New objects in mathematician

C. Rubtsov
rubcov@usa.net

The simples receptions of research and research of new mathematical objects (operations, numbers, functions etc.). The operations "easier than addition" and "of reflexive multiplication" are describe. The brief information about Δ-numbers located for $(-\infty)$ is given the common information about ω-images and ω-images are represented, are generated courses of practical application of new mathematical objects for more deep knowledge of the material world, enclosing us. The present paper are fragments from the monography of the author [1] (http://www.crosswinds.net/russia/~rubcov or http://oasis.fortunecity.com/andes/205/index.htm).

1. Introduction

The research of new mathematical objects is connected to attempt of reconstruction of foundations mathematics. Thus for a basis not audit of these foundations is taken, bat - rushing to expand known representations about operations and objects. In particular, the definition of hypothetical operation "is easier than addition" carries on to statement of a topicality of the element $(-\infty)$ and extension of a field of real numbers. The study of spectra of fields of numbers located for $(-\infty)$, allows to unify their properties and to consider interior identity of various sets of numbers in aggregate with appropriate operations. Classifying simples operations with two operands, the new operations and functions are obtained.

The searching of a commutative operation similar on a structure to exponentation, creates premises for shaping the concept ω-images. The realization of this concept within the framework of a homeomorphism ω-procedures illustrates a possibility ω-figurative transformation of global ordered gangs of objects and ratios (including, separate methods and whole sections of mathematics). All this strengthens an applied content of mathematics and moves apart the boundaries of mathematical modelling of processes which are flowing past in a material medium.

2. The operation "is easier than addition" and new numbers

Analyzing reduced structures of arithmetical operations $\left(\underbrace{a+a+a+\ldots+a}_{n}=a\cdot n,\ \underbrace{a\cdot a\cdot a\cdot\ldots\cdot a}_{n}=a^{n},\ a\in\mathbf{R},\ n\in\mathbf{Z}\right)$, it is possible to suppose operations easier than addition $\left(\underbrace{a\circ a\circ a\circ\ldots\circ a}_{n}=a+n\right)$ and raising in a superdegree $\left(\underbrace{a^{a^{a^{\cdot^{\cdot^{\cdot^{a}}}}}}}_{n}={}^{n}a\right)$. Reliability of such assumption which proves to be true by many facts, one of is a validity of the iterative formulas for evaluations of values of an operand at a realization of inverse operation. It is known [2], that

$$x\cdot x=c\ \Rightarrow\ x=\sqrt{c}\approx\frac{{}^{c}\!/_{a}+a}{2}, \qquad (1)$$

where $a-$approximate value of the radical from number c $(c\in\mathbf{R})$.

By analogy to the formula (1), taking into account a rank of operation, we shall note [1, 3]:

$$x^{x}=c\ \Rightarrow\ x=\text{ʃ}\overline{c}\approx\sqrt{a\cdot\log_{a}c},\ (c\geq 1.7,\ a\geq 1.6); \qquad (2)$$

$$x\circ x=c\ \Rightarrow\ x=\frac{c}{2}\approx(c-a\circ a)-2, \qquad (3)$$

where $a-$approximate values x, " $\circ$ " – the operation "is easier than addition".

The definition 1. By operation "zero - operation" ("is easier than addition") the expression satisfying to the following conditions is named:

$$a \circ b = \begin{cases} a+1, & a > b; \\ b+1, & b > a; \\ a+2 = b+2, & b = a; \\ a, & b = (-\infty); \\ b, & a = (-\infty). \end{cases}$$

Operation inverse " $\circ$ " we shall designate " Δ ".

In work [3] is remarked, that $\lim\limits_{n\to+\infty} {}^{n}\hat{\jmath}\overline{x} = \sqrt[x]{x}$; $\lim\limits_{n\to+\infty} \sqrt[n]{x} = \frac{x}{x}$; $\lim\limits_{n\to+\infty} \frac{x}{n} = x - x$; $\lim\limits_{n\to+\infty} x - n = x \Delta x$. These equalities characterize the neutral (zero) elements of operations. In particular, for " $\circ$ " a unit element is $(-\infty)$ [1].

Table 1. Operations with two operands $(n \in \mathbf{Z} \wedge i \in \{1, 2, 3\})$ [1,3]

$n \setminus i$	**1**	**2**	**3**
...	...	...	...
0	$a \circ b = c$	$c \Delta b = a$	$c \Delta a = b$
1	$a + b = c$	$c - b = a$	$c - a = b$
2	$a \cdot b = c$	$c/b = a$	$c/a = b$
3	$a^{b} = c$	$\sqrt[b]{c} = a$	$\log_{a} c = b$
4	${}^{b}a = c$	${}^{b}\hat{\jmath}\overline{c} = a$	$\mathrm{slog}_{a} c = b$
...	...	...	...

The definition 2. The solutions of an equation $x \circ a = (-\infty)$ $(a \in \mathbf{R})$ are named Δ-numbers.

Really, $x = (-\infty)\Delta a = \Delta a$ if to note without a unit element.

Let's compare the definition 2 to the definitions opposite (negative) and inverses (fractional) numbers:

$$x + a = 0 \;\; (a > 0) \;\Rightarrow\; x = -a;$$

$$x \cdot a = 1 \;\; (a > 1) \;\Rightarrow\; x = a^{-1} = \frac{1}{a} = 1{:}a,$$

where $0; 1-$ accordingly unit elements.

In the past the opposite numbers were designated [4] "$0m,a$" (that is "$0-a$"), and the inverses of number until now write with a unit element. In work [1] the labels without a unit element, i.e. $\frac{1}{a}=:a$, and $x=-\infty\Delta a=\Delta a$ are accepted. (The set Δ-numbers is designated $\mathbf{R}_\Delta$).

In a table 1 the operations with inclusion new ingredients are represented direct $(i=1)$ and inverses $(i=2,3)$. For example, from $\mathrm{slog}_a c=b \Rightarrow {}^b a=c$, i.e. $\mathrm{slog}_a c-$ "superlog" of number c on a foundation a, and $y={}^x a$, $y={}^n\sqrt[\wedge]{x}$, $y=\mathrm{slog}_a x-$new functions.

The more detailed exposition Δ-numbers (their axiomatics, property etc.) is reduced in [1]. Let's mark only some properties.

1. For anyone's $a,b\in\mathbf{R}$ $\Delta a+\Delta b=a+b$, $\Delta a+b=\Delta(a+b)=a+\Delta b$ (rule of signs). Is comparable: $(-a)\cdot(-b)=a\cdot b$, $(-a)\cdot b=-(a\cdot b)=a\cdot(-b)$.
2. For anyone's $a,b\in\mathbf{Z}$ $(\Delta a,\Delta b)\in\mathbf{R}_\Delta$, $(\Delta a)\cdot b=a\cdot b$ $(b\in\mathbf{N}_2)$, $(\Delta a)\cdot b=\Delta(a\cdot b)$ $(b\in\mathbf{N}_1)$, $a\cdot(\Delta b)=\Delta(a\cdot b)$, $\Delta a\cdot\Delta b=\Delta((\Delta a)\cdot b)$, (4), where $\mathbf{N}_2,\mathbf{N}_1-$ accordingly set of even and odd numbers.
3. For $(a,b)\in\mathbf{R}$ the equalities [1] take place:

$$a^{\Delta b}=-\left(a^b\right);\quad \log(-a)=\Delta\log a, \tag{5}$$

i.e. the set Δ-numbers is a real branch of values of logs of negative numbers.

The note 1. In this connection it is interesting to return to long-term dispute of the L. Euler and J. d'Alembert [4] concerning equality $2\cdot\ln(-1)=0$. The interpretation of this equality from new positions is those: $(\ln(-1))\cdot 2=(\Delta 0)\cdot 2=0\cdot 2=0$, as $\ln(-1)=\Delta 0\in\mathbf{R}_\Delta$, and the commutability of multiplication on $\mathbf{R}_\Delta$ agrees (4) is broken.

With introduction $\mathbf{R}_\Delta$ the extension of a field of real numbers for the actual element $(-\infty)$ is realized.

In [1] the existence infinite of a spectrum of sets of numbers of a type "delta" is shown.

3. Reflexive multiplication ($\odot$) and other mathematical operations in ω-figurative form

The searching of commutative operation similar to exponentation, was one of the stimulating factors promoting creation of the concept ω-images. Basic dominating postulates of this concept [1]:

- In similar on a structure abstract ω-spaces the images of objects are admissible one-to-one;
- Operations of reflection from ω_i in ω_j $(i \neq j)$ reduces in transformation of object in the correspondence with some function of connection;
- Exist infinite spectra ω-images of mathematical objects.

Let all known mathematical objects, ratio and the connections between them make space ω_0. Let's generate space, identical to it, ω_1. Applying an exponential function of connection $f_c = k^x$ $(k \neq 1)$, we shall note some ω-images. The number a $(a \in \mathbf{R})$ from ω_1 will be reflection in ω_0 as $a' = k^a$. By designating operation of reflection $\backslash \omega_1 \to \omega_0 \backslash$, we shall receive:

$$a + b \backslash \omega_1 \to \omega_0 \backslash k^{a+b} = k^a \cdot k^b = a' \cdot b';$$

$$a \cdot b \backslash \omega_1 \to \omega_0 \backslash k^{a \cdot b} = \left(k^a\right)^{\log_k k^b} = k^a \odot k^b = a' \odot b';$$

$$a - b \backslash \omega_1 \to \omega_0 \backslash k^{a-b} = k^a / k^b = a'/b';$$

$$a/b \backslash \omega_1 \to \omega_0 \backslash k^{a/b} = \left(k^a\right)^{1/\log_k k^b} = k^a \triangle k^b = a' \triangle b';$$

$$a^b \backslash \omega_1 \to \omega_0 \backslash a^{\to b} = \underbrace{a \odot a \odot a \odot \ldots \odot a}_{(\log_k b) \in \mathbf{Z}};$$

$$\log_a b \backslash \omega_1 \to \omega_0 \backslash \mathrm{ilog}_a b \text{ etc.},$$

where a', b' – images in ω_0 numbers a and b; "$\bullet$" – multiplication being an image of addition "+"; "$\odot$" – reflexive multiplication (commutative exponentation) – image of multiplication, and "$\triangle$" – reflexive division – image of division.

Let's reduce relations, which follow from a sense of operations "$\odot$" and "$\triangle$":

$(:a)\odot(:b) = a\odot b \quad (a > 1,\, b > 1)$ by analogy with $(-a)\cdot(-b) = a\cdot b \quad (a > 0,\, b > 0)$, $a\odot k = a$, $a^{\to k} = a$, $a^{\to 1} = k$, $a\odot 1 = 1$, $a\odot 0 = 0 \quad (a \neq 0)$, $a^{\to b1}\odot a^{\to b2} = a^{\to(b1\cdot b2)}$, $a^{\to b1}\triangle a^{\to b2} = a^{\to\left(\frac{b1}{b2}\right)}$, $\left(a^{\to k1}\right)^{\to k2} = a^{\to(k1\odot k2)}$, $\sqrt[\to b]{a} = a^{\to(\triangle b)}$, $\sqrt[\to b]{a^{\to b}} = a$, $a^{\to(k1\triangle k2)} = \sqrt[\to k2]{a^{\to k1}}$, $\mathrm{ilog}_a a^{\to b} = b$, $\mathrm{ilog}_a b = \triangle\,\mathrm{ilog}_b a$, $\mathrm{ilog}_a(k_1\odot k_2) = \mathrm{ilog}_a k_1\cdot \mathrm{ilog}_a k_2$, $\mathrm{ilog}_a(k_1\triangle k_2) = \dfrac{\mathrm{ilog}_a k_1}{\mathrm{ilog}_a k_2}$, $\mathrm{ilog}_a b^{\to k} = k\odot\,\mathrm{ilog}_a b$, $\mathrm{ilog}(b)\triangle\mathrm{ilog}\left(a^{\to k}\right) = \triangle k\odot\,\mathrm{ilog}_a b$, $\mathrm{ilog}_a b = \mathrm{ilog}_c b\triangle\,\mathrm{ilog}_c a$, $a\odot b^c = a^c\odot b$, $a\odot(b\cdot c) = (a\odot b)\cdot(a\odot c)$, $a\odot\left(\dfrac{b}{c}\right) = \dfrac{a\odot b}{a\odot c}$, $(a\cdot c)\triangle b = (a\triangle b)\cdot(c\triangle b)$, $a\triangle b^c = \sqrt[c]{a}\triangle b$ etc..

Though the operations "$\odot$" and "$\triangle$" qualitatively differ from "$\bullet$" and "$\div$", the relations on their basis are adequate on a construction to similar relations because of "$\bullet$" and "$\div$".

The operation "$\triangle$" defines a set of numbers of a type $\{\triangle a\} \equiv \left\{k^{1/\log_k a},\, k \neq 1\right\}$.

Let's remark, that $\Delta a + \Delta b = a + b$, $(-a)\cdot(-b) = a\cdot b$, $(:a)\odot(:b) = a\odot b$, i.e., synthesizing objects and operations, it is possible to find the invariant forms.

The composite likeness ω-figurative structures fundamentally is investigated in work [1]. Let's mark the separate facts:

а) $\ldots\mathbf{R}_{\Delta}\setminus\omega_1\to\omega_0\setminus\mathbf{R}_{-}\setminus\omega_1\to\omega_0\setminus\mathbf{R}_f = \{:a\}\setminus\omega_1\to\omega_0\setminus\{\triangle a\}\ldots$;

б) $f(x_1,\ldots,x_n)\setminus\omega_1\to\omega_0\setminus k^{f(\log_k x_1,\ldots,\log_k x_n)}$;

$f(x_1,\ldots,x_n)\setminus\omega_i\to\omega_0\setminus\left(i+\mathrm{slog}_k f\left(\left(-i+\mathrm{slog}_k x_1\right)_k,\ldots,\left(-i+\mathrm{slog}_k x_n\right)_k\right)\right)_k$;

в) $f'(x) = \lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x} \quad \backslash \omega_1 \to \omega_0 \backslash \quad {}^{'}\!f(x) =$

$$= \lim_{\Delta x \to 1} \frac{f(x \cdot \Delta x)}{f(x)} \dot{\triangle} (\Delta x);$$

$$f'(x) \backslash \omega_{-1} \to \omega_0 \backslash \ \overset{\circ}{f}(x) = \lim_{\delta x \to (-\infty)} \left(\log_k \left(k^{f\left(\log_k\left(k^x + k^{\delta x}\right)\right)} - k^{f(x)} \right) - \delta x \right);$$

${}^{'}\!f(x) = k^{x \cdot f'(x)/f(x)}$; $f'(x) = k^{\overset{\circ}{f}(x) - f(x) + x}$. (Reducing a rank of operations $\{\times ; \div\} \to \{+ ; -\}$ and substituting objects $\left\{ f'(x);\ {}^{'}\!f(x) \right\} \to \left\{ \overset{\circ}{f}(x);\ f'(x) \right\}$, we obtain identical on a structure of the formula).

Besides identified ω-transformations of operations (in particular, $\left\{ +; - ; \times ; \div ; \sum_n, f'; \int f\,dx; \ldots \right\} \ \backslash \omega_1 \to \omega_0 \backslash \left\{ \times ; \div ; \odot ; \dot{\triangle} ; \prod_n ; k^{x \cdot (\ln f)'} ; \exp\left(\int \frac{\log_k f}{x} dx \right); \ \ldots \right\}$) reduce in shaping the modified calculuses.

As an example of the facts from such analyses we shall note ω-images of the formula B. Taylor.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a) \cdot (x-a)^n}{n!} \quad \backslash \omega_1 \to \omega_0 \backslash \quad \prod_{n=0}^{\infty} \left({}^{(n)}\!f(a) \right)^{\left(\log_k \left(\frac{x}{a} \right) \right)^n / n!},$$

for $\backslash \omega_{-1} \to \omega_0 \backslash \ \log_k \sum\limits_{n=0}^{\infty} k^{\overset{\circ}{(n)}\!f(a) + n \cdot \log_k \left(k^x - k^a \right) - \log_k n!}$, $k \neq 1$.

In work [1] the ω-images of the numerical methods of a solution of the differential equations are reduced also which fix perspective directions ω-figurative research of a physical reality.

4. Courses of practical application of operation "$\circ$"

With introduction of operation of a type "$\circ$" the premises for exact mathematical exposition of passage of object from one condition in another are created. For example, for mathematical modelling of transient "rest - driving" and on the contrary use impulse and step-functions [2]. However, the mathematical sentences, in which these functions are applied, are necessary to consider as heuristic and requiring in strict substantiations. Approximations of these functions continuous, the representations by an integral the J. Fourier or replacement by integrals of the T. Stieltjes at introduction of generalized concepts of "functions" and "differential" (theory of distributions of the P. Laurent of the H. Schwarz) sometimes do not give qualitative outcomes [2].

The operations of a type "$\circ$" allow ideally to describe any situation connected to an impulse behaviour of object. Introducing auxiliary functional components as special expressions, it is possible to note various "rugged" functions. Examples of such components are reduced below.

1). Auxiliary function:

$$XSU(x;a)=(x\circ a)-a-1.$$

$$XSU(x;a)=\begin{cases}0, & x<a;\\ 1, & x=a;\\ x-a, & x>a.\end{cases}$$

2). Auxiliary function:

$$ST(x;a)=x-(x\circ a)+a+1.$$

$$ST(x;a)=\begin{cases}x, & x<a;\\ a-1, & x=a;\\ a, & x>a.\end{cases}$$

3). Function of inclusion:

$$ON(x;a;c)=c\cdot\operatorname{sgn}(XSU(x;a)) \text{ or at } c=1:$$

$$ON(x;a;1)=ON(x;a).$$

$$ON(x;a;c)=\begin{cases}0, & x<a;\\ c, & x\geq a.\end{cases}$$

4). Function of a cut-off:

$$OFF(x;\, a;\, c) = c - ON(x;\, a;\, c) \text{ or at } c = 1:$$

$$OFF(x;\, a;\, 1) = OFF(x;\, a).$$

$$OFF(x;\, a;\, c) = \begin{cases} c, & x < a; \\ 0, & x \geq a. \end{cases}$$

5). Step-function:

$$S(x;\, a;\, b;\, c) = ON(x;\, a;\, c) \cdot OFF(x;\, b) \text{ or at } c = 1:$$

$$S(x;\, a;\, b;\, 1) = S(x;\, a;\, b).$$

$$S(x;\, a;\, b;\, c) = \begin{cases} 0, & x < a; \\ c, & a \leq x < b; \\ 0, & x \geq b. \end{cases}$$

6). Auxiliary function:

$$GAS(x;\, a) = XSU(x;\, a) \cdot (ST(x;\, a) - a).$$

$$GAS(x;\, a) = \begin{cases} 0, & x < a; \\ -1, & x = a; \\ 0, & x > a. \end{cases}$$

7). Module x:

$$|x| = (1 - (-x \circ 0)) \cdot ((x \circ 0) - 2) + ((x \circ 0) - 1) \cdot (2 - (-x \circ 0)), \text{ either}$$

$$|x| = (1 - (-x \circ 0)) \cdot ((x \circ 0) - 2) + XS(x;\, 1) \cdot (2 - (-x \circ 0)).$$

$$|x| = \begin{cases} -x, & x < 0; \\ 0, & x = 0; \\ x, & x > 0. \end{cases}$$

8). Maximum of two numbers $a \text{ and } b$:

$$\max(a;\, b) = (a \circ b) - 1 + GAS(a - b;\, 0), \text{ either}$$

$$\max(a;\, b) = (a \circ b) - 2 + |\operatorname{sgn}(a - b)|.$$

$$\max(a;\, b) = \begin{cases} a, & a > b; \\ b, & a < b; \\ a = b, & a = b. \end{cases}$$

9). Minimum of two numbers $a \text{ and } b$:

$$\min(a;\, b) = -\max(-a;\, -b).$$

10). Function of a sign:

$$\operatorname{sgn}(x) = \frac{|x|}{x} + GAS(x;0).$$

$$\operatorname{sgn}(x) = \begin{cases} -1, & x < 0; \\ 0, & x = 0; \\ 1, & x > 0. \end{cases}$$

Inference

The operations of a type "$\circ$" and inverses by it can be used at designing complex numbers of a new nature. From a table 1 follows, that the study of new operations and numbers can be continued.

Δ-numbers allow deeper to perceive such global concepts as "infinity" and "a minus infinity ". The large interest is represented ω- images of various physical substance, ω- images are not investigated yet...

References

1. Rubtsov C.A. New mathematical objects, - Belgorod: BelGTASM, Kiev: NPP INFORMAVTOSIM. - 1996, p. 251 (In Russian)
2. G.Korn, T.Korn, Mathematical Handbook (for scientists and engineers), McGraw - Hill Bool Company, N.Y., S.Fr., Toronto, London, Sydney, 1968, p. 832
3. Rubtsov C.A. Algorithms ingredients in a set of algebraic operations // Cybernetics., - 1989. - № 3, p. 111-112 (In Russian)
4. D.J.Struik, Abriss der Geshichte der Mathematik, Veb deutscher Verlag der Wissenschaften, Berlin, 1963