Resonance Frequencies of a Transformer

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1 Introduction

In the engineering sciences, one is often confronted with resonance of solid structures or of systems involving fluid - structure interaction. Resonance occurs, when a physical system is subject to an oscillating force with frequency close to a natural frequency of the system. In such situations, even small external forces may destabilize the system, e.g., by amplifying the oscillations. Therefore, in the course of engineering of a new product one must be aware of this phenomenon and design the product so, that resonance makes as little problems as possible.

This paper is concerned with the problem of computing the natural frequencies of large electric power transformers. In large power transformer stations there are alternating Lorentz-forces acting on the conducting parts of the construction. The forces are caused by the alternating current in the coils of the transformer. These forces cause vibration of the casing, and therefore additional noise and mechanical stresses. Furthermore, energy is lost due to this vibration.

Therefore it is necessary to minimize the vibration by a special construction of the casing. For this purpose, the eigenfrequencies of the transformer are of a huge interest, the computation of which is the goal of this paper.

The reader will, however, see, that the methods and mathematical models developed and used here are applicable to a wide array of other problems involving solid-fluid interaction.

The structure of the paper is the following. In Chapter 2, we describe in more details the problem. The models are constructed and described in Chapter 3. The next chapter is concerning with numerical approximations and implementation of the models. Here we also present the numerical results. The final thoughts and conclusions are summarized in Chapter 5.

2 The Problem

A typical electrical power transformer consists of several coils, immersed in special oil, and a casing closing the construction and shielding the electromagnetic field. When an alternating current is applied to one of the coils, it induces an alternating current in the other coil, the voltage being dependent on the ratio of number of turns of wire in the both coils (by Faraday's law).

The alternating current in the coils causes an alternating magnetic field (Biot-Savart law), that, in it's turn, produces eddy-currents (circulating currents) in all the conducting parts of the transformer, including the casing.

Now, the eddy-currents in the casing are subject to the Lorentz force density $F = j \times B$, where j is the current density and B denotes the magnetic field. Since B is oscillating, so is F, and the frequency of the forces is twice as much as the one of the current in the coils (50 Hz resp. 60 Hz in US). Therefore the casing has to be designed in such a way that the resonance frequencies do not come close to 100-120 Hz.

To compute the natural frequencies (eigenfrequencies), it is not enough to model only the casing. Since the transformer is filled with oil, we have a coupled system for whose eigenfrequencies we are searching. By intuition we expect, that the presence of a dense fluid should decrease the natural frequencies of a system since mass is added.

3 The Models

In this section we are constructing different models of the mechanical behavior of the transformer. By a model we understand a well-posed system of equations with appropriate initial, boundary and interface conditions.

A general strategy for construction of models for fluid-solid inteaction is the following: start with a model of the solid structure and then add the equations of fluid dynamics. Most of the models can be formulated in both two and three dimensions. This strategy leads us not only to models already described in literature, but also to interesting hybrids like the *three-beams* and the *five plates* models that, to our best knowledge, are original.

The section is organized as follows. We start with describing lower dimensional approximations of the casing (the beam and plate models). In the next subsection we provide an unusual model for fluid dynamics, namely, the Lagrangian description of Euler's equations, which turns out to be very convenient for eigenfrequency computations. Finally, we provide a well-known model, based on a three-dimensional model of linear elasticity.

3.1 Beam/Plate Equations

We observe that the walls of the casing are very thin: the thickness comprising only a fraction of a centimeter, while the other dimensions being in meters. In such cases, the approximation by plates or beams often performs better than models of three-dimensional elasticity theory.

3.1.1 Three beams model

Let us consider a 2D cross-section of the transformer. The casing now consists of three parts, each of which can be modelled by using a beam equation.

Assuming, that only normal displacements are possible, we derive the condition, that the corners of the casing must be fixed.

The problem of what to do about the interaction of the beams can be avoided if we convince us that both configurations showed in figure 2 are in fact equivalent, if we describe the deformation in local coordinates.

Each of this casing-part can now be modelled as a single beam. We have the following equation for a beam, which describes the vibration (known as Euler-Bernoulli beam equation):

$$EI\frac{\partial^4 W}{\partial x^4} = p.$$

Here, E is the Young's modulus, I is the area momentum of inertia of the beam's cross-section, p is the force density working on the beam, and W is the normal displacement.

In the corners we analysed two possibilities to connect the single beams. The first one is to fix the corners, so that rotation is the only allowed movement, and to assume right angles, i.e. continuity of the first derivative of W. Since rotation is possible, we have continuity of momentum, too. The second possibility is to assume that the behaviour of the beams above is similar to a single beam with two fixed inner points, i.e. one for each corner.

The new problem is much simpler and the model can be constructed directly from the beam equation, with the additional assumption of two fixed interior points.

3.1.2 Five plates model

We assume that the casing of the transformer is a cube with the bottom face clamped in the floor. Each of the remaining five faces is modelled by a plate equation, giving rise to the name of the model.

In the most straight-forward plate model, we assume that the deformations of the plate are occuring only in the normal direction of the plate. If the displacement is denoted by W, then the following plate bending equation (also called Lagrange equation) can be used:

$$D\nabla^4 W + \rho h\frac{\partial^2 W}{\partial t^2} = p,$$

where ∇^4 is the biharmonic operator, p is the load (force density) on the plate, and $D = Eh^3/12(1-\nu^2)$ is called the flexural rigidity of the plate. Here E is the Youngs modulus of elasticity, ν is the Poisson-ratio, and h is the thickness of the plate.

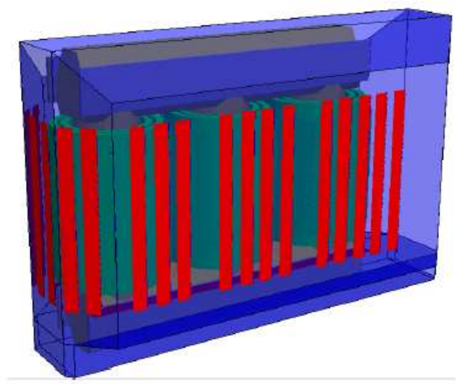

Figure 1: The Transformer

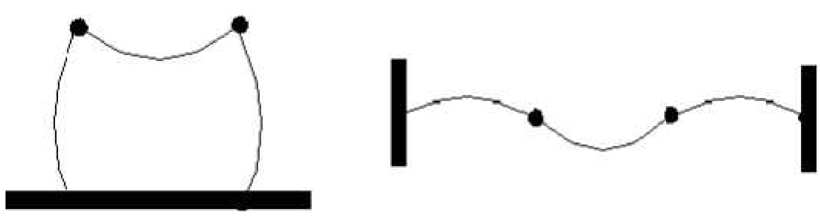

Figure 2: 3 beams model

Additionally, we must consider the interaction between the plates. If we asume that only normal deformations occur, then it is not difficult to see, that the edges of the cube must be considered as fixed. Indeed: by continuity requirements, any nonzero displacement of an edge leads to tangential displacement of the adjacent plate.

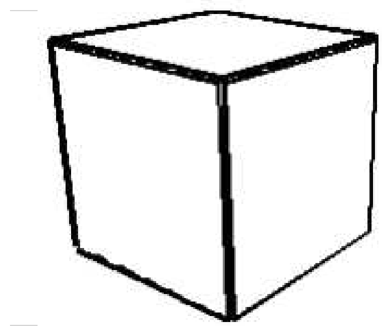

Figure 3: Simple Cube-Casing

To complete this model for the casing, we need to figure out how to model the interaction of two adjacent plates. We make a simple model for this interaction, assuming that the dihedral angles between any two plates remain constant, i.e., right, and that the bending moments are continuous across the edges.

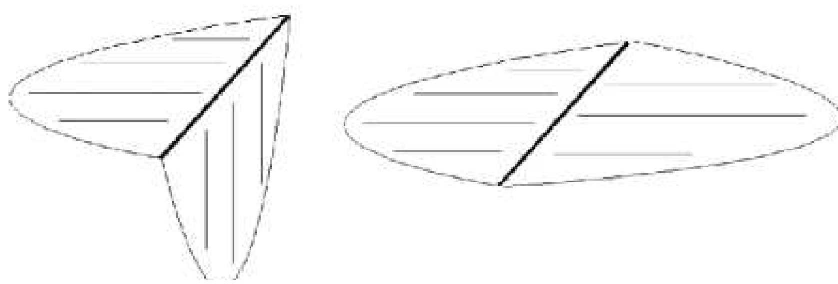

Figure 4: Locally interaction of plates

3.2 Equations for Fluid Dynamics : Lagrangian approach

As all physical equations, the equations of fluid dynamics may be given in different coordinate systems. However, there are two main approaches in description of fluids: to treat the fluid as a field (*Eulerian* approach) or to treat the fluid as an ensemble of particles (*Lagrangian* approach).

In Eulerian description, the field variables (e.g., velocity, pressure) are associated with the current position of the fluid particles, e.g., $p(x,t)$ denotes pressure near the point x in time t. This approach leads to material derivatives $(u \cdot \nabla)$, which make the equations nonlinear.

In the alternative - Lagrangian approach - one follows the fluid particles, e.g., $p(x,t)$ denotes pressure near the point, where the particle, initially situated at x, is located in the time t.

In virtually all standard textbooks on fluid dynamics, the Eulerian description is chosen. Nevertheless, in some applications the Lagrangian description is the more convenient choice (e.g., modelling dynamics of particles immersed in a fluid).

We provide two reasons, why we choose the Lagrangian approach. First: we observe that deformations of the casing cause deformations of the fluid domain: $\Omega_F = \Omega_F(t)$. The Lagrangian description, however, is a remedy: the fluid particles are associated with the unperturbed fluid domain $\Omega_F = \Omega_F(0)$. Another reason is the linearity of equations in the Lagrangian description, that will help us to evaluate the obtained models. Still, there is a third reason: as we shall see, the equations in Lagrangian formulation are very easy to handle.

Let $p(x,t)$ denote the fluid pressure and $u(x,t)$ denote the displacement (*not* velocity) in Lagrangian coordinates. For an incompressible, inviscid fluid in the absence of external forces, the equations are given by (see also [4]):

$$\begin{aligned} \nabla p &= \rho \ddot{u}, \\ \operatorname{div}(\dot{u}) &= 0, \end{aligned}$$

where ρ is the fluid density, and the dots stand for partial time derivatives.

3.3 Hydroelasticity Equations

In this subsection, we present a system of equations for the modelling of interaction between a solid body and an inviscid incompressible fluid as used in [4].

The variables and parameters we are using are:

u_F	fluid displacement,
p	fluid pressure,
ρ_F	fluid density,
u_S	solid displacement,
ρ_S	solid density,
$\epsilon(u_s)$	strain tensor in solid: $\epsilon(u)_{i,j} = (\partial_i u + \partial_j u)/2$,
$\sigma(u_S)$	stress tensor.

The equations are:

$$\begin{aligned} \operatorname{div}(\sigma(u_S)) = \rho_S \ddot{u}_S & \quad \text{in}\, \Omega_S, \\ \sigma(u_S)n + p \cdot n = 0 & \quad \text{on}\, \Gamma_I, \\ u_S \cdot n = u_F \cdot n & \quad \text{on}\, \Gamma_I, \\ \nabla p = \rho_F \ddot{u}_F & \quad \text{in}\, \Omega_F, \\ \operatorname{div}(\dot{u}_F) = 0 & \quad \text{in}\, \Omega_F, \\ \sigma(u_S)n = 0 & \quad \text{on}\, \Gamma_N, \\ u_S = 0 & \quad \text{on}\, \Gamma_D. \end{aligned}$$

The first equation represents the equilibrium of forces, the next two describe the interaction between the solid and the fluid. The fourth and fifth equations describe the motion and incompressibility of the fluid. The ground Γ_D is fixed, and the outer boundary Γ_N is free. This is represented by the last two equations.

We add the linear Hooke's law:

$$\sigma(u_S) = D\epsilon(u_S),$$

where D is a constant fourth-order tensor. For isotropic materials, we have

$$\sigma_{i,j} = \lambda_S \sum_{k=1}^{n} \epsilon_{k,k}\delta_{i,j} + 2\mu_S \epsilon_{i,j},$$

where $\lambda_S = \nu_S E(1-2\nu_S)^{-1}(1+\nu_S)^{-1}$ and $\mu_S = E(1+\nu_S)^{-1}/2$ are the Lame coefficients.

3.3.1 Equations for Eigenfrequencies

The ansatz:

$$\begin{aligned} u_S(x,t) &= u_S(x)e^{i\omega t}, \\ u_F(x,t) &= u_F(x)e^{i\omega t}, \\ p(x,t) &= p(x)e^{i\omega t} \end{aligned}$$

leads to the following time-independent system of equations:

$$\begin{aligned}
-\mathrm{div}\,(D\epsilon(u_S)) = \omega^2\rho_S u_S \quad & \text{in}\,\Omega_S,\\
\sigma(u_S)n + p\cdot n = 0 \quad & \text{on}\,\Gamma_I,\\
u_S\cdot n = \tfrac{1}{\omega^2\rho_F}\nabla p\cdot n \quad & \text{on}\,\Gamma_I,\\
\mathrm{div}(\tfrac{1}{\omega^2\rho_F}\nabla p) = 0 \quad & \text{in}\,\Omega_F,\\
\sigma(u_S)n = 0 \quad & \text{on}\,\Gamma_N,\\
u_S = 0 \quad & \text{on}\,\Gamma_D.
\end{aligned}$$

Note that we have eliminated the fluid displacement variable u_F, which can be recovered by $u_F = -\nabla p/(\omega^2\rho_F)$.

By multiplication of the two differential equations defined on domains by test functions, integration over the applicable domains, integration by parts and application of the other relations at the different parts of boundary, we arrive at the following variational formulation:

Find $u \in V_S := \{u \in [H^1(\Omega_S)]^n \;:\; \mathrm{tr}_{\Gamma_D} u = 0\}$ *and* $p \in V_F := H^1(\Omega_F)$ *such that*

$$\int_{\Omega_S} D\epsilon(u):\epsilon(v)\mathrm{d}x + \int_{\Gamma_I} pv\cdot n\mathrm{d}s = \omega^2\int_{\Omega_S}\rho_S uv\mathrm{d}x \quad \forall v\in H_S,$$

$$\tfrac{1}{\omega^2}\int_{\Omega_F}\tfrac{1}{\rho_F}\nabla p\nabla q\mathrm{d}x + \int_{\Gamma_I} u\cdot nq\mathrm{d}s = 0 \quad \forall q\in H_F.$$

We define bilinear forms:

$$A(u,v) = \int_{\Omega_S} D\epsilon(u):\epsilon(v)dx,$$

$$B(v,p) = \int_{\Gamma_I} pv\cdot nds,$$

$$C(p,q) = \int_{\Omega_F}\frac{1}{\rho_F}\nabla p\nabla q dx,$$

$$M(u,v) = \int_{\Omega_S}\rho_S uv dx,$$

for $u,v\in V_S$ and $p,q\in V_F$. Using this notation, the weak formulation can be written in the following form:

Find $u\in V_S$ *and* $p\in V_F := H^1(\Omega_F)$ *such that*

$$\begin{array}{ll}
A(u,v)+B(v,p) = \omega^2 M(u,v) & \forall v\in V_S,\\
\frac{1}{\omega^2}C(p,q)+B(u,q) = 0 & \forall q\in V_F.
\end{array}\tag{1}$$

We also state the eigenfrequency problem:

Find $\omega > 0$ *such that the variational problem (1) has nontrivial solutions* $(u, p) \neq (0, 0)$.

Since this problem is linear, we cannot hope that the problem (1) will be well-posed for all ω because any linear combination of two solutions is a solution itself. In fact, the eigenfrequency problem is to find such $\omega \in \mathbb{R}$, for which the problem (1) has no *unique* solutions.

Later in the implementation we will specify a finite dimensional space in which we will be able to solve this equations, and in fact the fluid variables can be eliminated.

4 Numerics and Implementation

4.1 FEM for Hydroelasticity Equations

To solve numerically the hydroelastic equation system, we approximate the variational formulation (1) in finite dimensional Hilbert spaces. We choose some finite dimensional subspaces $V_h^S \subset V^S$ and $V_h^F \subset V^F$ (see below), and pose the following problem.

Find $u_h \in V_h^S$ *with* $u_h|_{\Gamma_D} = 0$ *and* $p_h \in V_h^F$ *such that*

$$A(u_h, v_h) + B(v_h, p_h) = \omega^2 M(u_h, v_h), \tag{2}$$

$$C(p_h, q_h) + \omega^2 B(u_h, q_h) = 0 \tag{3}$$

holds for all $(v_h, q_h) \in V_h^S \times V_h^F$.

Now we shall define the subspaces V_h^F, V_h^S. We start with sub-dividing the domain $\Omega = \Omega_S \cup \Omega_F$ into triangular elements: $\overline{\Omega} = \bigcup_{i=1}^{n_T} \overline{T_i}$ so that the resulting triangulation is regular (see [8] for a definition) *and* none of the open triangular elements T_i contains a common point with *both* domains Ω_S, Ω_F, or equivalently, for all of the triangles the intersection with the boundary Γ_I is empty. Since Γ_I is piecewise linear, such triangulations are possible.

Thus we can write $\overline{\Omega_S} = \bigcup_{i \in I_S} \overline{T_i}$ and $\overline{\Omega_F} = \bigcup_{i \in I_F} \overline{T_i}$, with $I_S \cap I_F = \emptyset$ and $I_S \cup I_F = \{1, \ldots, n_T\}$.

We denote the space of all first-order polynomials on a domain $G \subset \mathbb{R}^n$, by $P_1(G)$. Now we define the subspaces V_h^F, V_h^S as the spaces of continuous functions that are first-order polynomials on each of the triangles. More precisely,

$$\begin{aligned} V_h^F &:= \{q_h \in C^0(\Omega_F) \ : \ q_h|_{T_i} \in P_1(T_i),\ \forall i \in I_F\}, \\ V_h^S &:= \{v_h \in [C^0(\Omega_S)]^n \ : \ v_h|_{T_i} \in [P_1(T_i)]^n,\ \forall i \in I_S\}, \end{aligned}$$

where n is the dimension of Ω, i.e., $n = 2$ or $n = 3$ depending on the dimension of the problem.

Next we choose bases for the constructed subspaces. Namely, we choose the nodal basis comprised of the famous "hat functions" (see [8]). Thus,

$$\begin{aligned} V_h^F &= \operatorname{span}\{q_i \,:\, i = 1, \ldots, m_F\}, \\ V_h^S &= \operatorname{span}\{v_i \,:\, i = 1, \ldots, m_S\}. \end{aligned}$$

The basic idea of the finite element method is the following. Let us expand the approximated solution in the basis functions: $u_h = \sum_{i=1}^{m_S} u_i v_i$ and $p_h = \sum_{i=1}^{m_F} p_i q_i$, where u_i and p_i are real numbers. We substitute these expressions in (2) and (3), and "test" the obtained relations only with pairs of basis functions. Then, the bilinearity of the forms A, B, C and M leads us to an algebraic system of equations with respect to the coefficients u_i and p_i.

Let us define

$$\begin{aligned} a_{i,j} &:= A(v_i, v_j) \text{ for } i, j = 1, \ldots, m_S, \\ b_{i,j} &:= B(v_i, q_j) \text{ for } i = 1, \ldots, m_S,\ j = 1, \ldots, m_F, \\ c_{i,j} &:= C(q_i, q_j) \text{ for } i, j = 1, \ldots, m_F, \\ m_{i,j} &:= M(v_i, v_j) \text{ for } i, j = 1, \ldots, m_S, \end{aligned}$$

and $A = (a_{i,j})$, $B = (b_{i,j})$, $C = (c_{i,j})$, $M = (m_{i,j})$.

Thus, the system obtained from the finite element discretization of the hydroelastic equations is the eigenvalue problem

$$\begin{aligned} Au + Bp &= \omega^2 Mu, \\ Cp + \omega^2 B^\top u &= 0. \end{aligned} \tag{4}$$

Since A is positive-definite and C is positive-indefinite, with kernel space $\operatorname{Ker} C = \operatorname{span}\left\{(1, \ldots, 1)^\top\right\}$, we conclude the following remark.

Remark 1 *The generalized eigenvalue problem (4) has a solution $\omega = 0$, $u = 0$ and $p = (1, \ldots, 1)^\top$. The eigenvalue $\omega = 0$ is of algebraic multiplicity one.*

The problem (4) is hard to solve numerically, since it involves matrices which are not symmetric and positive-definite. Therefore it makes sense to transform it to a more convenient form. The singularity of C does not allow us to express p in terms of u from the second equation. However, a slight perturbation of C, namely, $C + \delta H$, with $H > 0$, would make it regular.

For justification of such regularization, we must prove that the roots of the polynomial in ω

$$P_\delta(\omega) = \det \begin{pmatrix} A - \omega^2 M & B \\ \omega^2 B^\top & C + \delta H \end{pmatrix}$$

lie close to the roots of the polynomial $P_0(\omega)$, if $|\delta|$ is small enough. For a proof of this fact, the reader is referred to the fourth page of [3].

For H we choose the symmetric positive-definite matrix defined by the application of the bilinear form

$$\chi(u,v) := \int_{\Omega_F} u(x)v(x)\mathrm{d}x$$

to the finite element basis functions defined on Ω_F.

Remark 2 *Matrix $C+\delta H$ is the stiffness matrix for the differential operator $-\Delta + \delta I$ acting on V_F. Thus, the perturbation of matrix C is equivalent to a regular perturbation of the differential equation.*

Now we choose $\delta > 0$ small enough and denote $\tilde{C} := C+\delta H$. By replacing C by $\tilde{C}$ in (4), we can express p from the second equation:

$$p = -\omega^2 \tilde{C}^{-1} B^\top u.$$

Substitution in the first equation yields

$$Au = (M + B\tilde{C}^{-1}B^\top)\omega^2 u. \tag{5}$$

Observe that both A and $M + B\tilde{C}^{-1}B^\top$ are symmetric positive-definite matrices.

Note that by Remark 1 and the results in [3] (cf. Page 4), we must expect that the problem (5) will have one eigenvalue close to zero. Since the zero eigenvalue in (4) does *not* correspond to a nontrivial vibration mode, we must exclude the small eigenvalue from the results.

Remark 3 *To impose the Dirichlet boundary condition on Γ_D, we add a penalty term to the matrix A, namely, a matrix with entries*

$$K \int_{\Gamma_D} v_i(x)v_j(x)\mathrm{d}x,$$

where K is a large number. Note that the added matrix is symmetric and non-negative.

We have seen, that the application of Finite Element Method to the time-independent hydroelastic equations leads to a generalized eigenvalue problem.

4.1.1 Inverse Power Iteration for a Generalized Eigenvalue Problem.

We have arrived at a generalized eigenvalue problem of the type

$$Ax = \lambda Mx, \tag{6}$$

where $A, M \in \mathbb{R}^{n\times n}$ are symmetric positive-definite matrices. The goal is to find such $\lambda \in \mathbb{R}$, for which there exist $x \in \mathbb{R}^n \setminus \{0\}$ solving (6). Such values of λ are called *generalized eigenvalues* of the matrix pair (A, M). We observe that if A and M are positive-definite, then by multiplying (6) from the left with M^{-1}, we obtain a standard eigenvalue problem for a positive-definite matrix, thus we expect, that there are n generalized eigenvalues (counting the algebraic multiplicities), all of them positive.

There are many methods for finding the eigenvalues of a matrix. However, in our particular problem we need to find only few smallest eigenvalues.

To find these eigenvalues, we apply an iterative method, called *inverse power iteration*. This method attracted our attention because it is easy to implement and understand. Due to the rather moderate size of the matrices (n not being larger than some thousands), we are satisfied with the rate of convergence of this simple method, and are not looking for more sophisticated methods. However, we note that this is not the fastest method available (cf. [7]).

We do not provide here a complete analysis of this method, but we explain the idea, how the method works. The generalized eigenvalue problem is still transformed to a standard eigenvalue problem for a single matrix, namely by multiplication of (6) from left with A^{-1}. We shall denote $S = A^{-1}M$.

For simplicity, we assume that some largest eigenvalues of S are distinct: $\lambda_1 > \cdots > \lambda_k \geq \lambda_{k+1} \geq \cdots \geq \lambda_n \geq 0$. Furthermore, we use the convention that the eigenvectors are normed with respect to the norm induced by the matrix M. Rewrite the problem in the form

$$Sx = \lambda^{-1}x.$$

We start with finding the smallest generalized eigenvalue of the pair (A, M) λ_1 and the corresponding generalized eigenvector $x_1 : Ax_1 = \lambda_1 Mx_1$. Observe that λ_1 is also the largest eigenvalue of S, and x_1 is the corresponding eigenvector. To find it, we do the following iteration: start with a randomly chosen vector $v_0 \in \mathbb{R}^n \setminus \{0\}$ and define recursively

$$v_m = Sv_{m-1}/(v_{m-1}^\top M v_{m-1}) \tag{7}$$

for $m = 1, 2, \ldots$. Now write S in its Jordan normal form: $S = TJT^{-1}$, so $S^k = TJ^kT^{-1}$. If we expand the initial guess v_0 in terms of columns x_j of T, which are nothing but eigenvectors of S, i.e.,

$$v_0 = \sum_{j=1}^{n} k_j x_j,$$

we see that, with $k = (k_1, \ldots, k_n)^\top$, and

$$w_m = TJ^m T^{-1} \sum_{j=1}^{n} k_j x_j = TJ^m T^{-1} T k = TJ^m k,$$

the sequence of $v_m = w_m/(w_{m-1}^\top M w_{m-1})$ evidently converges to a vector lying in the span of $\{x_1\}$, unless $k_1 = 0$. (Indeed: in the product $J^m k$, as $m \to \infty$, the dominating term will be the one corresponding to the maximal diagonal entry of J. Hence, the normed version of the product $TJ^m k$ will converge to the same column of T, which is the eigenvector x_1.) However, by choosing v_0 randomly, we have generically $k_1 \neq 0$. Thus the vector sequence (v_m) converges to an eigenvector x_1.

The eigenvalue λ_1 can be found from (6), by multiplying it from the left side with $x^\top$ and substituting x with x_1:

$$\lambda_1 = \frac{x_1^\top A x_1}{x_1^\top M x_1}. \tag{8}$$

We have computed a generalized eigenvector of (A, M) corresponding to the smallest eigenvalue. A slight modification of this algorithm allows us to find the other eigenvalues, in the order of increasing magnitude. The key idea is to project the approximation of the "next" eigenvector (computed by (7)) M-orthogonally onto the orthogonal complement of the subspace spanned by the set of already computed eigenvectors, i.e., using the formula (9) given below instead of (7).

The inverse power method is "canned" in the following algorithm. Given a regular matrix A and a symmetric positive-definite matrix M of equal dimensions $n \times n$. We want to find $k \leq n$ smallest generalized eigenvalues of (A, M) and the corresponding eigenvectors.

Algorithm 1 *Compute* $S = A^{-1}M$.
For $l = 0, 1, \ldots k-1 \leq n$ *do*

1. *Choose* $v_0^{l+1} \in \mathbb{R}^n \setminus \{0\}$ *randomly.*

2. *For* $m = 1, 2, \ldots$ *compute*

$$w_m^{l+1} = S w_{m-1}^{l+1} / ({w_{m-1}^{l+1}}^\top M w_{m-1}^{l+1}),$$

and

$$v_m^{l+1} = w_m^{l+1} - \sum_{j=1}^{l} (x_j^\top M w_m^{l+1}) x_j. \tag{9}$$

Repeat it until a criterion for convergence is satisfied, or until a limit of number of iterations is reached.

3. *Define* $x_{l+1} = v_m^{l+1} / ({v_m^{l+1}}^\top M v_m^{l+1})$.

4. *Compute* $\lambda_{l+1} = \frac{x_{l+1}^\top A x_{l+1}}{x_{l+1}^\top M x_{l+1}}$.

The arguments, why the sequence of w_m^{l+1} converges to an eigenvector of S for all $l = 1, \ldots, k$ are similar as we have shown for the case $l = 0$ above.

A detailed proof of convergence and estimate of performance of the power methods may be found in any good textbook on numerical linear algebra, e.g., in [2].

4.1.2 Results

In this section, we describe a model problem with a simple geometry, which can easily be handled by the methods presented in the above chapters. We present and discuss the obtained results, as well as observe the advantages and limitations of the proposed method.

We consider a two-dimensional model of a closed steel vessel filled with incompressible fluid (oil). The thickness of the walls is set to 10 centimeters, and the distance between opposite walls is set to 1 meter (see Figure 4.1.2).

We suppose that the natural vibrations of this system are limited in the plane of drawing. We pose the task to find the natural frequencies of vibrations of this solid-fluid system.

The mechanical behaviour of the system in question is governed by the hydroelastic equations in two dimensions. To numerically approximate this system of equations, we employ the finite element method described in the previous sections.

For the material properties, we use values found in engineering handbooks: steel density $\rho_S = 7.8 \cdot 10^3 \text{kg/m}^3$, Young modulus $E = 2 \cdot 10^{11}\text{Pa}$, Poisson ratio $\nu = 0.29$; oil density $\rho_F = 0.9 \cdot 10^3 \text{kg/m}^3$.

The triangulation of the domain is done by using NETGEN software (see [10]). We use two different triangulations: one consisting of 146 triangles (further denoted by T_1), and the other consisting of 1188 triangles (denoted by T_2).

The results - four smallest angular eigenfrequencies (in rad/s) - are summarized in the following table.

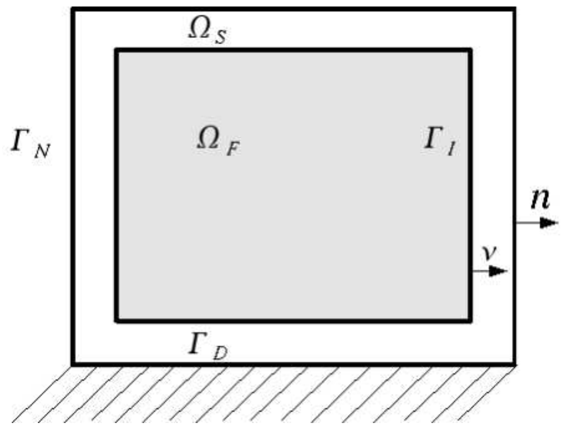

Figure 5: Domain

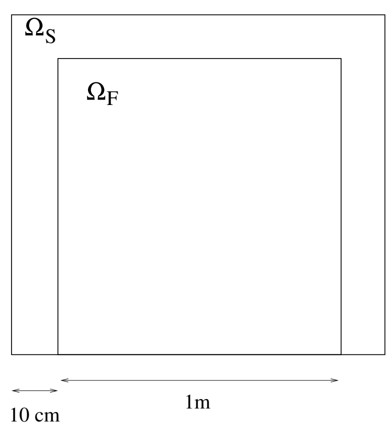

Figure 6: The geometry of the model problem

Triangulation	ω_1	ω_2	ω_3	ω_4
T_1	791	2831	4710	5778
T_2	480	1765	2977	5166

Frequency λ (in 1/s) can be obtained from the angular frequency ω by the linear relation $\lambda = \omega/(2\pi)$.

Numerical experiments with a greater variety of meshes should be performed to draw conclusions about convergence of the method with respect to mesh size.

4.2 Numerics for the Plate/Beam Model

We describe now the implementation of the two-dimensional model combining the three beams model and the Euler's equation in Lagrange coordinates. The analysis for the 3 dimensional five plates model is similar.

The domain we are considering is simple and can be discretized in order to use a scheme of finite differences.

Our model reads as follows.

Equation for the beams:

$$\frac{\partial^4 W}{\partial \tau^4} = \omega^2 \beta W + p,$$

where τ is the tangential coordinate, and β contains all the other physical constants used in the Euler-Bernoulli equation. For the fluid, we take the equations stated in the subsection 3.2, use the ansatz that the physical magnitudes u and p are time-harmonic and eliminate the displacement u, obtaining the Laplace equation

$$\Delta p = 0.$$

In the interface we have an additional condition:

$$\nabla p \cdot n = \omega^2 W.$$

In our particular problem, the geometry is simple. We are considering a square for the 2d transformer. In this case is easy to apply the method of finite differences, using the additional condition of fixed corners and the rotation of torques for the adjacent beams.

Consider the dimensions of the domain to be 1. We will perform a discretization of the model using N discretization intervals per side. Thus, the

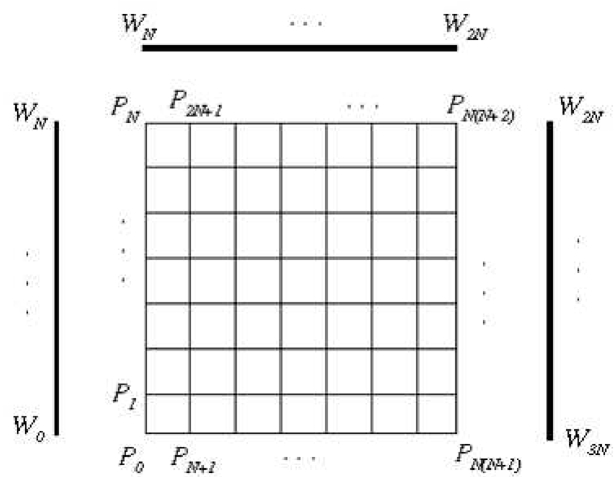

Figure 7: Square domain for the fluid and three intervals for the beams

three beams contain $3N$ intervals (N for each beam). The deformation of the beams can be represented using a vector:

$$W = (W_0, W_1, ..., W_N, ..., W_{2N}, ..., W_{3N}).$$

Note that the corners correspond to the points with numbers N, $2N$. So the values of W_0, W_N, W_{2N} and W_{3N} are zero.

The fluid domain (look at the picture) can be also discretized using the same number of intervals in each horizontal layer. The pressure field can be expresed in a single vector:

$$P = (p_0, ...p_{N(N+2)}).$$

In the finite difference method the derivatives are substituted for differences in the following way:

$$\frac{df}{dx} \approx \frac{f_{i+1} - f_i}{h},$$

where h is the discretization step (we use uniform meshes throughout this section).

For higher order derivatives or for partial derivatives it is also possible to use similar discretization in a very direct way. In the beam-fluid equations, the following differential operators need to be discretized:

$$\frac{d^4W}{dx^4} \approx \frac{W_{i+2} - 4W_{i+1} + 6W_i - 4W_{i-1} + W_{i-2}}{h^4},$$

$$\Delta p \approx \frac{p_{i+1,j} + p_{i-1,j} + p_{i,j+1} + p_{i,j-1} - 4p_{i,j}}{h^2},$$

$$\nabla p \cdot n \approx \frac{p_{\text{boundary}} - p_{\text{interior}}}{h}.$$

The model equations are defined in two domains, a one-dimensional domain for beam equations and a two-dimensional domain for fluid equation. Using the discretization explained above, the difference equations can be expressed in a matrix notation in the following way:

$$\begin{pmatrix} A & B^t \\ B & C \end{pmatrix} \begin{pmatrix} W \\ p \end{pmatrix} = \omega^2 \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} W \\ p \end{pmatrix}.$$

Here the matrix A represents the beam equation, the matrix B and B^t represents the interaction between fluid and solid in the boundary, and the matrix C represents the Laplace equation in the fluid domain for the pressure. All for the finite difference scheme.

Like in the hydroelasticity model, once the displacements field where found, it is possible to substitute the values to obtain the pressure vector an recover all the information. In the figure are shown the modes and pressure field in the fluid for a two-dimensional model of three beams with fluid in the interior.

The equation can be transformed to a generalized eigenvalue problem,

$$AW = \omega^2 (M + B^T C^{-1}) W.$$

This problem can be solved by the method given in section 4.1.1. It is also possible to recover the information related to the pressure from the deformations of the beams.

The graphic in the bottom shows the first eigenfrequencies for the same three beams model, one with fluid (circles) and the second without fluid (stars). As expected, the frequencies are reduced by the fluid.

4.2.1 Five plates without fluid

In the same way is it possible to model the casing in three dimensions by using the five plate model for the casing. The analysis is similar, but the assembling of the matrix is a little awkward. The next pictures shows the first and third modes for a cube clamped in the bottom, however with no fluid inside.

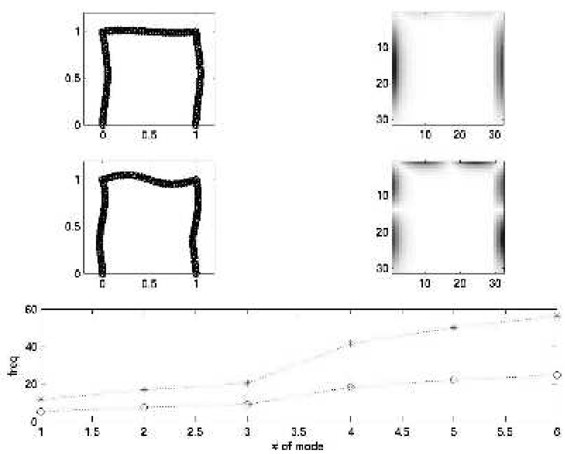

Figure 8: Modes, Pressure Field and First eigenfrequencies for 3 beam model with (o) and without (*) fluid.

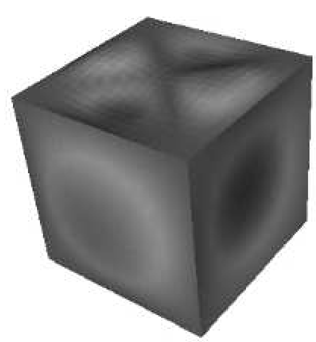

Figure 9: First mode for the 5 plate model

5 Conclusions

Working on the problem of finding the natural (resonance) frequencies of a transformer, treated as a solid structure filled with an incompressible liquid, we have considered several ways of modelling such a system as well as numerical methods for evaluation of the constructed models. These methods are by no means limited to applications on transformers, but are general enough to be employed in modelling various hydroelastic systems.

The liquid has been treated as an ideal (i.e., incompressible, with negligible viscosity) fluid. However, the model of fluid can be adapted according to the accuracy requirements for the model, without major changes of the other parts of this model.

In the finite element formulation, it is possible to exclude all vectorial fluid variables from the equations, leaving the pressure as the only fluid variable. This makes it possible to reduce the dimensions of the linear system to be solved, and thus treat computationally large problems in reasonable time.

For the solid we have used two types of models. The first is a beam or plate model. The obvious advantage of these models is that a 3D problem is reduced to a 1- or 2D problem respectively. A drawback of these simple models is the difficulties to adequately model a casing with complicated shape, leading to model errors. Such models are approximated numerically by finite difference methods.

The second model for the solid we have applied is the full 3D linear elasticity model. For numerical approximation of this model, we have used a finite element method. The main advantage of this model is that it is physically more correct description of solid bodies. A drawback is the neccessity to discretize a 3D domain. If the solid body is very thin in one direction, like a wall of transformer, then very fine discretization is required to ensure the shape regularity of the spatial mesh. Due to this problem, we have chosen a 2D model problem with geometry, that is easy to discretize.

Therefore, to simulate the 3D problem with a high accuracy, models combining shell theory (for thin structures) and elasticity may be preferable (cf. [9]).

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