

## UNIT -III ELECTRIC AND MAGNETIC FIELD IN MATERIALS

### 3.1 Poisson's and Laplace's Equation

From point form of Gauss's law  $\nabla \cdot \mathbf{D} = \rho_v \rightarrow (1)$

$$\text{But } \mathbf{D} = \epsilon \mathbf{E}$$

Sub the value of 'D' in (1) we get  $\nabla \cdot (\epsilon \mathbf{E}) = \rho_v$

$$\nabla \cdot \mathbf{E} = \frac{\rho_v}{\epsilon} \rightarrow (2)$$

we know that  $\mathbf{E} = -\nabla V$

Sub. this value in (2) we get

$$\nabla \cdot (-\nabla V) = \frac{\rho_v}{\epsilon}$$

$$\nabla \cdot (\nabla V) = -\frac{\rho_v}{\epsilon} \rightarrow (3)$$

$$\nabla \cdot \nabla V = \frac{\partial}{\partial x} \mathbf{a}_x \cdot \left( \frac{\partial V}{\partial x} \right) \mathbf{a}_x + \frac{\partial}{\partial y} \mathbf{a}_y \cdot \left( \frac{\partial V}{\partial y} \right) \mathbf{a}_y + \frac{\partial}{\partial z} \mathbf{a}_z \cdot \left( \frac{\partial V}{\partial z} \right) \mathbf{a}_z$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

$$\nabla^2 V = -\frac{\rho_v}{\epsilon} \rightarrow (4)$$

Equation (4) is said to be Poisson's Equation

If  $\rho_v = 0$  indicating zero volume charge density  $\therefore \nabla^2 V = 0 \rightarrow (5)$

This is Laplace's equation.  $\nabla^2$  is Laplacian of 'V'

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad [\text{Cartesian}]$$

$$\nabla^2 V = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial V}{\partial \rho} \right) + \frac{1}{\rho^2} \left( \frac{\partial^2 V}{\partial \phi^2} \right) + \frac{\partial^2 V}{\partial z^2} \quad [\text{Cylindrical}]$$

$$\nabla^2 V = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2}$$

[Spherical]

### 3.2 Electric Polarization

The dipole moment 'P' is  $P = qd \rightarrow (1)$

q -> positive one of the two bound charges comprising the dipole.

d -> vector from negative to positive charge.

If there are 'n' dipoles per unit volume, then for a volume ' $\Delta V$ ', there are ' $n\Delta V$ ' dipoles. The dipole moment is obtained by vector sum.

$$P_{\text{total}} = \sum_{i=1}^{n\Delta V} p_i \rightarrow (2)$$

Polarization 'P' is defined as the dipole moment per unit volume.

$$P = \lim_{\Delta V \rightarrow 0} \frac{1}{\Delta V} \sum_{i=1}^{n\Delta V} p_i \quad \text{C/m}^2 \rightarrow (3)$$

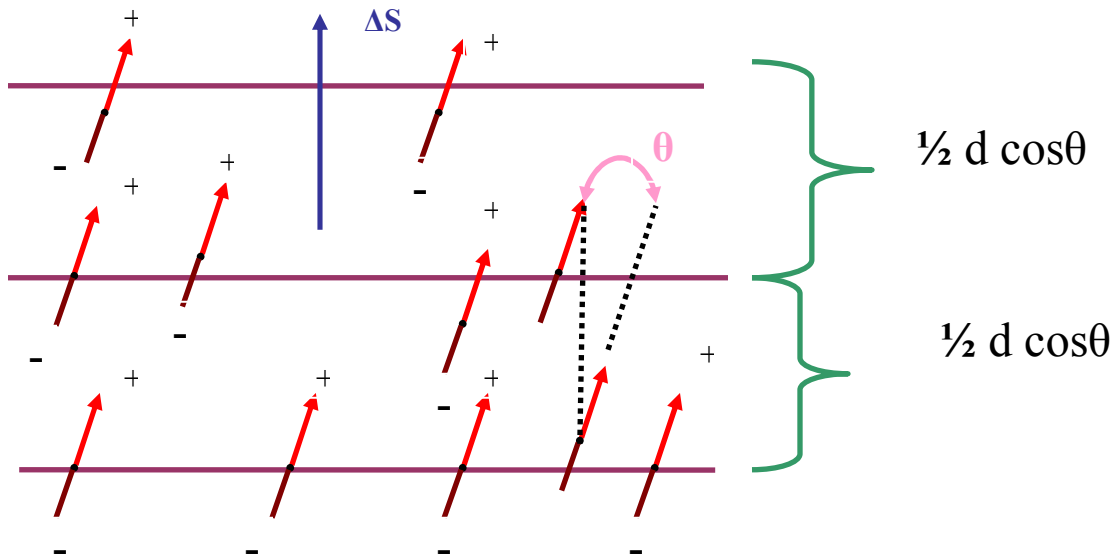


Figure shows non-polar molecules form dipole moments 'p'. There is a net transfer of bound charges across  $\Delta s$ .

The total molecules per unit volume =  $n$  molecules/m<sup>3</sup>.

The net total charge which crosses the elemental surface in  $\uparrow$  direction is

$$nqd \cos \theta \bullet \Delta s$$

$$\Delta q_b = nqd \bullet \Delta s \quad \rightarrow (4)$$

$q_{b \rightarrow}$  bound charge not a free charge

where  $p = nqd$  (dipole moment per unit volume)

$$\Delta q_b = p \bullet \Delta s \quad \rightarrow (5)$$

Net increase in bound charge within a closed surface

$$q_b = -\oint_S p \bullet ds \quad \rightarrow (6)$$

-ve sign indicates the direction  $\Delta S$  is outwards.

By using Gauss law, the total enclosed charge is  $q_T = \oint \epsilon_0 \mathbf{E} \cdot d\mathbf{s} \rightarrow (7)$

Where  $q_T = q_b + q$

$q_b \rightarrow$  bound charge

$q \rightarrow$  free charge

$$q = q_T - q_b$$

$$= \oint \epsilon_0 \mathbf{E} \cdot d\mathbf{s} - [-\oint \mathbf{p} \cdot d\mathbf{s}]$$

$$= \oint \epsilon_0 \mathbf{E} \cdot d\mathbf{s} + \oint \mathbf{p} \cdot d\mathbf{s}$$

$$q = \oint (\mathbf{P} + \epsilon_0 \mathbf{E}) \cdot d\mathbf{s} \rightarrow (8)$$

$$q = \oint \mathbf{D} \cdot d\mathbf{s} \rightarrow (9)$$

$$(8), (9) \Rightarrow \mathbf{D} = \mathbf{P} + \epsilon_0 \mathbf{E} \rightarrow (10)$$

Equation (10) gives the electric flux density for dielectric material.

### 3.3 Nature of dielectric materials:

- A dielectric in an electric field can be viewed as free space arrangement of microscopic electric dipoles which are composed of +ve & -ve charges whose centre do not coincide.
- Dipoles are not free charges & they cannot contribute to the conduction process. Rather they are bound in space by atomic & molecular forces & can only shift positions slightly in response to external fields. They are called bound charges in contrast to the free charges that determine conductivity.

- The characteristic which all dielectric materials have in common is their ability to store electric energy. Their storage takes place by means of shift in relative position of the internal bound +ve & -ve charges against the normal molecular & atomic forces.
- The actual mechanism of the charge displacement differs in various dielectric materials.
  - a) Some molecules termed as polar molecules have a permanent displacement existing between centers of gravity of +ve & -ve charges and each pair of charges act as a dipole.
  - b) Non-polar molecules do not have the dipole moment until a field is applied.

### 3.4 Definition of capacitance:

#### Capacitors

A capacitor is an electrical device consists of two conductors separated by an insulating (or) dielectric medium.

By definition, **the capacitance of capacitor** is the ratio of the charge on one of its conductors to the potential difference between them.

$$C = \frac{q}{V} \text{ Farad (or) coulombs/volts}$$

### 3.5 Capacitance of various geometrics using Laplace equation:

#### (1) Cartesian co-ordinate system:

Let us assume that potential 'v' is a function of 'x' only.

$$\therefore \text{Laplace equation } \frac{\partial^2 v}{\partial x^2} = 0 \quad \rightarrow (1)$$

$$V = Ax + B \quad \rightarrow (2)$$

$$\text{At } X = 0, V = 0 \quad \Rightarrow B = 0 \quad \rightarrow (3)$$

$$\text{At } X = d, V = V_0 \quad \Rightarrow A = \frac{V_0}{d} \quad \rightarrow (4)$$

$$\therefore V = \frac{V_0}{d} X \quad \rightarrow (5)$$

To find capacitance value:

**Step 1:**

$$\begin{aligned} E &= -\nabla V \\ &= -\nabla \left( \frac{V_0 X}{d} \right) \\ &= -\frac{\partial}{\partial X} \left( \frac{V_0 X}{d} \right) \mathbf{a}_x \end{aligned}$$

$$E = -\frac{V_0}{d} \mathbf{a}_x$$

**Step 2:**

$$D = \epsilon E$$

$$D = -\frac{\epsilon V_0}{d} \mathbf{a}_x$$

**Step 3:**

$$D_N = \rho_s$$

$$D_N = -\left( \frac{\epsilon V_0}{d} \right) = \rho_s$$

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**Step 4:**

$$\begin{aligned}q &= \int_s \rho_s ds \\&= -\int \frac{\epsilon V_0}{d} ds \\q &= -\left(\frac{\epsilon V_0}{d}\right)s\end{aligned}$$

Capacitance  $c = \frac{|q|}{V_0} = \frac{\epsilon S}{d}$

$s \rightarrow$  Area of plates

$d \rightarrow$  Distance between plates

This is equation of capacitance of a parallel plate capacitor.

**2) Cylindrical co-ordinate system:**

In cylindrical co-ordinates assume variation w.r.t. ' $\rho$ ', Laplace equation becomes

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial v}{\partial \rho} \right) = 0 \quad \rightarrow (1)$$

$$\frac{\rho \partial v}{\partial \rho} = A$$

$$\frac{\partial v}{\partial \rho} = A / \rho$$

$$v = A \ln \rho + B \quad \rightarrow (2)$$

Upper Limit  $v = v_0$  at  $\rho = a$

Lower Limit  $v = 0$  at  $\rho = b$

Applying Upper boundary condition (2)  $\Rightarrow v_0 = A \ln a + B \quad \rightarrow (3)$

Applying Lower boundary condition  $0 = A \ln b + B \quad \rightarrow (4)$

$$(4) \Rightarrow B = -A \ln b$$

Substitute 'B' value in equation (3)

$$v_0 = A \ln a - A \ln b$$

$$v_0 = A \ln(a/b) \quad \rightarrow (5)$$

$$A = \frac{v_0}{\ln(a/b)}$$

Substitute 'A' and 'B' value in equation (2)

$$v = \frac{v_0}{\ln(a/b)} \ln \rho - \frac{v_0 \ln b}{\ln(a/b)}$$

$$v = \frac{v_0}{\ln(a/b)} \ln(\rho/b)$$

$$v = v_0 \frac{\ln(\rho/b)}{\ln(a/b)} \quad \rightarrow (6)$$

To find capacitance:

**Step 1:**

$$E = -\nabla v$$

$$E = -\left[ \frac{\partial v}{\partial \rho} \mathbf{a}_\rho \right]$$

$$E = -\left[ \frac{\partial}{\partial \rho} \left[ \frac{v_0 \ln \rho}{\ln(a/b)} \right] \mathbf{a}_\rho - \frac{\partial}{\partial \rho} \left[ \frac{v_0 \ln b}{\ln(a/b)} \right] \right]$$

$$E = -\frac{v_0}{\rho \ln(a/b)} \mathbf{a}_\rho$$

$$E = \frac{v_0}{\rho \ln(b/a)} \mathbf{a}_\rho$$

**Step 2:**

$$D = \epsilon E$$

$$D = \frac{\epsilon v_0}{\rho \ln(b/a)} \mathbf{a}_\rho$$

**Step 3:**

$$D_N = \frac{\epsilon v_0}{a \ln(b/a)}$$

**Step 4:**

$$D_N = \rho_s = \frac{\epsilon v_0}{a \ln(b/a)}$$

Step 5:

$$q = \int \rho_s ds$$

$$q = \int \frac{\epsilon v_0}{a \ln(b/a)} A$$

$$= \frac{\epsilon v_0}{a \ln(b/a)} A$$

$$= \frac{\epsilon v_0}{a \ln(b/a)} (2\pi a)L$$

$$q = \frac{2\pi\epsilon L}{\ln(b/a)} v_0$$

$$C = \frac{q}{v_0} = \frac{2\pi\epsilon L}{\ln(b/a)}$$

The above equation gives the capacitance of co-axial conductor with inner conductor radius 'a' and outer conductor radius 'b'.

3) **Spherical co-ordinate system:**

In spherical co-ordinate assume variation w.r.t. 'r' Laplace equation becomes

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial v}{\partial r} \right) = 0$$

$$\frac{\partial}{\partial r} \left( r^2 \frac{\partial v}{\partial r} \right) = 0$$

$$r^2 \frac{\partial v}{\partial r} = A$$

$$\frac{\partial v}{\partial r} = \frac{A}{r^2}$$

$$v = -\left( \frac{A}{r} \right) + B \quad \rightarrow (1)$$

Assume boundary conditions  $v = 0$  at  $r = b$  and  $v = v_0$  at  $r = a$

Apply upper boundary conditions

$$0 = -\left( \frac{A}{b} \right) + B$$

$$B = \frac{A}{b}$$

Apply lower boundary conditions

$$v_0 = -\left( \frac{A}{a} \right) + B$$

Substitute 'B' value in above equation

$$v_0 = -\left(\frac{A}{a}\right) + \frac{A}{b}$$

$$A\left(-\left(\frac{1}{a}\right) + \frac{1}{b}\right) = v_0$$

$$A\left(\frac{-b+a}{ab}\right) = v_0$$

$$A = v_0\left(\frac{ab}{a-b}\right) \rightarrow (2)$$

$$B = v_0\left(\frac{a}{a-b}\right) \rightarrow (3)$$

$$(1) \Rightarrow v = -v_0\left(\frac{ab/a-b}{r}\right) + v_0\frac{a}{a-b}$$

$$= v_0\left(\frac{-ab+ar}{r(a-b)}\right)$$

$$= v_0\frac{a(r-b)}{r(a-b)}$$

$$= v_0\left(\frac{1(1-b/r)}{1-b/a}\right)$$

$$= v_0\left(\frac{b(1/b-1/r)}{b(1/b-1/a)}\right)$$

$$v = v_0 \left( \frac{1/r - 1/b}{1/a - 1/b} \right) \rightarrow (4)$$

To find capacitance:

**Step 1:**

$$E = -\nabla v$$

$$= -\frac{\partial v}{\partial r} \mathbf{a}_r$$

$$= -v_0 \frac{(-1/r^2)}{1/a - 1/b} \mathbf{a}_r$$

$$E = \frac{v_0 / r^2}{1/a - 1/b} \mathbf{a}_r$$

**Step 2:**

$$D = \epsilon E$$

$$D = \frac{\epsilon v_0}{r^2 (1/a - 1/b)} \mathbf{a}_r$$

**Step 3:**

$$D_N = \rho_s = \frac{\epsilon v_0}{r^2 (1/a - 1/b)}$$

**Step 4:**

$$q = \int \rho_s ds$$

$$\begin{aligned}
&= \int \frac{\epsilon V_0}{r^2(1/a - 1/b)} ds \\
&= \frac{\epsilon V_0}{r^2(1/a - 1/b)} s \\
&= \frac{\epsilon V_0}{r^2(1/a - 1/b)} (4\pi r^2) \\
&= \frac{4\pi\epsilon V_0}{(1/a - 1/b)}
\end{aligned}$$

$$c = \frac{q}{V_0} = \frac{4\pi\epsilon}{1/a - 1/b}$$

The above equation gives the capacitance of a spherical capacitor.

**3.6** Electrostatic energy and energy density (or) energy density in an electrostatic field:

$$c = \frac{q}{V} \rightarrow (1)$$

$$q = cV \rightarrow (2)$$

$$V = \frac{q}{c} \rightarrow (3)$$

$$V = \frac{\text{workdone}}{\text{unit charge}}$$

$$v = \frac{dw}{dq}$$

$$dw = v dq$$

$$(3) \Rightarrow dw = \frac{q}{c} dq$$

$$w = \int \frac{q}{c} dq$$

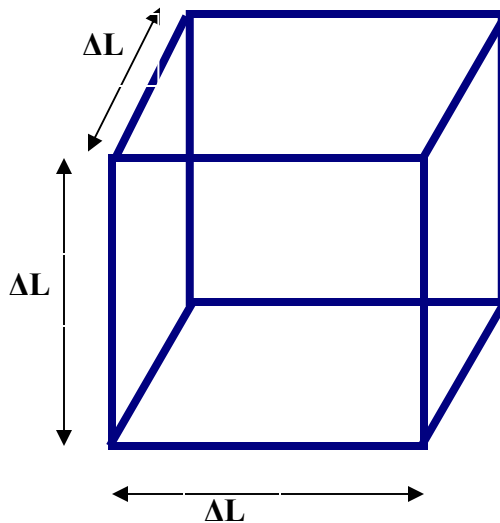
$$w = \frac{q^2}{2c} \rightarrow (4)$$

$$(2) \Rightarrow w = \frac{1}{2} \frac{c^2 v^2}{c}$$

$$w = \frac{1}{2} c v^2 \rightarrow (5)$$

This equation gives the energy stored in capacitor.

Energy density:



$$\Delta c = \frac{\epsilon A}{d}$$
$$= \frac{\epsilon (\Delta L)^2}{\Delta L}$$

$$\Delta c = \epsilon \Delta L \quad \rightarrow (6)$$

$$\Delta v = E \Delta L \quad \rightarrow (7)$$

$$\Delta w = \frac{1}{2} \Delta c \Delta v^2 \quad \rightarrow (8)$$

$$(6), (7) \Rightarrow \Delta w = \frac{1}{2} \epsilon \Delta L \times E^2 \Delta L^2$$

$$= \frac{1}{2} \epsilon E^2 (\Delta L)^3$$

$$\frac{\Delta w}{\Delta V} = \frac{1}{2} \epsilon E^2$$

$$w = \frac{1}{2} \epsilon E^2 \quad \text{J/m}^3 \quad \rightarrow (9)$$

**The above equation gives Energy stored per unit volume otherwise called as energy density.**

$$w = \frac{1}{2} DE \quad \rightarrow (10)$$

## Relation between 'w' and 'W' (i.e.) Energy 'w' and Energy Density 'W'

$$\Delta w = W \Delta v$$

$$w = \int W dv$$

$$w = \int_{\text{vol}} \frac{1}{2} \epsilon E dv$$

$$w = \int_{\text{vol}} \frac{1}{2} D E dv$$

### 3.7 Electric current (I):

Electric charge in motion constitute current. The unit is ampere (A) **defined as the rate of movement of charge passing a given reference point of one coulomb per second.**

$$I = \frac{dq}{dt}$$

### 3.8 Current density (J):

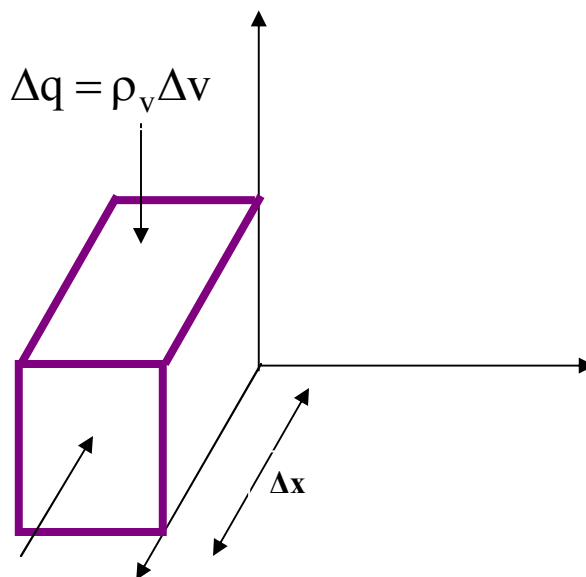


Figure shows an increment of charge  $\Delta q = \rho_v \Delta s \Delta L$  which moves a distance  $\Delta x$  in at time  $\Delta t$ .

$$\begin{aligned}\Delta I &= \frac{\Delta q}{\Delta t} \\ &= \frac{\rho_v \Delta v}{\Delta t} \\ &= \frac{\rho_v \Delta s \Delta L}{\Delta t} \\ &= \rho_v \Delta s \frac{\Delta L}{\Delta t} \\ &= \rho_v \Delta s V\end{aligned}$$

$$J = \frac{\Delta I}{\Delta s} = \rho_v v$$

$$J = \rho_v v$$

The current 'I' is conventional current.

$\rho_v \rightarrow$  volume charge density  $\text{C} / \text{m}^3$

$v \rightarrow$  velocity

$J = \rho_v v \rightarrow$  **Conventional current density.**

It is related linearly to volume charge density as well as to velocity.

### 3.9 Point form of Ohm's law:

Let  $V_d \rightarrow$  drift velocity and it is linearly related to the electric field intensity by the mobility of the electron in the given material.

$$V_d = \mu E$$

$$V_d = -\mu_e E \quad \rightarrow (1)$$

$\mu_e \rightarrow$  Mobility of electrons  $m^2 / V_s$

- ve sign is due to electron velocity is in direction opposite to the direction of 'E'.

$$J = \rho_v V$$

$$J = \rho_e V$$

$$(1) \Rightarrow J = -\rho_e \mu_e E \quad \rightarrow (2)$$

The - ve value of ' $\rho_e$ ' and the minus sign lead to a current density 'J' that is in the same direction as the electric field intensity 'E'.

$$J = \sigma E \quad \rightarrow (3)$$

where  $\sigma = -\rho_e \mu_e \quad \rightarrow (4) \quad s/m \text{ (or)} \quad \Omega/m.$

$\sigma \rightarrow$  Conductivity

**It may be expressed in terms of the charge density and the electron mobility.**

**Equation (3) is called as the point form of ohm's law.**

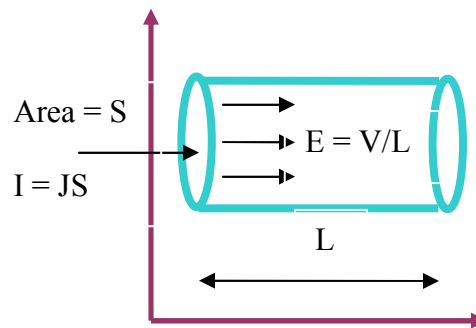
### 3.10 Ohm's Law:

$$V = IR$$

It states that the potential difference (or) voltage 'V' between the ends of conductor is equal to the product of its resistance 'R' and current 'I'.

### 3.11 Resistance(R):

Figure below shows uniform current density 'J' and electric field intensity 'E' in a cylindrical region of length 'L' and cross sectional area 'S' where  $V=IR$ .



$$E = \frac{V}{L} \rightarrow (1)$$

$$J = \frac{I}{s} \rightarrow (2)$$

$$J = \sigma E \rightarrow (3)$$

$$(2) = (3) \Rightarrow \frac{I}{s} = \sigma E$$

$$(1) \Rightarrow \frac{I}{s} = \sigma \frac{V}{L}$$

$$V = \frac{L}{\sigma S} I$$

$$\frac{V}{I} = R = \frac{L}{\sigma S}$$

$$R = \frac{L}{\sigma S} \quad \text{Resistance of a given material in ohms}$$

$L \rightarrow$  length of material

$\sigma \rightarrow$  Conductivity

$S \rightarrow$  Area

### 3.12 Continuity equation for current

The principle of conservation of charge states that charges can be neither created nor destroyed although equal amount of +ve & -ve charge may be simultaneously created, obtained by separation, destroyed or lost by recombination.

The continuity equation follows from this principle.

The current through the closed surface is

$$I = \oint \mathbf{J} \cdot d\mathbf{s} \quad \rightarrow (1)$$

This outward flow of +ve charge must be balanced by decrease of +ve charge within the closed surface. If the charge inside the closed surface is denoted by 'q<sub>i</sub>' then

the rate of decrease is  $-\left(\frac{dq_i}{dt}\right)$ , the principle of conservation of charge requires

$$I = \oint_s \mathbf{J} \cdot d\mathbf{s} = -\frac{dq_i}{dt} \rightarrow (2)$$

**Equation (2) is the integral form of the continuity equation** and the differential form or point is obtained by using the divergence theorem to change the surface integral into a volume integral.

$$\oint_s \mathbf{J} \cdot d\mathbf{s} = \int_{\text{vol}} (\nabla \cdot \mathbf{J}) dv$$

$$(2) \Rightarrow \int_{\text{vol}} (\nabla \cdot \mathbf{J}) dv = \oint_s \mathbf{J} \cdot d\mathbf{s} = -\frac{dq_i}{dt}$$

$$\int_{\text{vol}} (\nabla \cdot \mathbf{J}) dv = -\frac{d}{dt} \int_{\text{vol}} \rho_v dv$$

$$\int_{\text{vol}} (\nabla \cdot \mathbf{J}) dv = -\frac{d}{dt} \int_{\text{vol}} \rho_v dv$$

$$= -\int_{\text{vol}} \frac{\partial \rho_v}{\partial t} dv$$

$$(\nabla \cdot \mathbf{J}) = -\left(\frac{\partial \rho_v}{\partial t}\right)$$

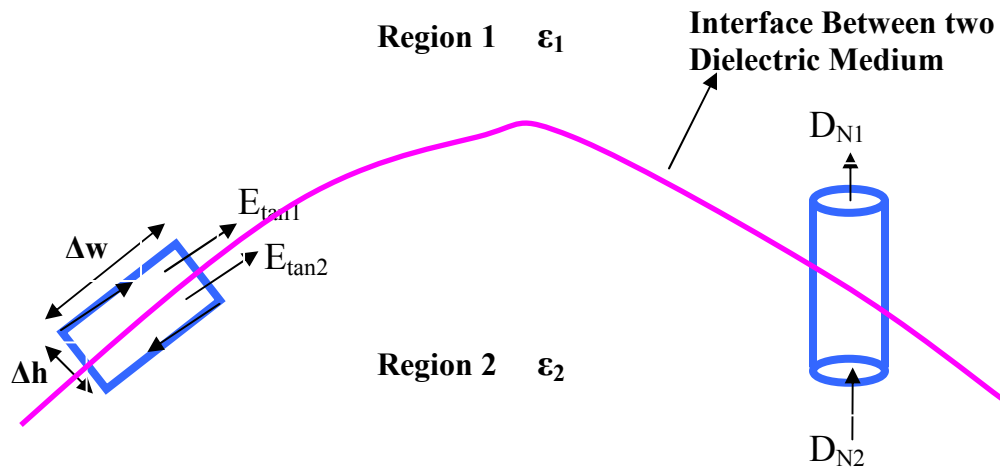
$$\nabla \cdot \mathbf{J} = -\left(\frac{\partial \rho_v}{\partial t}\right) \rightarrow (3)$$

Equation (3) gives the point form (or) differential form of the continuity equation. This equation indicates that current (or) charges per second, diverging form a

small volume per unit volume as equal to the time rate of decrease of charge per unit volume at every point.

### 3.13 Boundary conditions for electric fields:

#### (I) Boundary conditions for dielectrics



Consider an interface between two dielectrics having permittivity  $\epsilon_1$  and  $\epsilon_2$  and occupying region 1 and 2 as shown in fig above

#### (i) Boundary conditions on tangential component:

$$\oint \mathbf{E} \cdot d\mathbf{L} = 0$$

Around the small closed path

$$E_{\tan 1} \Delta w = E_{\tan 2} \Delta w = 0$$

$$E_{\tan 1} = E_{\tan 2} \rightarrow (1)$$

$$\frac{D_{\tan 1}}{\epsilon_1} = \frac{D_{\tan 2}}{\epsilon_2}$$

$$\frac{D_{\tan 1}}{\epsilon_{r1} \epsilon_0} = \frac{D_{\tan 2}}{\epsilon_{r2} \epsilon_0}$$

$$\frac{D_{\tan 1}}{D_{\tan 2}} = \frac{\epsilon_1}{\epsilon_2} \rightarrow (2)$$

**Note1:** From (1), **Tangential electric field intensity is continuous across boundary** whereas from (2) **Tangential flux density is discontinuous across the boundary.**

**(ii) Boundary conditions on normal component:**

$$\oint \mathbf{D}_s \cdot d\mathbf{S} = q$$

Since the sides are very short and the flux leaving the top and bottom surface is the difference

$$D_{N1} \cdot \Delta s - D_{N2} \cdot \Delta s = \Delta q$$

$$D_{N1} \cdot \Delta s - D_{N2} \cdot \Delta s = \rho_s \Delta s$$

$$D_{N1} - D_{N2} = \rho_s \rightarrow (3)$$

As there are no free charge available for perfect dielectrics. Therefore at the interface ' $\rho_s$ ' is zero.

$$D_{N1} - D_{N2} = 0$$

$$D_{N1} = D_{N2} \rightarrow (4)$$

$$\epsilon_1 E_{N1} = \epsilon_2 E_{N2}$$

$$\frac{E_{N1}}{E_{N2}} = \frac{\epsilon_2}{\epsilon_1} \rightarrow (5)$$

**Note 2:** From (4), **the normal component of the flux density is continuous across the charge – free boundary between two dielectrics.** From (5), **the normal component of the electric field intensity ‘E’ is discontinuous.**

(iii) Change in vectors ‘D’ and ‘E’:

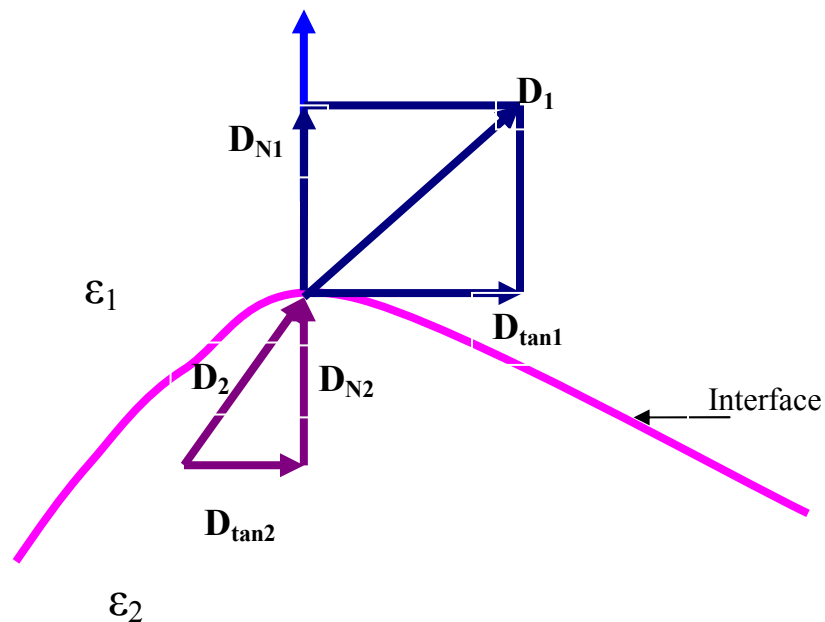


Figure shows refraction of ‘D’ at a dielectric interface for the case shown

$$\epsilon_1 > \epsilon_2, E_1 < E_2, D_1 > D_2.$$

Let  $D_1, E_1$  make an angle  $\theta_1$  with a normal to the surface.

$$\text{We know that } D_{N1} = D_{N2} \rightarrow (1)$$

$$D_{N1} = D_1 \cos \theta_1$$

$$D_{N2} = D_2 \cos \theta_2$$

$$(1) \Rightarrow D_{N1} = D_1 \cos \theta_1 = D_2 \cos \theta_2 = D_{N2} \quad \rightarrow (2)$$

$$\frac{D_{\tan 1}}{D_{\tan 2}} = \frac{\epsilon_1}{\epsilon_2}$$

$$\frac{D_{\tan 1}}{D_{\tan 2}} = \frac{D_1 \sin \theta_1}{D_2 \sin \theta_2} = \frac{\epsilon_1}{\epsilon_2} \quad \rightarrow (3)$$

$$(3) \Rightarrow \epsilon_2 D_1 \sin \theta_1 = \epsilon_1 D_2 \sin \theta_2 \quad \rightarrow (4)$$

$$\frac{(4)}{(2)} \Rightarrow \epsilon_2 \tan \theta_1 = \epsilon_1 \tan \theta_2$$

$$\frac{\tan \theta_1}{\tan \theta_2} = \frac{\epsilon_1}{\epsilon_2} \quad \rightarrow (5)$$

The magnitude of  $D_2$  from (2), (4)

$$D_2 \sin \theta_2 = \frac{\epsilon_2}{\epsilon_1} D_1 \sin \theta_1 \quad \rightarrow (5)$$

$$(2) \Rightarrow D_2 \cos \theta_2 = D_1 \cos \theta_1$$

$$(4)^2 + (2)^2 \Rightarrow D_2^2 = \left( \frac{\epsilon_2}{\epsilon_1} \right)^2 D_1^2 \sin^2 \theta_1 + D_1^2 \cos^2 \theta_1$$

$$D_2 = \sqrt{D_1 \cos^2 \theta_1 + \left(\frac{\epsilon_2}{\epsilon_1}\right)^2 \sin^2 \theta_1} \rightarrow (6)$$

$$E_2 = \frac{D_2}{\epsilon_2} = \frac{D_1}{\epsilon_2} \sqrt{\cos^2 \theta_1 + \left(\frac{\epsilon_2}{\epsilon_1}\right)^2 \sin^2 \theta_1}$$

$$E_2 = \frac{\epsilon_1 E_1}{E_2} \sqrt{\cos^2 \theta_1 + \left(\frac{\epsilon_2}{\epsilon_1}\right)^2 \sin^2 \theta_1}$$

$$E_2 = E_1 \sqrt{\left(\frac{\epsilon_1}{\epsilon_2}\right)^2 \cos^2 \theta_1 + \sin^2 \theta_1} \rightarrow (7)$$

(6), (7) Allow to find quickly field on one side of a boundary if we know the field on the other side.

**(II) Boundary conditions for conductor:**

**(i) Boundary conditions on tangential components:**

For perfect dielectrics,

$$E_{\tan 1} = E_{\tan 2} \rightarrow (1)$$

$$\frac{D_{\tan 1}}{D_{\tan 2}} = \frac{\epsilon_1}{\epsilon_2} \rightarrow (2)$$

If medium '2' is a conductor, the field  $E_{\tan 2}$  in medium '1' must be zero under static conditions.

$$(1) \Rightarrow E_{\tan 1} = 0 \rightarrow (3)$$

$$D_{\tan 1} = 0 \rightarrow (4)$$

(ii) **Boundary conditions on normal components of perfect conductors**

Boundary conditions on normal components of perfect dielectrics.

$$D_{N1} - D_{N2} = \rho_s \rightarrow (5)$$

If medium '2' is a conductor  $D_{N2} = 0$

$$D_{N1} = \rho_s$$

$$D_{N2} = 0 \rightarrow (6)$$

$$E_{N1} = \frac{\rho_s}{\epsilon_1} \rightarrow (7)$$

### 3.14 Definition of Inductance:

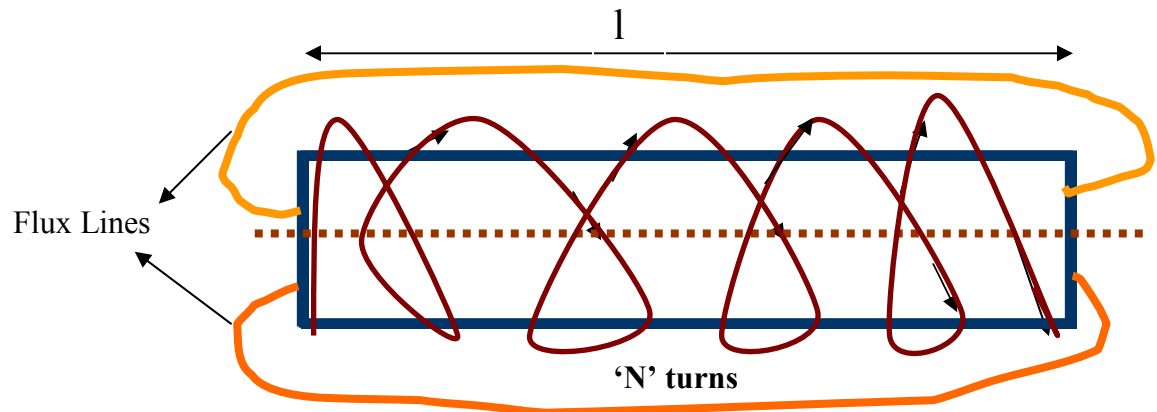
**Inductance (or self inductance) 'L' is defined as ratio of the total flux linkages to the current when they link.**

$$L = \frac{N\phi}{I} \quad \text{Henry}$$

$I \rightarrow$  Current in N-turn coil produces the total flux ' $\phi$ ' and ' $N\phi$ ' flux linkages.

$N\phi \rightarrow$  Flux linkages  $\rightarrow$  is defined as the product of no. of turns ' $N$ ' and flux ' $\phi$ ' linking each of them.

### 3.15 Inductance of solenoids:



If ' $B$ ' is the flux density and ' $A$ ' is the cross sectional area of solenoid, then the total flux linkage of a large solenoid is

$$\Lambda = N\phi \quad \rightarrow (1)$$

$$\Lambda = NBA \quad \rightarrow (2)$$

where  $\phi = BA$

For long solenoid,

$$B = \frac{\mu_0 NI}{l} \rightarrow (4)$$

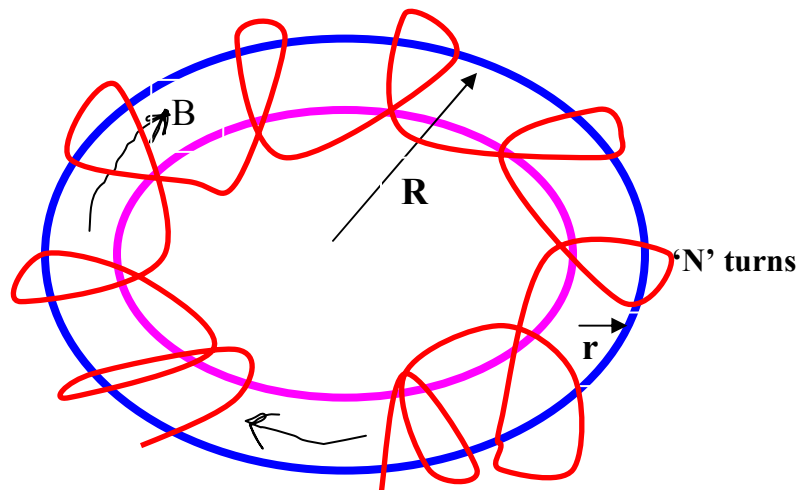
$$(2) \Rightarrow \Lambda = \frac{N\mu_0 NI}{l} A$$

$$\Lambda = \frac{\mu_0 N^2 I}{l} A$$

$$\frac{\Lambda}{I} = \frac{\mu_0 N^2 A}{l}$$

$$L = \frac{\Lambda}{I} = \frac{\mu_0 N^2 A}{l} \quad \text{Henry}$$

### 3.16 Inductance of toroid



$R \rightarrow$  radius of toroid

$r \rightarrow$  radius of coil

**Toroid**  $\rightarrow$  If a long solenoid is bent into the form of a ring and thereby closed on itself, it becomes a toroidal solenoid or simply a toroid.

The flux linkage is

$$\Lambda = N\phi$$

$$= NBA$$

$$\Lambda = N \frac{\mu NI}{l} A$$

$$\Lambda = \frac{\mu N^2 I}{l} A$$

where  $l = 2\pi R$

$$\therefore \Lambda = \frac{\mu N^2 I}{2\pi R} A$$

$$L = \frac{\Lambda}{I} = \frac{\mu N^2}{2\pi R} A$$

For air core toroid  $\mu = \mu_0$

$$L = \frac{\mu_0 N^2 A}{2\pi R}$$

where  $A = \pi r^2$

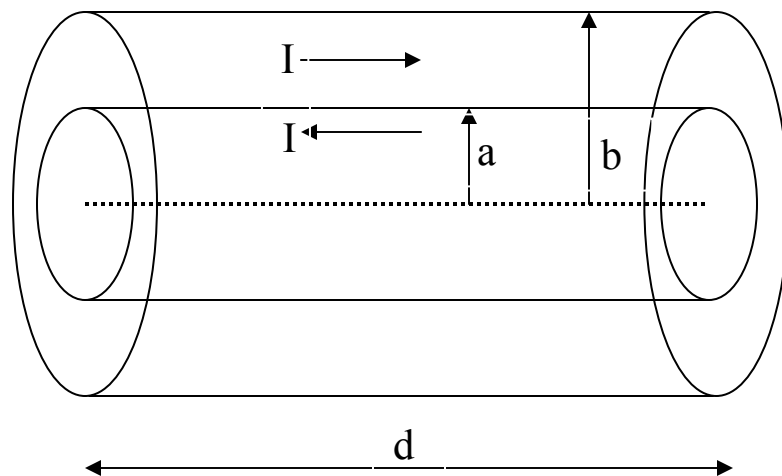
$A \rightarrow$  Cross sectional area of the coil.

$$L = \frac{\mu_0 N^2}{2\pi R} (\pi r^2)$$

$$L = \frac{\mu_0 N^2 r^2}{2R}$$

### 3.17 Inductance of the co-axial cable:

Consider a co-axial transmission line constructed of conducting cylinders of radius 'a' and 'b' as shown in figure. The current in the inner conductor is 'I' but the current in the outer conductor is of the same magnitude but opposite direction.



The flux density at any radius 'ρ'

$$B = \mu H$$

$$B = \frac{\mu I}{2\pi\rho} a_\rho \quad \rightarrow (1)$$

The total flux linkage per metre length

$$\Lambda = \int_{\rho=a}^b B \bullet d\rho a_\rho \quad \text{wb / m}$$

$$= \int_a^b \frac{\mu I}{2\pi\rho} a_\rho \bullet d\rho a_\rho$$

$$= \frac{\mu I}{2\pi} \int_a^b \frac{d\rho}{\rho}$$

$$= \frac{\mu I}{2\pi} [\ln \rho]_a^b$$

$$\lambda = \frac{\mu I}{2\pi} \ln\left(\frac{b}{a}\right) \quad \text{wb / m}$$

$$\lambda = \frac{d\mu I}{2\pi} \ln\left(\frac{b}{a}\right) \quad \text{weber}$$

d → Length of coaxial cable

$$L = \frac{\Lambda}{I} = \frac{\mu d}{2\pi} \ln\left(\frac{b}{a}\right) \quad \text{Henry} \quad \rightarrow (1)$$

$$\frac{L}{d} = \frac{\mu}{2\pi} \ln\left(\frac{b}{a}\right) \quad \text{H/m} \quad \rightarrow (2)$$

For air filled cable,

$$\frac{L}{d} = \frac{\mu_0}{2\pi} \ln\left(\frac{b}{a}\right) \quad \text{H/m}$$

**Equation (1) gives the inductance of a co-axial cable and**

**Equation (2) gives inductance per metre length of a co-axial cable**

### 3.18 Definition of mutual inductance

Mutual inductance between circuits 1 and 2  $M_{12}$  is defined in terms of mutual flux linkages

$$M_{12} = \frac{N_2 \phi_{12}}{I_1} \quad \text{Henry} \quad \rightarrow (1)$$

$\phi_{12} \rightarrow$  Signifies the flux produced by  $I_1$  which links the path of the filamentary current  $I_2$ .

$N_2 \rightarrow$  is the number of turns in circuit '2'.

$$M_{12} = M_{21} \quad \rightarrow (2)$$

### 3.19 Energy and Energy density in magnetic fields

#### Energy stored in a steady magnetic field 'B'

Voltage across the inductor is given by

$$V = L \frac{dI}{dt} \rightarrow (1)$$

Magnetic energy delivered to the load (inductor)

$$w_H = \int \text{power } dt$$

$$= \int VI dt$$

$$(1) \Rightarrow w_H = \int L \frac{dI}{dt} I dt$$

$$= \int LI dI$$

$$w_H = \frac{LI^2}{2} \quad \text{Joules}$$

**This Equation gives the energy stored in a steady magnetic field.**

#### Energy density:

$$\Delta w_H = \frac{1}{2} \Delta L \Delta I^2 \rightarrow (2)$$

$$\Delta I = H \Delta L \rightarrow (3)$$

Substitute (3) in (1),

$$\Delta w_H = \frac{1}{2} \Delta L H^2 \Delta L^2 \quad \rightarrow (4)$$

$$\Delta L = \mu \Delta l \quad \rightarrow (5)$$

Substitute (5) in (4)

$$\begin{aligned} \Delta w_H &= \frac{1}{2} \mu \Delta l \quad H^2 \Delta l^2 \\ &= \frac{1}{2} \mu H^2 \Delta l^3 \\ &= \frac{1}{2} \mu H^2 \Delta V \quad \text{Where } \Delta V = \Delta l^3 \end{aligned}$$

$$\frac{\Delta w_H}{\Delta V} = w_H = \frac{1}{2} \mu H^2 \quad \text{J/m}^3$$

$$w_H = \frac{1}{2} \mu H^2 \quad \text{J/m}^3 \quad \rightarrow (6)$$

**The energy per unit volume (or) energy density  $w_H$  of the magnetic field at the point is given by (6).**

**Energy in terms of energy density**

$$\begin{aligned} w_H &= \int_{\text{vol}} w_H \, dv \\ &= \int_{\text{vol}} \frac{1}{2} \mu H^2 \, dV \end{aligned}$$

$$w_H = \frac{1}{2} \int \mu H^2 dV \rightarrow (7)$$

$$w_H = \frac{1}{2} \int \mathbf{B} \cdot \mathbf{H} dV \rightarrow (8)$$

$$w_H = \frac{1}{2} \int \frac{B^2}{\mu} dV \rightarrow (9)$$

### 3.20 Nature of magnetic materials

#### 1) Non-magnetic

Materials in which the magnetic effect is so weak are to be non-magnetic materials.

Ex: vacuum.

#### 2) Diamagnetic

These materials magnetic effects are so weak. Although the spin and orbit magnetic moments cancel each other in the absence of external magnetic field. But an applied field causes the spin moment to slightly exceed the orbital moment, resulting in a small net magnetic moment which opposes the applied field 'B'.

Ex: Bismuth.

#### 3) Paramagnetic

These materials in which orbit and spin magnetic moments are unequal, resulting in a net magnetic moment for the atom, even with no applied field, when an external field is applied, the atomic dipoles tend to like up with the field so that the magnetic moment of the sample is increased in proportion to the number of atoms in the sample.

#### **4) Ferromagnetic**

In these substances, there is a quantum effect known as “exchange coupling” between adjacent atoms in the crystal lattice of the material. However at temperature above a critical value, known as the curie temperature, the exchange coupling disappears and the materials become an ordinary paramagnetic type.

Ex: Iron, Nickel and cobalt.

#### **5) Anti-ferromagnetic**

In these materials like magnetic moments of adjacent atoms align in opposite directions so that the net magnetic moment of a specimen is nil even in the presence of an applied field.

#### **6) Ferromagnetic**

The magnetic moments of the adjacent atoms in ferromagnetic substance are aligned opposite, but the moments are not equal so that there is a net magnetic moment, which is less than ferromagnetic.

#### **7) Super paramagnetic**

This consists of ferromagnetic particles suspended in a dielectric binder. Each particle may contain many magnetic domains, but exchange forces cannot penetrate to adjacent particles.

Ex: Tapes used in audio, video and data recording systems.

### 3.21 Magnetization and permeability

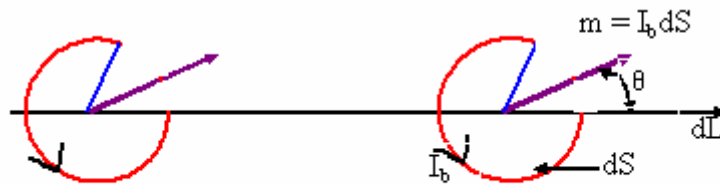


Figure shows a section 'dL' of a closed path along which the magnetic dipoles have been partially defined by some external magnetic field.

#### 1) Magnetic dipole moment 'm' ( $\text{Am}^2$ )

The bound current ' $I_b$ ' circulates about a path enclosing a differential area  $dS$  establishing a dipole moment ( $\text{Am}^2$ )

$$m = I_b ds \quad (\text{Am}^2) \quad \rightarrow (1)$$

If there are 'n' magnetic dipoles per unit volume and consider a volume ' $\Delta V$ ', then the sum.

$$m_{\text{total}} = \sum_{i=1}^{n\Delta v} m_i \quad \rightarrow (2)$$

#### 2) Magnetization (M) (A/m)

Magnetization 'M' defined as the magnetic dipole moment per unit volume.

$$M = \lim_{\Delta v \rightarrow 0} \frac{1}{\Delta V} \sum_{i=1}^{n\Delta v} m_i \quad \text{A/m} \quad \rightarrow (2)$$

3) Permeability ( $\mu$ ) H/m:

$$I_b = \oint \mathbf{M} \cdot d\mathbf{L} \quad \rightarrow (4)$$

Ampere's circuits law in terms of total current

$$\oint \mathbf{H} \cdot d\mathbf{L} = I_T \quad \rightarrow (5)$$

$$\oint \frac{\mathbf{B}}{\mu_0} \cdot d\mathbf{L} = I_T \quad \rightarrow (6)$$

$$I_T = I_b + I$$

where I is the total free current enclosed by the closed path.

$$I = I_T - I_b$$

$$(4), (6) \Rightarrow I = \oint \frac{\mathbf{B}}{\mu_0} \cdot d\mathbf{L} - \oint \mathbf{M} \cdot d\mathbf{L}$$

$$I = \oint \left( \left( \frac{\mathbf{B}}{\mu_0} \right) - \mathbf{M} \right) \cdot d\mathbf{L}$$

$$\mathbf{H} = \left( \frac{\mathbf{B}}{\mu_0} \right) - \mathbf{M}$$

$$\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}) \quad \rightarrow (7)$$

$\mu_0 \rightarrow$  Free space permeability ( $4\pi * 10^{-7}$  H/m)

For linear isotropic media,

$$\mathbf{M} = \chi_m \mathbf{H} \quad \rightarrow (8)$$

$\chi_m \rightarrow$  Magnetic susceptibility (unit less)

$\therefore$  substitute (8) in (7)

$$B = \mu_0(H + \chi_m H)$$

$$B = \mu_0(1 + \chi_m)H \rightarrow (9)$$

$$B = \mu_0\mu_r H \rightarrow (10)$$

Comparing (9),(10) where  $\mu_r = 1 + \chi_m \rightarrow (11)$

$\mu_r \rightarrow$  Relative permeability

$$\mu = \mu_0\mu_r \rightarrow (12)$$

Equation (12) is defined as permeability of magnetic medium. Unit: H/m

$$(10) \Rightarrow B = \mu H \rightarrow (13)$$

Equation (13) gives the relationship between 'B' and 'H'

### 3.22 Magnetic Boundary Conditions

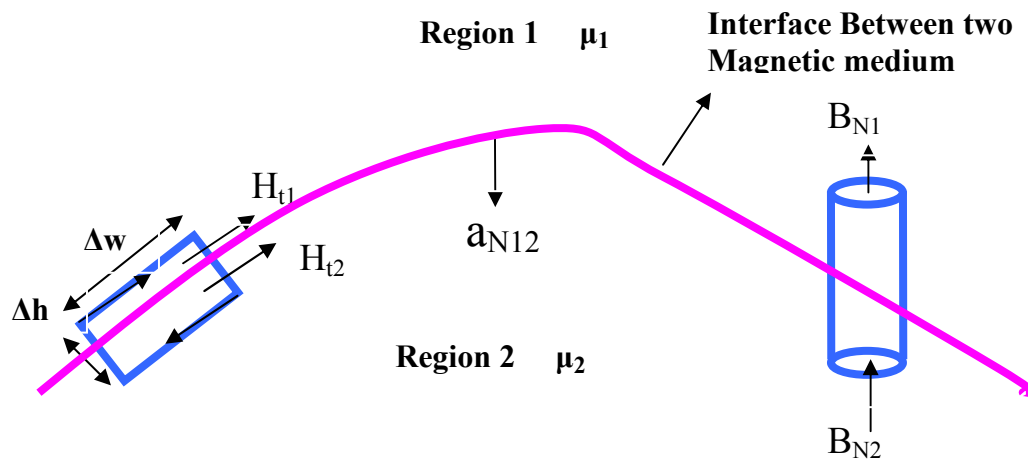


Figure shows a Gaussian surface and a closed path are constructed at the boundary between media 1 and 2, having permeability of  $\mu_1$  and  $\mu_2$  respectively.

**(1) Boundary conditions on normal components**

It is determined by considering the small cylindrical Gaussian surface.

Gauss law for the magnetic field,

$$\oint_s \mathbf{B} \cdot d\mathbf{s} = 0$$

$$B_{N1} \Delta s - B_{N2} \Delta s = 0$$

$$B_{N1} = B_{N2} \rightarrow (1)$$

$$\mu_2 H_{N2} = \mu_1 H_{N1}$$

$$H_{N2} = \frac{\mu_1}{\mu_2} H_{N1} \rightarrow (2)$$

Note 1: From Equation (1), **normal component of the magnetic flux density 'B' is continuous.**

Note 2: From Equation (2), **the normal component of magnetic field intensity 'H' is discontinuous.**

**(2) Boundary conditions on tangential components**

It is determined by considering the small closed rectangular path.

Ampere's circuital law,

$$\oint \mathbf{H} \cdot d\mathbf{L} = I$$

$$H_{t1} \Delta L - H_{t2} \Delta L = k \Delta L$$

$$H_{t1} - H_{t2} = k \rightarrow (3)$$

$$\left(\frac{B_{t1}}{\mu_1}\right) - \left(\frac{B_{t2}}{\mu_2}\right) = k \rightarrow (4)$$

Note 3: The directions are specified more exactly by using the product to identify the tangential components.

$$(H_1 - H_2) \times \mathbf{a}_{N12} = k$$

$$(H_{t1} - H_{t2}) \times \mathbf{a}_{N12} = k$$

Where  $\mathbf{a}_{N12}$  is the unit normal at the boundary directed from region 1 to region 2.

**(3) Boundary condition on the normal component of the Magnetization**

$$M_{N2} = \chi_{m2} \frac{\mu_1}{\mu_2} H_{N1}$$

$$M_{N2} = \frac{\chi_{m2} \mu_1}{\chi_{m1} \mu_2} M_{N1}$$

**(4) Boundary condition on the tangential component of Magnetization**

$$H_{t2} = H_{t1} - k$$

$$M_{t2} = \chi_{m2} H_{t2}$$

$$M_{t2} = \chi_{m2} H_{t1} - \chi_{m2} k$$

$$M_{t2} = \frac{\chi_{m2}}{\chi_{m1}} M_{t1} - \chi_{m2} k$$