \bar{G} + gt + \tilde{G}_t$$

   and

$$\tilde{G}_t = \rho_G \tilde{G}_{t-1} + \varepsilon_{G,t}, \quad -1 < \rho_G < 1$$

Firms: Given perfect competition**,**

$$\begin{aligned} w_t &= F_L\left(K_t, L_t\right) = (1-\alpha) K_t \\ r_t &= F_K\left(K_t, L_t\right) - \delta = \alpha\left(\frac{A_t L_t}{K_t}\right)^{1-\alpha} - \delta \end{aligned}$$

3.4 Household Behavior

Case1: One Period Life and No initial Wealth
Household maximization becomes

$$\begin{array}{ll} \max\limits_{c_t,\ell_t} & u\left(c_t,\ell_t\right)=\ln c_t+b\ln\left(1-\ell_t\right) \\ s.t. & c_t\le w_t\ell_t \end{array}$$

so that the Lagrangian would be

$$\mathcal{L}=\ln c_t+b\ln\left(1-\ell_t\right)+\lambda\left(w_t\ell_t-c_t\right)$$

with FOCs being

$$\begin{array}{rcl} \dfrac{\partial\mathcal{L}}{\partial c_t}=0 & \Rightarrow & \dfrac{1}{c_t}-\lambda=0 \\ & \Rightarrow & \lambda=c_t^{-1} \\ \dfrac{\partial\mathcal{L}}{\partial \ell_t}=0 & \Rightarrow & \dfrac{-b}{1-\ell_t}+w_t\lambda=0 \\ & \Rightarrow & \dfrac{b}{1-\ell_t}=\dfrac{w_t}{c_t}=\ell_t \\ & \Rightarrow & \ell_t=\dfrac{1}{1+b} \end{array}$$

so that

$$c_t=\frac{w_t}{1+b}$$

Rmk: The reason for the independence between ℓ_t and w_t is log-linear cancel the income and substitution effect.
Case2: Two period life and no endowment
Household maximization becomes

$$\begin{array}{ll} \max\limits_{c_t,c_{t+1},\ell_t,\ell_{t+1}} & u\left(c_t,c_{t+1},\ell_t,\ell_{t+1}\right)=\ln c_t+b\ln\left(1-\ell_t\right)+e^{-\rho}\left[\ln c_{t+1}+b\ln\left(1-\ell_{t+1}\right)\right] \\ s.t. & c_t+s_t\le w_t\ell_t \\ & c_{t+1}\le w_{t+1}\ell_{t+1}+\left(1+r_{t+1}\right)s_t \end{array}$$

so that the Lagrangian would be

$$\begin{array}{rcl} \mathcal{L} & = & \ln c_t+b\ln\left(1-\ell_t\right)+e^{-\rho}\left[\ln c_{t+1}+b\ln\left(1-\ell_{t+1}\right)\right] \\ & & +\lambda\left(w_t\ell_t+\dfrac{1}{1+r_{t+1}}w_{t+1}\ell_{t+1}-c_t-c_{t+1}\right) \end{array}$$

with FOCs being

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_t} &= 0 \quad \Rightarrow \quad \frac{1}{c_t} - \lambda = 0 \\
\frac{\partial \mathcal{L}}{\partial c_{t+1}} &= 0 \quad \Rightarrow \quad \frac{e^{-\rho}}{c_{t+1}} - \frac{\lambda}{1+r_{t+1}} = 0 \\
\frac{\partial \mathcal{L}}{\partial \ell_t} &= 0 \quad \Rightarrow \quad \frac{-b}{1-\ell_t} + w_t \lambda = 0 \\
\frac{\partial \mathcal{L}}{\partial \ell_{t+1}} &= 0 \quad \Rightarrow \quad \frac{-e^{-\rho} b}{1-\ell_{t+1}} + \frac{w_{t+1}\lambda}{1+r_{t+1}} = 0
\end{aligned}$$

which implies Euler equation to be

$$\frac{c_{t+1}}{c_t} = e^{-\rho}\left(1+r_{t+1}\right)$$

and intertemporal substitution equation would be

$$\frac{1-\ell_t}{1-\ell_{t+1}} = \frac{1}{e^{-\rho}\left(1+r_{t+1}\right)} \frac{w_{t+1}}{w_t}$$

which implies relative labor supply depends on real interest rate and real wage.
Case3: Infinite life and no endowment with uncertainty
ousehold maximization becomes

$$\begin{aligned}
\max_{\{c_t\}_{t=0}^{\infty}, \{\ell_t\}_{t=0}^{\infty}} U &= \sum\nolimits_{t=0}^{\infty} e^{-\rho t}\left[\ln c_t + b \ln\left(1-\ell_t\right)\right] \frac{N_t}{H} \\
&= \sum\nolimits_{t=0}^{\infty} e^{-\rho t}\left[\ln c_t + b \ln\left(1-\ell_t\right)\right] \frac{1}{H} e^{\bar{N}+nt} \\
s.t. \quad & c_t + s_t e^{-n} \leq w_t \ell_t + \left(1+r_t\right) s_{t-1}
\end{aligned}$$

Define maximum function to be

$$V\left(s_{t-1}, r_t\right) = \frac{e^{\bar{N}}}{H} \max_{c_t, \ell_t, s_t} \ln c_t + b \ln\left(1-\ell_t\right) + e^{-\rho} E_t V\left(s_t, r_{t+1}\right)$$

$$s.t. \quad s_t \leq e^n\left[w_t \ell_t + \left(1+r_t\right) s_{t-1} - c_t\right]$$

or

$$V\left(s_{t-1}, r_t\right) = \frac{e^{\bar{N}}}{H} \max_{c_t, \ell_t} \ln c_t + b \ln\left(1-\ell_t\right) + e^{-\rho} E_t V\left[w_t \ell_t + \left(1+r_t\right) s_{t-1} - c_t, r_{t+1}\right]$$

By Maximum principle, the FOC are

$$\begin{aligned}
c_t &: \quad \frac{1}{c_t} = e^{-\rho} E_t\left[V_1\left(s_t, r_{t+1}\right)\right] \\
\ell_t &: \quad \frac{b}{1-\ell_t} = e^{-\rho} E_t\left[V_1\left(s_t, r_{t+1}\right)\right] w_t
\end{aligned}$$

which implies consumption and lesiure tradeoff follows:

$$\frac{c_t}{1-\ell_t}=\frac{w_t}{b}$$

By envelop theorem,

$$V_1\left(s_{t-1},r_t\right)=e^{-\rho}E_t\left[V_1\left(s_t,r_{t+1}\right)\left(1+r_t\right)\right]$$

Now note that

$$\frac{1}{c_{t+1}}=e^{-\rho}E_{t+1}\left[V_1\left(s_t,r_{t+1}\right)\right]$$

so

$$\begin{aligned}E_t\left[\frac{1}{c_{t+1}}\left(1+r_{t+1}\right)\right] &= E_t\left[e^{-\rho}E_{t+1}\left[V_1\left(s_t,r_{t+1}\right)\right]\left(1+r_{t+1}\right)\right]\\ &= E_t\left[V_1\left(s_t,r_{t+1}\right)\right]\\ &= \frac{1}{c_t}e^{\rho}\end{aligned}$$

or

$$\frac{1}{c_t}=e^{-\rho}E_t\left[\frac{1}{c_{t+1}}\left(1+r_{t+1}\right)\right]$$

which is special case of Euler's equation.

3.5 Special case of the model

Additional Assumption:

1. Full depreciation $\delta=1$
2. No government $G_t=0$

By these two conditions, we have

$$\begin{aligned}K_{t+1} &= Y_t-C_t\\ 1+r_t &= \alpha\left(\frac{A_tL_t}{K_t}\right)^{1-\alpha}=\alpha\frac{Y_t}{K_t}\end{aligned}$$

Rule of thumb method: Assume for all t,

$$s_t\equiv 1-\frac{C_t}{Y_t}\equiv 1-\frac{c_tN_t}{Y_t}=\hat{s}$$

then as $c_t=\left(1-s_t\right)Y_t/N_t$ and $1/c_t=e^{-\rho}E_t\left[\left(1+r_{t+1}\right)/c_{t+1}\right]$, we have

$$\begin{aligned}-\ln\left[\left(1-s_t\right)Y_t/N_t\right] &= -\rho+\ln E_t\left[\left(1+r_{t+1}\right)/c_{t+1}\right]\\ &= -\rho+\ln E_t\left[\frac{\left(1+r_{t+1}\right)}{\left(1-s_t\right)Y_{t+1}/N_{t+1}}\right]\end{aligned}$$

or

$$\begin{aligned}-\ln(1-s_t)-\ln Y_t+\ln N_t &= -\rho+\ln E_t\left[\frac{(1+r_{t+1})K_{t+1}}{K_{t+1}(1-s_t)Y_{t+1}/N_{t+1}}\right]\\ &= -\rho+\ln E_t\left[\frac{\alpha Y_{t+1}}{s_tY_t(1-s_t)Y_{t+1}/N_{t+1}}\right]\\ &= -\rho+\ln E_t\left[\frac{\alpha N_{t+1}}{s_tY_t(1-s_t)}\right]\end{aligned}$$

Hence

$$-\ln(1-s_t)-\ln Y_t+\ln N_t=-\rho+\ln\alpha+\ln N_t+n-\ln s_t-\ln Y_t+\ln E_t\left[\frac{1}{(1-s_t)}\right]$$

or

$$\ln s_t-\ln(1-s_t)=-\rho+n+\ln\alpha+\ln E_t\left[\frac{1}{(1-s_t)}\right]$$

In equailibrium, $s_t=\hat{s}$, so we have

$$\ln\hat{s}-\ln(1-\hat{s})=-\rho+n+\ln\alpha+\ln\frac{1}{(1-\hat{s})}$$

or

$$\ln\hat{s}=-\rho+n+\ln\alpha$$

and hence

$$\hat{s}=\alpha e^{n-\rho}$$

which confirms the correctness of rule of thumb as saving rate is constant.
From consumption-lesiure tradee-off expression,

$$\frac{c_t}{1-\ell_t}=\frac{w_t}{b_t}$$

we have

$$\ln c_t-\ln(1-\ell_t)=\ln w_t-\ln b_t$$

Now by $c_t=(1-\hat{s})Y_t/N_t$ and $w_t=(1-\alpha)Y_t/\ell_tN_t$, we have

$$\ln(1-\hat{s})+\ln Y_t-\ln N_t-\ln(1-\ell_t)=\ln(1-\alpha)+\ln Y_t-\ln\ell_t-\ln N_t$$

or

$$\ln\ell_t-\ln(1-\ell_t)=\ln(1-\alpha)-\ln(1-\hat{s})-\ln b$$

and hence

$$\ell_t=\frac{1-\alpha}{1-\alpha+b(1-\hat{s})}\equiv\hat{\ell}$$

where independence is again from log-linear utility.

Time series of shock variables:
From production function $Y_t=K_t^\alpha(A_tL_t)^{1-\alpha}$, we have

$$\ln Y_t=\alpha\ln K_t+(1-\alpha)(\ln A_t+\ln L_t)$$

As $K_t = \hat{s}Y_{t-1}$, $L_t = \hat{\ell}N$, $\ln A_t = \bar{A} + gt + \tilde{A}_t$ and $\ln L_t = \bar{N} + nt$, we have

$$\ln Y_t = \alpha \ln \hat{s} + \alpha \ln Y_{t-1} + (1-\alpha)\left(\bar{A} + gt + \tilde{A}_t + \bar{N} + nt\right)$$

Define $\tilde{Y}_t = \ln Y_t - E_t \ln Y_t$, we have

$$\tilde{Y}_t = \alpha \tilde{Y}_{t-1} + (1-\alpha)\tilde{A}_t$$

or

$$\tilde{A}_{t-1} = \frac{1}{1-\alpha}\left(\tilde{Y}_{t-1} + \alpha \tilde{Y}_{t-2}\right)$$

Now, given $\tilde{A}_t = \rho_A \tilde{A}_{t-1} + \varepsilon_{A,t}$, we have

$$\begin{aligned}
\tilde{Y}_t &= \alpha \tilde{Y}_{t-1} + (1-\alpha)\left(\rho_A \tilde{A}_{t-1} + \varepsilon_{A,t}\right) \\
&= \alpha \tilde{Y}_{t-1} + \rho_A\left(\tilde{Y}_{t-1} - \alpha \tilde{Y}_{t-2}\right) + (1-\alpha)\varepsilon_{A,t} \\
&= \left(\alpha + \rho_A \tilde{Y}_{t-1}\right) - \alpha\rho_A \tilde{Y}_{t-2} + (1-\alpha)\varepsilon_{A,t}
\end{aligned}$$

so that $\tilde{Y}_t$ is AR(2) process.

3.6 Solving model in general case

Log-linearize around steady state:

$$\begin{aligned}
\tilde{C}_t &\simeq a_{CK}\tilde{K}_t + a_{CA}\tilde{A}_t + a_{CG}\tilde{G}_t \\
\tilde{L}_t &\simeq a_{LK}\tilde{K}_t + a_{LA}\tilde{A}_t + a_{LG}\tilde{G}_t
\end{aligned}$$

Method of Undetermined coefficient:
From Intratemporal FOCs, $c_t/(1-\ell_t) = w_t/b$ and $w_t = (1-\alpha)(K_t/A_tL_t)^\alpha A_t$, we have

$$\ln c_t - \ln(1-\ell_t) = \ln\left(\frac{1-\alpha}{b}\right) + (1-\alpha)\ln A_t + \alpha \ln K_t - \alpha \ln L_t$$

By first-order Taylor series approximation around steady state, we have

$$\begin{aligned}
&(\ln c_t - \ln c^*) - \frac{\ell^*}{1-\ell^*}(\ln \ell_t - \ln \ell^*) = (1-\alpha)\tilde{A}_t + \alpha \ln \tilde{K}_t - \alpha \ln \tilde{L}_t \\
\Rightarrow\quad &\tilde{c}_t - \frac{\ell^*}{1-\ell^*}\tilde{\ell}_t = (1-\alpha)\tilde{A}_t + \alpha \ln \tilde{K}_t - \alpha \ln \tilde{L}_t
\end{aligned}$$

In steady state, $\tilde{c}_t = \tilde{C}_t$ and $\tilde{\ell}_t = \tilde{L}_t$, we have

$$\begin{aligned}
&a_{CK}\tilde{K}_t + a_{CA}\tilde{A}_t + a_{CG}\tilde{G}_t + \left(\frac{\ell^*}{1-\ell^*} + \alpha\right)\left(a_{LK}\tilde{K}_t + a_{LA}\tilde{A}_t + a_{LG}\tilde{G}_t\right) \\
= \quad &(1-\alpha)\tilde{A}_t + \alpha \ln \tilde{K}_t
\end{aligned}$$

Hence,

$$\begin{aligned} a_{CK} + \left(\frac{\ell^*}{1-\ell^*} + \alpha\right) a_{LK} &= \alpha \\ a_{CA} + \left(\frac{\ell^*}{1-\ell^*} + \alpha\right) a_{LA} &= 1-\alpha \\ a_{CG} + \left(\frac{\ell^*}{1-\ell^*} + \alpha\right) a_{LG} &= 0 \end{aligned}$$

From intertemporal FOC, $1/c_t = e^{-\rho} E_t \left[\left(1 + r_{t+1}\right)/c_{t+1}\right]$, $\tilde{C}_t \simeq a_{CK}\tilde{K}_t + a_{CA}\tilde{A}_t + a_{CG}\tilde{G}_t$ and $K_{t+1} = K_t + Y_t - C_t - G_t - \delta K_t$, we have

$$\tilde{K}_t \simeq b_{KK}\tilde{K}_t + b_{KA}\tilde{A}_t + b_{KG}\tilde{G}_t$$

3.7 Implication

Numerical analysis results.

3.8 Empirical Application: The Persistence of Output Fluctuations

Shocks in RBC model are driven by technology type. However, technological shcok should have permanent component.
Nelson and Plosser Test is find whether there is persistence of output fluctuation.

$$\Delta \ln y_t = a + b\left\{\ln y_t - \left[\alpha + \beta\left(t-1\right)\right] + \varepsilon_t\right\}$$

where H_0 is unit root and H_1 is trend stationary. The empirical result is cannot reject using Dicky-Fuller unit root test.
Campell and Mankiw Test is to find out the size of the persistence of shock.
However, these tests subject to two problems:

1. Statistical problem: due to short horizon of data, it is hardly easy to define what is temporary
2. theoretical problem: provide little information about driving forces of the fluctuation

3.9 Emprical Application: Calibrating RBC Model

To choose parameter values on the basis of microeconomic evidence and then to compare the model's predictions concerning the variances and covariance of various series with those data.

3.10 Extension and Limition

Extension:

1. Indivisble labor
2. distortionary tax
3. multiple secotrs and sector specific shocks

Objections:

1. size of technology shocks
2. intertemporal substitution of labor
3. omission of monetary disturbance
4. no significant propagation mechanism of shocks

Appendix A

Appendix I EC750

A.1 Annuity Payment in Blanhard Model

Why annuity payment is equal to probability of death times the assets?
Simple proof: Insurance company is paying cohort s per living person at rate $pw(s,t)$ so the annuity payment must be same.
Rigorous proof: Insurance company collects $w(s,t)$ from person of cohort s if he dies at time t and has to make annuity payment at proportion α of $w(s,t)$ for cohort s if he is living at time t. Therefore, the expected inflow should be

$$\begin{aligned}
& \int_0^\infty \int_0^v e^{-ru}\alpha w(s,u)\,du \frac{1}{p}e^{-pv}dv \\
= \; & \int_0^\infty \int_0^v e^{-ru}\alpha w(s,u)\frac{1}{p}e^{-pv}dudv \\
= \; & \int_0^\infty \int_u^\infty e^{-ru}\alpha w(s,u)\frac{1}{p}e^{-pv}dvdu \\
= \; & \int_0^\infty e^{-ru}\alpha w(s,u)\int_u^\infty \frac{1}{p}e^{-pv}dvdu \\
= \; & \frac{\alpha}{p^2}\int_0^\infty e^{-ru}w(s,u)\,e^{-pu}du
\end{aligned}$$

and the expected inflow would be

$$\begin{aligned}
& \int_0^\infty e^{-rv}w(s,v)\frac{1}{p}e^{-pv}dv \\
= \; & \frac{1}{p}\int_0^\infty e^{-rv}w(s,v)\,e^{-pv}dv
\end{aligned}$$

Since insurance market is competitive so that all annuity contract is actuarial fair, the expected inflow should be same as expected outflow at the time of issuance, so we have

$$\frac{\alpha}{p^2}\int_0^\infty e^{-ru}w(s,u)\,e^{-pu}du = \frac{1}{p}\int_0^\infty e^{-rv}w(s,v)\,e^{-pv}dv$$

which implies

$$\alpha = p$$

Hence, the insurance annuity payment would be $pw\left(s,t\right)$.

A.2 Social Planner's Learning-by-doing Model

The social planner is going to maximize $C\left(t\right)$ over time period, however, as population is constant, $L\left(t\right)=L$ for all t, it is same to maximize $c\left(t\right)$ over the same horizon:

$$\begin{aligned} & \max_{c(t)} \int_{t=1}^{\infty} e^{-\rho t}\frac{c\left(t\right)^{1-\sigma}-1}{1-\sigma}dt \\ s.t. \quad & \dot{K} = F\left(K,KL\right)-C-\delta K \end{aligned}$$

where the constraint can be rewritten as

$$\begin{aligned} \dot{k} &= F\left(k,K\right)-c-\delta k \\ &= kF\left(1,L\right)-c-\delta k \\ &= kf\left(L\right)-c-\delta k \end{aligned}$$

Now the Hamiltonian would be

$$H = e^{-\rho t}\frac{c\left(t\right)^{1-\sigma}-1}{1-\sigma}+e^{-\rho t}\lambda\left[f\left(L\right)k-c-\delta k\right]$$

with F.O.C.s are

$$\begin{aligned} \frac{\partial H}{\partial c} = 0 \quad &\Rightarrow \quad c^{-\sigma}=\lambda \\ \frac{de^{-rt}\lambda}{dt} = \frac{\partial H}{\partial k} \quad &\Rightarrow \quad -\rho e^{-\rho t}\lambda+e^{-\rho t}\dot{\lambda}=e^{-\rho t}\lambda\left[f\left(L\right)-\delta\right] \\ &\Rightarrow \quad \dot{\lambda}=\lambda\left[f\left(L\right)-\rho-\delta\right] \end{aligned}$$

so that

$$\frac{\dot{c}}{c}=\frac{1}{\sigma}\left[f\left(L\right)-\rho-\delta\right]$$

A.3 Complete Derivation of Rich Idea-Based Growth Model

Environment:

1. Final good production function: $Y_t=\left[\left(\int_0^{A_t} x_{it}^{\theta}di\right)^{1/\theta}\right]^{\alpha} H_{Y_t}^{1-\alpha}$
2. Captial accumulation equation: $\dot{K}_t = Y_t - C_t - \delta K_t$

3. Idea produciton function: $\dot{A}_t = \nu H_{A_t}^{\lambda} A_t^{\phi}$
4. Capital good constraint: $\int_0^{A_t} x_{it} di = K_t$
5. Human captial constraint: $H_{A_t} + H_{Y_t} = H_t$
6. Labor force: $L_t = (1 - \ell_{h_t}) N_t$
7. Human capital: $H_t = h_t L_t$
8. Mincerian human capital accumulation: $h_t = e^{\psi \ell_{h_t}}$
9. Population growth: $N_t = N_0 e^{nt}$

Household problems:

$$\begin{aligned} & \max_{c_t, \ell_{h_t}, \ell_t} \int_0^{\infty} N_t \frac{c_t^{1-\varepsilon} - 1}{1-\varepsilon} e^{-\rho t} dt \\ \text{s.t.} \qquad & \dot{\nu}_t = (r-n)\nu + w_t h_t \ell_t - c_t \\ & h_t = e^{\psi \ell_{h_t}} \\ & 1 = \ell_{h_t} + \ell_t \\ & N_t = N_0 e^{nt} \\ & \lim_{t \to \infty} \nu_t \exp\left\{ - \int_0^t (r_s - n) ds \right\} \geq 0 \end{aligned}$$

which can be simplify to the case that

$$\begin{aligned} & \max_{c_t, \ell_{h_t}} \int_0^{\infty} \frac{c_t^{1-\varepsilon} - 1}{1-\varepsilon} e^{-(\rho - n)t} dt \\ s.t \quad \dot{\nu}_t \quad &= \quad (r-n)\nu + w_t e^{\psi \ell_{h_t}} (1 - \ell_{h_t}) - c_t \end{aligned}$$

with Hamiltonian

$$H = e^{-(\rho - n)t} \left\{ \frac{c_t^{1-\varepsilon} - 1}{1-\varepsilon} + \lambda \left[(r-n)\nu + w_t e^{\psi \ell_{h_t}} (1 - \ell_{h_t}) - c_t \right] \right\}$$

so that the F.O.C.s are

$$\begin{aligned} \frac{\partial H}{\partial c_t} &= 0 \\ -\frac{\partial H}{\partial \nu_t} &= e^{-(\rho - n)t} \dot{\lambda} - (\rho - n) e^{-(\rho - n)t} \lambda \\ \frac{\partial H}{\partial \ell_{h_t}} &= 0 \end{aligned}$$

where the first two imply the standard condition

$$\frac{\dot{c}}{c} = \frac{1}{\varepsilon} (r - \rho)$$

and the last condition implies

$$\lambda w_t \left[-e^{\psi \ell_{h_t}} + e^{\psi \ell_{h_t}} \left(1 - \ell_{h_t}\right) \psi \right] = 0$$

or

$$\ell_{h_t} = 1 - \frac{1}{\psi}$$

Final good problem:

$$\max_{\{x_{it}\}, H_{y_t}} \pi_i = \left(\int_0^{A_t} x_{it}^{\theta} di \right)^{\alpha/\theta} H_{Y_t}^{1-\alpha} - H_{Y_t} w_t - \int_0^{A_t} p_{it} x_{it} di$$

with F.O.C.s

$$\begin{aligned}
\frac{\partial \pi_i}{\partial H_{Y_t}} = 0 \quad &\Rightarrow \quad \left(\int_0^{A_t} x_{it}^{\theta} di \right)^{\alpha/\theta} H_{Y_t}^{-\alpha} - w_t = 0 \\
&\Rightarrow \quad (1-\alpha) \frac{Y_t}{H_{y_t}} = w_t \\
\frac{\partial \pi_i}{\partial x_{it}} = 0 \quad &\Rightarrow \quad p_{it} = \frac{\alpha}{\theta} \left(\int_0^{A_t} x_{it}^{\theta} di \right)^{\alpha/\theta - 1} H_{Y_t}^{1-\alpha} \theta x_{it}^{\theta - 1} \\
&\Rightarrow \quad x_{it} = \left[\frac{\alpha y}{\int_0^{A_t} x_{it}^{\theta} di} \frac{1}{p_i} \right]^{1/(1-\theta)}
\end{aligned}$$

so that

$$\begin{aligned}
\frac{dx_{it}}{dp_{it}} \frac{p_{it}}{x_{it}} &= \frac{1}{1-\theta} \left[\frac{\alpha y}{\int_0^{A_t} x_{it}^{\theta} di} \frac{1}{p_i} \right]^{1/(1-\theta) - 1} \frac{\alpha y}{\int_0^{A_t} x_{it}^{\theta} di} \left(-\frac{1}{p_{it}^2} \right) \frac{p_{it}}{x_{it}} \\
&= -\frac{1}{1-\theta}
\end{aligned}$$

Capital good problem:

$$\max_{p_{it}} \left(p_{it} - r - \delta\right) x_{it}\left(p_{it}\right)$$

with F.O.C.s is

$$\begin{aligned}
& x_{it}\left(p_{it}\right) + \left(p_{it} - r - \delta\right) \frac{dx_{it}\left(p_{it}\right)}{dp_{it}} = 0 \\
\Rightarrow \quad & \frac{x_{it}\left(p_{it}\right)}{p_{it}} \left[p_{it} + \left(p_{it} - r - \delta\right) \frac{p_{it}}{x_{it}\left(p_{it}\right)} \frac{dx_{it}\left(p_{it}\right)}{dp_{it}} \right] = 0 \\
\Rightarrow \quad & \frac{x_{it}\left(p_{it}\right)}{p_{it}} \left[p_{it} + \left(p_{it} - r - \delta\right) \left(-\frac{1}{1-\theta} \right) \right] = 0 \\
\Rightarrow \quad & \frac{x_{it}\left(p_{it}\right)}{p_{it}} \left[p_{it} \left(\frac{-\theta}{1-\theta} \right) + (r + \delta) \left(\frac{1}{1-\theta} \right) \right] = 0
\end{aligned}$$

so that since $x_{it}\left(p_{it}\right)\neq 0$, we have

$$p_{it}=\frac{1}{\theta}\left(r+\delta\right)$$

which also implies

$$x_{it}=\bar{x}_t$$

Now, consider

$$\begin{aligned}
p_{it} &= \frac{\alpha}{\theta}\left(\int_0^{A_t} x_{it}^{\theta}di\right)^{\alpha/\theta-1} H_{Y_t}^{1-\alpha}\theta x_{it}^{\theta-1} \\
&= \frac{\alpha}{\theta}\bar{x}_t^{\alpha-\theta}A^{\alpha/\theta-1}H_{Y_t}^{1-\alpha}\theta\bar{x}_t^{\theta-1} \\
&= \alpha\bar{x}_t^{\alpha-1}A^{\alpha/\theta-1}H_{Y_t}^{1-\alpha} \\
&= \alpha K_t^{\alpha-1}A^{\alpha/\theta-1+1-\alpha}H_{Y_t}^{1-\alpha} \\
&= \alpha\frac{Y_t}{K_t}
\end{aligned}$$

so that combine with the fact that $p_{it}=\left(r+\delta\right)/\theta$, we have

$$r=\alpha\theta-\frac{Y_t}{K_t}-\delta$$

Therefore, going back to the household condition:

$$\frac{\dot{c}}{c}=\frac{1}{\varepsilon}\left(r-\rho\right)$$

or in per capita terms

$$\frac{\dot{\tilde{c}}}{\tilde{c}}=\frac{1}{\varepsilon}\left(r-\rho-\varepsilon g\right)$$

and in steady state $\dot{\tilde{c}}=0$ and $r=\alpha\theta-Y_t^*/K_t^*-\delta$, we have

$$0=\frac{1}{\varepsilon}\left[\left(\alpha\theta-\frac{Y_t^*}{K_t^*}-\delta\right)-\rho-\varepsilon g\right]$$

or

$$\frac{K_t^*}{Y_t^*}=\frac{\alpha\theta}{\delta+\rho+\varepsilon g}$$

and since

$$\frac{K_t^*}{Y_t^*}=\frac{\tilde{k}_t^*}{\tilde{y}_t^*}=\frac{n+\delta+g}{s_K}$$

we have

$$s_K=\frac{\alpha\theta\left(n+\delta+g\right)}{\delta+\rho+\varepsilon g}$$

Research:

$$\max_{H_{A_t}} p_{A_t}\bar{\nu}H_{A_t}-w_tH_{A_t}$$

where

$$\bar{\nu} = v_t H_{At}^{\lambda-1} A_t$$

is given value of productivity of research with researcher production function

$$\dot{A}_t = v_t H_{At}$$

The F.O.C. is

$$p_{A_t} \bar{\nu} = w_t$$

From firms problem, $w_t = (1-\alpha) Y_t / H_{Y_t}$,

$$p_{A_t} \frac{\dot{A}_t}{H_{A_t}} = (1-\alpha) \frac{Y_t}{H_{Y_t}}$$

or

$$\frac{H_{A_t}}{H_{Y_t}} = \frac{s_A^* H_t}{1 - s_A^* H_t} = \frac{s_A^*}{1 - s_A^*} = \frac{p_A \dot{A}_t}{(1-\alpha) Y_t}$$

By arbitrage condition, interest rate differential between bonds and purchase capital should be the same:

$$r_t = \frac{\pi_{it}}{p_{A_t}} + \frac{\dot{p}_{A_t}}{p_{A_t}}$$

and taking $p_{A_t} = \pi / (r - m_t)$ where $m_t = \dot{p}_{A_t} / p_{A_t}$,

$$\begin{aligned} \pi_{it} &= \left(\frac{1}{\theta} - 1\right) (r + \delta) x_{it} \\ &= \left(\frac{1}{\theta} - 1\right) \left(\alpha\theta - \frac{Y_t}{K_t} - \delta + \delta\right) \bar{x}_t \\ &= \sigma\theta \frac{Y_t}{K_t} \frac{K_t}{A_t} = \sigma\theta \frac{Y}{A} \end{aligned}$$

so that we have

$$\frac{s_A^*}{1 - s_A^*} = \frac{\frac{\sigma\theta}{r - m_t} \frac{Y_t}{A_t} \dot{A}_t}{(1-\alpha) Y_t}$$

A.4 Proof for return of scale equal to AC/MC

Suppose the firm has production function $F(K, L)$ and production objective Q facing labor cost w and captial cost r. Production function has t degree of returns implies

$$F(\alpha K, \alpha L) = \alpha^t F(K, L)$$

Differentiate both sides with respect to α, and then set $\alpha = 1$, we have

$$K F_K + L F_L = t F(K, L)$$

and

$$t = \frac{KF_K + LF_L}{F(K, L)}$$

Now, consider the cost minimizing problem,

$$\begin{array}{cc} \min\limits_{K,L} & wL + rK \\ s.t. & F(K, L) \geq Q \end{array}$$

Using Lagrangaing multipler λ, we have F.O.C.s

$$\begin{aligned} w + \lambda F_L &= 0 \\ r + \lambda F_K &= 0 \\ F(K, L) - Q &= 0 \end{aligned}$$

By envelop theorem, the marginal cost would be

$$MC = \lambda$$

and average cost is

$$AC = \frac{wL + rK}{Q}$$

so that the ratio is

$$\begin{aligned} \frac{AC}{MC} &= \frac{wL + rK}{Q\lambda} \\ &= \frac{LF_L + KF_K}{F(K, L)} \\ &= t \end{aligned}$$

Appendix B

Appendix II EC751

Afterword

Acknowledgements

Bibliography