Nine-month Report

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PhD project: **The extended Yang-Baxter structure for the Calogero-Moser model**

Contents

1 Courses and seminars

I did my MSc (by research) at the University of York under the supervision of Dr. Stefan Weigert. I have studied quantum entanglement which is a core property in quantum information science. My MSc project was mainly focused on how to measure the amount of entanglement for a given state. We have proposed a new kind of method which is called PT[1]-entanglement measure. However, this alternative method is still elusive as to whether it can be a candidate of entanglement measure or not. After having completed the MSc degree, I moved to the University of Leeds to work with Prof. Frank Nijhoff. Since I moved to Leeds, I have attended two courses as follows
PHYS5200M01 Quantum Information Science. Prof. V Vedral
MAGIC023 Integrable systems. Dr. Marta Mazzocco,
and I have also attended four seminars

- Applied Mathematics seminar
- Integrable systems seminar
- Informal integrable lunch meeting
- Postgraduates seminar

Integrable System is a new area of research for me. This requires me to read a large amount of material, e.g. the lecture notes on "Discrete Systems and Integrability" MATH5490 by F. W. Nijhoff[1] and C. M. Field's PhD thesis on "On the quantization of integrable discrete-time systems" [2]. I have studied the PT-symmetric quantum mechanics which is initiated by Bender [27]. I did a calculation for PT-symmetric quantum systems by using the path integral approach. The detail of this calculation will be found later in this report. I have also looked at a paper by Degasperis and Ruijsenaars [33] in which a new family of Hamiltonians for the harmonic oscillator was found. In this context, we are interested in finding the discrete-time version of the system. Finally, I had an opportunity to join the Quantum Information School, Spring 2008, at Bristol between 17-19 March 2008.

2 Research

I am interested in investigating the connection between discrete-time integrable systems and quantum computation processes. Common to the subject is the pivotal role played by the unitary operator of the time evolution. In discrete-time systems the unit time step is generated by this unitary operator, while in quantum computation the unitary operator plays the role of quantum gates.

[1]PT stands for partial transpose.

At the early state of quantum computation, Benioff [3, 4] considered a finite lattice of spin-1/2 systems of Turing Machines. The discrete-time unitary operator has been constructed to perform computation in each time step. In the case of statistical mechanics and quantum statistic, integrable systems are sometimes referred to as solvable models. Thus, there is an expectation that they can be used to create a model of quantum computation processes, or at least as more sophisticated paradigms for a mathematical approach to quantum computation. Of course, it is still not clear whether explicit connections exist or not.

There are many notions of integrability used in literature, especially in the classical case. I will focus specifically on the notion of integrability in the sense of Liouville, to be explained in the next section. On the quantum level the analogous definitions are more subtle, e.g. the discussion in [5]. A common tool, nonetheless, in many circumstances is the notion of Lax pair and Lax equations, which I also discuss in what follows, together with the notion of classical R-matrix structures. However, for the time being we will focus on the classical situation.

It is my interest to study the quantization of the discrete-time Calogero-Moser model as part of my research project both from the point of view of canonical quantization as well as through the path integral. Possibly, this will involve the complexified approach to quantization, which is PT-symmetric quantum theory. As a toy model, I have looked at the path integral for the PT-symmetric harmonic oscillator in both the continuous-time and discrete-time versions. However, in this case, the PT-symmetric quantum theory and the standard quantum theory are related through a similarity transformation. The connection between integrable systems and quantum information science, which is a long-term aim of the research, will be discussed in the last section.

2.1 Completely integrability Hamiltonian systems

Suppose we have a system of N degrees of freedom. Let $\mathbf{p} = (p_1, p_2, ..., p_N)$ and $\mathbf{q} = (q_1, q_2, ..., q_N)$ be the momentum and coordinates vectors in N-dimensional phase space. In conventional Newtonian case we have a Hamiltonian of the form:

$$H(\mathbf{p}, \mathbf{q}) = \frac{1}{2}\mathbf{p}^2 + V(\mathbf{q}), \tag{2.1}$$

where $\mathbf{p}^2 = \sum_{i=1}^{N} p_i^2$ and $V(\mathbf{q})$ is the potential of all particle interact pairwise. However, the general form of Hamiltonian will be encountered in the context of integrable systems, i.e. the Calogero-Moser models. The Hamilton's equations corresponding to the Hamiltonian Eq. (2.1) have the form

$$\dot{q}_i = \{H, q_i\} = \frac{\partial q_i}{\partial t}, \tag{2.2}$$

$$\dot{p}_i = \{H, p_i\} = \frac{\partial p_i}{\partial t}, \tag{2.3}$$

A system under consideration is called completely integrable if there exist variable $J_j(q,p)$ and $\phi_k(q,p)$ of action-angle type. We can express through action-angle variables the coordinates and momentum, and we can integrate the equation of motion of the system.

Suppose we have a system with N degrees of freedom with functions $I_1, I_2, ..., I_N$, it is called completely integrable if it satisfies the following requirements (integrability in the sense of Liouville theorem) [6]:

- The functions are invariants: $\{H, I_k\} = 0 \;, \quad k = 1, 2, ..., N$
- The functions are in involution with respect to the Poisson bracket, i.e.

$$\{I_j, I_k\} = \sum_{l=1}^{N} \left\{ \frac{\partial I_j}{\partial p_l}\frac{\partial I_k}{\partial q_l} - \frac{\partial I_j}{\partial q_l}\frac{\partial I_k}{\partial p_l} \right\} = 0$$

- The functions are functionally independent throughout the phase space

If the system has the number of commuting Poisson brackets less than the number of degrees of freedom, the system is referred to quasi-integrable or quasi-solvable.

The definition of integrability for the discrete-time integrable systems is entirely analogous to that of Liouville in the continuous-time case. Veselov [7] showed that the discrete-time system is integrable if there exists the functions $I_1, I_2, ..., I_N$ with the following requirements:

- The functions are invariants under the mapping, i.e. $I_i(q,p) = I_i(\hat{q},\hat{p})$ where $(q,p) \mapsto (\hat{q},\hat{p})$.
- The functions are in involution with respect to the Poisson bracket, i.e.

$$\{I_j, I_k\} = \sum_{l=1}^{N} \left\{ \frac{\partial I_j}{\partial p_l}\frac{\partial I_k}{\partial q_l} - \frac{\partial I_j}{\partial q_l}\frac{\partial I_k}{\partial p_l} \right\} = 0$$

- The functions are functionally independent throughout the phase space.

However, in the first place, we will focus on the definition of integrability in the continuous-time situation (in the sense of Liouville).

Historically, it was noticed that many of the completely integrable systems can be expressed in the Lax equation

$$\frac{dL}{dt} = [L, M], \tag{2.4}$$

where L and M are $N \times N$ matrices and their elements depend on the coordinates q_j and momenta p_j. In order to obtain L and M matrices, there is no trivial method. An example of an explicit form of L and M matrices, which will be given in the next section, can be found in the Calergero-Moser models. The Lax equation gives the same equation of motion that can be obtained from Hamiltonian equations. From Eq. (2.4), $L(t)$ undergoes a unitary transformation

$$L(t) = U^{-1}(t)L(0)U(t), \tag{2.5}$$

and we can show that

$$\begin{aligned}\frac{dL}{dt} &= -U^{-1}(t)\frac{dU(t)}{dt}U^{-1}(t)L(0)U(t) + U^{-1}(t)L(0)\frac{dU(t)}{dt} \\ &= LM - ML = [L, M],\end{aligned} \tag{2.6}$$

with $M = U^{-1}\frac{dU(t)}{dt}$. Then all the Hamiltonians of the form $Tr(L^k)$, which are the coefficients of the characteristic polynomial $P(\lambda) = det(L - \lambda I)$ where λ is the eigenvalue, are constant of the motion

$$\frac{d}{dt}Tr(L^k(t)) = \frac{d}{dt}Tr(U^{-1}(t)L^k(0)U(t)) = \frac{d}{dt}Tr(L^k(0)) = 0.$$

We can obtain conserved quantities from the Lax equation without referring to the Poisson structure. However, the notion of Liouville integrability requires the knowledge of Poisson structure together with the involution property of the conserved quantities. Babelon and Viallet [8] have shown that the Poisson structure of the Lax matrix can be expressed in the form

$$\left\{L_1 \underset{,}{\otimes} L_2\right\} = [R, L_1] - [R^*, L_2], \tag{2.7}$$

where $L_1 = L \otimes I$, $L_2 = I \otimes L$ and $R \equiv R_{12}$ is referred to a classical R-matrix (see in the next section).

Suppose we have Eqs. (2.4), (2.8) and any matrix representation $\left\{L_1^n \underset{,}{\otimes} L_2^m\right\}$, we can show that (see appendix A)

$$M_2^{(n)} \equiv -nTr_1\left(L_1^{n-1}R^*\right). \tag{2.8}$$

The computation of the Poisson structure of the Lax matrix will be found in the next section.

2.2 The classical R-matrix for the Calogero-Moser model

At the early state of integrable systems, the Calogero-Moser systems have been extensively studied and have been shown that the Caloger-Moser systems are completely integrable systems in the senses of classical and quantum approaches [9, 10]. But there was the difficulty of showing that the systems are completely integrable. Although knowing an R-matrix structure is not strictly needed, it does provide one with additional insights and make aspects of understanding the integrability easier. Then the Calogero-Moser provides good features in many senses to study.

Calogero [11] introduced and studied one-dimensional N-body problems which the pair potential are the sum of a quadratic plus an inversely quadratic term. The Hamiltonian under consideration is

$$\hat{H} = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\frac{\partial^2}{\partial q_i^2} + \sum_{i=2}^{N}\sum_{j=1}^{i-1}\left(\frac{m\omega^2}{4}(q_i - q_j)^2 + g(q_i - q_j)^{-2}\right), \qquad (2.9)$$

where $g > -\hbar^2/4m$ to prevent to collapse [12] and q_i are the coordinates of the ith particle. The energy spectrums and wave functions of Eq. (2.9) have been carried out. If we set $\omega = 0$, the Hamiltonian Eq. (2.9) becomes

$$H = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\frac{\partial^2}{\partial q_i^2} + g\sum_{i=2}^{N}\sum_{j=1}^{i-1}(q_i - q_j)^{-2}. \qquad (2.10)$$

This Hamiltonian describes the scattering states and has no discrete spectrum. The system Eq. (2.10) is sometimes referred to the rational Calogero model. The asymptotic behavior [11] of the wave function has been discussed for the scattering process but we will not go into that details. There is a huge literatures devoted to the study in both classical and quantum theory of CM models, and I have not got an opportunity to study all aspects. I just looked on those aspects which are useful for my own research. e.g. R-matrix.

In order to investigate the Yang-Baxter for the quantum R-matrix for the Calogero-Moser model, we start to study a paper by J. Avan and M. Talon [19] in which the classical R-matrix has been established.

Let us briefly explain how to derive the classical R-matrix. Following the paper [19], the Hamiltonian under consideration is

$$H = \sum_{i=1}^{n} p_i^2 + \sum_{i\neq j}\upsilon(q_i - q_j), \qquad (2.11)$$

where $\upsilon(q_i - q_j)$ is the two-body interactions which is classified into five types I [$\upsilon(q) = q^{-2}$], II [$\upsilon(q) = a^2 sinh^{-2} aq$], III [$\upsilon(q) = a^2 sin^{-2} aq$], IV [$\upsilon(q) = a\wp(aq, \omega_1, \omega_2)$, $\wp$ being the double-periodic Weierstrass function], V [$\upsilon(q) = q^{-2} + gq^2$], (p_i, q_i) are canonically conjugate variables for the ith particle. Therefore, we restrict our attention only on the potential type I (and V).

The L and M matrices for the system Eq. (2.11) with the potential type I are given by

$$L \equiv \sum_{i=1}^{n} p_i e_{ii} + i\sum_{i\neq j} w(q_i - q_j)e_{ij}, \qquad (2.12)$$

$$M = i\sum_{i\neq j} w'(q_i - q_j)e_{ij} + i\sum_{i\neq j} w^2(q_i - q_j)e_{ii}. \tag{2.13}$$

where $v(q_i - q_j) \equiv w^2(q_i - q_j)$ and the matrices e_{ij} are the standard elementary matrices whose entries are given by $(e_{ij})_{kl} = \delta_{lk}\delta_{jl}$.

In order to obtain the classical R-matrix, we have to consider the Poisson bracket structure for the Lax operator L

$$\left\{L \underset{,}{\otimes} L\right\} = \sum_{ij}\sum_{kl}\{p_i\delta_{ij} + iw_{ij}, p_k\delta_{kl} + iw_{kl}\}\, e_{ij} \otimes e_{kl}, \tag{2.14}$$

where $w_{ij} = w(q_i - q_j)$ and we know that $\{p_i, p_k\} = 0$ and $\{w_{ij}, w_{kl}\} = 0$. Then the remain teams are

$$\begin{aligned}\{p_i, w_{kl}\} &= w'_{kl}\{p_i, q_k - q_l\}, \\ &= w'_{kl}(-i\delta_{ik} + i\delta_{il}),\end{aligned} \tag{2.15}$$

and

$$\begin{aligned}\{w_{ij}, p_k\} &= w'_{ij}\{q_i - q_j, p_k\}, \\ &= w'_{ij}(i\delta_{ki} - i\delta_{kj}).\end{aligned} \tag{2.16}$$

where $'$ indicates the differentiation with respect to i and k, respectively.

We now insert Eqs. (2.15) and (2.16) into Eq. (2.14) and we obtain

$$\begin{aligned}\left\{L \underset{,}{\otimes} L\right\} &= i\sum_{ijkl}\left[w'_{kl}(\delta_{il}\delta_{ij} - \delta_{ik}\delta_{ij}) + w'_{ij}(\delta_{ki}\delta_{kl} - \delta_{kj}\delta_{kl})\right]e_{ij} \otimes e_{kl}, \\ &= i\sum_{kl} w'_{kl}(e_{ll} - e_{kk}) \otimes e_{kl} - i\sum_{ij} w'_{ij}e_{ij} \otimes (e_{ii} - e_{jj}).\end{aligned} \tag{2.17}$$

We know that i, j, k, l are dummy indices then Eq. (2.17) becomes

$$\left\{L \underset{,}{\otimes} L\right\} = i\sum_{i\neq j} w'_{ij}\left[(e_{ii} - e_{jj}) \otimes e_{ij} - e_{ij} \otimes (e_{ii} - e_{jj})\right]. \tag{2.18}$$

We know that we can represent the Poisson bracket structure in a generic R-matrix structure [8]

$$\left\{L \underset{,}{\otimes} L\right\} = [R, L \otimes I] - [R^*, I \otimes L], \tag{2.19}$$

where I is the identity matrix and $N^2 \times N^2$ matrix R is defined as

$$R = \sum_{ijkl}^{N} R^{ijkl} e_{ij} \otimes e_{kl}, \tag{2.20}$$

and R^* is the operator obtained by exchanging the indices in the way

$$R^* = \sum_{ijkl}^{N} R^{klij} e_{ij} \otimes e_{kl}. \tag{2.21}$$

Let us consider the first bracket in Eq. (2.19)

$$\begin{aligned} [R, L \otimes I] &= \sum_{abcd} \sum_{i} R^{abcd} p_i \, [e_{ab} \otimes e_{cd}, e_{ii} \otimes I], \\ &+ \sum_{abcd} \sum_{i \neq j} i R^{abcd} w_{ij} \, [e_{ab} \otimes e_{cd}, e_{ij} \otimes I]. \end{aligned} \tag{2.22}$$

We can show that

$$\begin{aligned} [e_{ab} \otimes e_{cd}, e_{ii} \otimes I] &= e_{ab} e_{ii} \otimes e_{cd} - e_{ii} e_{ab} \otimes e_{cd} \\ &= \delta_{ib} e_{ai} \otimes e_{cd} - \delta_{ia} e_{ib} \otimes e_{cd}, \end{aligned} \tag{2.23}$$

and

$$\begin{aligned} [e_{ab} \otimes e_{cd}, e_{ij} \otimes I] &= e_{ab} e_{ij} \otimes e_{cd} - e_{ij} e_{ab} \otimes e_{cd} \\ &= \delta_{ib} e_{aj} \otimes e_{cd} - \delta_{ja} e_{ib} \otimes e_{cd}. \end{aligned} \tag{2.24}$$

Inserting Eqs. (2.23) and (2.24) into Eq. (2.22), we obtain

$$\begin{aligned} [R, L \otimes I] &= \sum_{abcd} \sum_{i} R^{abcd} p_i \, (\delta_{ib} e_{ai} \otimes e_{cd} - \delta_{ia} e_{ib} \otimes e_{cd}) \\ &+ \sum_{abcd} \sum_{i \neq j} i R^{abcd} w_{ij} \, (\delta_{ib} e_{aj} \otimes e_{cd} - \delta_{ja} e_{ib} \otimes e_{cd}) \\ &= \sum_{aicd} R^{aicd} p_i e_{ai} \otimes e_{cd} - \sum_{ibcd} R^{ibcd} p_i e_{ib} \otimes e_{cd} \\ &+ \sum_{ai \neq jcd} i R^{aicd} w_{ij} e_{aj} \otimes e_{cd} - \sum_{j \neq ibcd} i R^{jbcd} w_{ij} e_{ib} \otimes e_{cd}. \end{aligned} \tag{2.25}$$

Next, we would like to calculate the second bracket in Eq. (2.19)

$$\begin{aligned} [R^*, I \otimes L] &= \sum_{abcd} \sum_{i} R^{cdab} p_i \, [e_{ab} \otimes e_{cd}, I \otimes e_{ii}] \\ &+ \sum_{abcd} \sum_{i \neq j} i R^{cdab} w_{ij} \, [e_{ab} \otimes e_{cd}, I \otimes e_{ij}] \end{aligned} \tag{2.26}$$

Using the same technique with Eqs. (2.23) and (2.24), we can show that

$$\begin{aligned} [R^*, I \otimes L] &= \sum_{abci} R^{ciab} p_i e_{ab} \otimes e_{ci} - \sum_{abid} R^{idab} p_i e_{ab} \otimes e_{id} \\ &+ \sum_{abci \neq j} i R^{ciab} w_{ij} e_{ab} \otimes e_{cj} - \sum_{abj \neq id} i R^{jdab} w_{ij} e_{ab} \otimes e_{id}. \end{aligned} \tag{2.27}$$

Combining Eqs. (2.25) and (2.27), we have

$$\begin{aligned}
\left\{L \underset{,}{\otimes} L\right\} &= \sum_{aicd} R^{aicd} p_i e_{ai} \otimes e_{cd} - \sum_{ibcd} R^{ibcd} p_i e_{ib} \otimes e_{cd} \\
&- \sum_{abci} R^{ciab} p_i e_{ab} \otimes e_{ci} + \sum_{abid} R^{idab} p_i e_{ab} \otimes e_{id} \\
&+ \sum_{ai\neq jcd} iR^{aicd} w_{ij} e_{aj} \otimes e_{cd} - \sum_{j\neq ibcd} iR^{jbcd} w_{ij} e_{ib} \otimes e_{cd} \\
&- \sum_{abci\neq j} iR^{ciab} w_{ij} e_{ab} \otimes e_{cj} + \sum_{abj\neq id} iR^{jdab} w_{ij} e_{ab} \otimes e_{id}. \qquad (2.28)
\end{aligned}$$

Rearranging the dummy indices, Eq. (2.28) becomes

$$\begin{aligned}
\left\{L \underset{,}{\otimes} L\right\} &= \sum_{adci} \Big\{ (p_i - p_a) R^{aicd} + (p_c - p_d) R^{cdai} + i \sum_k \big(R^{akcd} w_{ki} \\
&\quad - R^{kicd} w_{ak} - R^{ckai} w_{kd} + R^{kdai} w_{ck} \big) \Big\} e_{ai} \otimes e_{cd}. \qquad (2.29)
\end{aligned}$$

Equating Eqs. (2.17) and (2.29), we obtain the system

$$\begin{aligned}
&(p_i - p_a) R^{aicd} + (p_c - p_d) R^{cdai} + i \sum_k \big(R^{akcd} w_{ki} \\
&- R^{kicd} w_{ak} - R^{ckai} w_{kd} + R^{kdai} w_{ck} \big) \\
&= \delta_{ai} (\delta_{ac} - \delta_{ad}) i w'_{cd} - \delta_{cd} (\delta_{ac} - \delta_{ci}) i w'_{ai}. \qquad (2.30)
\end{aligned}$$

It is difficult to find the general solution of the system Eq. (2.30), and the solution is not unique. However, since we only need a particular solution for the R-matrix, Avan and Talon made a number of simplifying assumptions in the paper [19]. The first assumption is that R does not depend on p_i and the second one is that $R^{aacd} = 0$, $R^{ccii} = 0$, $R^{iiii} = 0$, $R^{aiai} = 0$ (please go to the reference in more details). To combining these two conditions, they obtained

$$R^{iaai} = -\frac{w'_{ai}}{w_{ai}},$$

$$R^{aada} = -R^{aaad} = c w_{ad},$$

where c is a constant which in this case is 1/2. We put these results into Eq. (2.20) and we arrive

$$R = \sum_{i\neq j} \frac{w'_{ij}}{w_{ij}} e_{ij} \otimes e_{ji} - \frac{1}{2} \sum_{i\neq j} w_{ij} e_{ii} \otimes (e_{ij} - e_{ji}). \qquad (2.31)$$

For the potential type I, we have

$$R = \sum_{i\neq j} \frac{1}{q_j - q_i} e_{ij} \otimes e_{ji} - \frac{1}{2} \sum_{i\neq j} \frac{1}{q_i - q_j} e_{ii} \otimes (e_{ij} - e_{ji}). \qquad (2.32)$$

The situation that Avan and Talon have considered is the non-relativistic CM case. However, in the relativistic situation the Poisson structure is not the same. Suris [20] stated that the main different between two cases is that the non-relativistic CM is described in term of linear Poisson bracket, whereas the relativistic CM is described in term of quadratic bracket. However, in case of discrete-time systems, e.g. mappings derived form KdV equation, the quadratic R-matrix structure is also involved [16]. The most general quadratic bracket is given by [20]

$$\left\{ L \underset{,}{\otimes} L \right\} = (L \otimes L)a_1 - a_2(L \otimes L) + (I \otimes L)s_1(L \otimes I) - (L \otimes I)s_2(I \otimes L), \tag{2.33}$$

where a_1, a_2, s_1, s_2 are the matrices satisfying conditions

$$a_1^* = -a_1 \ , \quad a_2^* = -a_2 \ , \quad s_2^* = s_1 \quad , \tag{2.34}$$

and

$$a_1 + s_1 = a_2 + s_2 = R. \tag{2.35}$$

The first condition guarantees the skew-symmetry of the Poisson Bracket Eq. (2.33), and the second one assures that the Hamiltonian flows with invariant Hamiltonian functions $H_n(q,p) = Tr(L^n)$ have the Lax form $\dot{L} = [M, L]$ with the M-matrix $M = Tr_2(RL_2^{n-1})$(see appendix A). Suris [20] has studied the hyperbolic and rational Ruijsenaars-Schneider models [21]. The remarkable properties of the Suris's results are that the matrices a_i, s_i, $i = 1, 2$ are dynamical depending only on the coordinates q_j, not on the momenta p_j. The most striking fact is that the structure found is an quadratization of a linear R-matrix bracket with

$$R = a_1 + s_1$$

which is the R-matrix Eq. (2.32) of the non-relativistic CM model found by Avan and Talon [19].

Later, the time-discretized version of classical Calogero-Moser has been studied by F. W. Nijhoff and G-D.Pang [13, 14]. They derived the discrete-time CM model from a semi-continuous Kadomtsev-Petviashvili equation leading to a finite-dimensional symplectic mapping. The discrete-time version of Lax pair and the classical R-matrix were created which is the same as for the continuum limit.

2.3 Path integral for a particle in PT-symmetric harmonic oscillator

Bender et al, [26, 27] has argued that hermitcity is not a natural physical requirement for Hamiltonians in quantum mechanics, but rather a mathematical criterium. This has become evident in his and collaborators' study of Hamiltonians that posses real eigenvalues. The matrix elements of the Hamiltonian can be complex in which the space-time reflection symmetry is preserved. They use the notation[2] $\hat{H} = \hat{H}^{\mathsf{PT}}$ instead of $\hat{H} = \hat{H}^{\dagger}$. A remarkable example of the quantum system is [27]

$$\hat{H} = \hat{p}^2 - (i\hat{x})^N, \tag{2.36}$$

where N is a continuous real parameter. They found that for $N \geq 2$ the eigenvalues of the Hamiltonian are real, positive and discrete. On the other hand, for $N < 2$, the spectrum is partly real and partly complex. However, there are a number of examples of PT-symmetric quantum systems.

A question arises at this point, whether there is any connection between Hermitian Hamiltonian and a non-Hermitian PT-symmetric Hamiltonian. Mostafazadeh [28, 29, 30] has shown that there exists a Hermitian operator ρ, in such a way

$$h = \rho^{-1} H \rho, \tag{2.37}$$

where h is Hermitian Hamiltonian and the operator ρ can be written in the form

$$\rho = e^{q/2}. \tag{2.38}$$

For a simple example, we consider

$$H = \frac{p^2}{2} + \frac{x^2}{2} + ix. \tag{2.39}$$

The q operator corresponds to the Hamiltonian Eq. (2.39) given by

$$q = -2p,$$

where p is the momentum and we can show that

$$h = e^p H e^{-p} = \frac{p^2}{2} + \frac{x^2}{2} + \frac{1}{2} \tag{2.40}$$

is a Hermitian Hamiltonian.

An other relation between a non-Hermitian and Hermitian Hamiltonian is that they have the same eigenvalues. Consider

$$H\Phi_n = E_n \Phi_n, \tag{2.41}$$

[2]where P is linear and has the effect of changing the sign of the momentum operator $\hat{p}$ and the position operator $\hat{x}$: $\mathsf{P}\hat{p}\mathsf{P} = -\hat{p}$ and $\mathsf{P}\hat{x}\mathsf{P} = -\hat{x}$. The time reversal T is antilinear and affects: $\mathsf{T}\hat{p}\mathsf{T} = -\hat{p}$, $\mathsf{T}\hat{x}\mathsf{T} = \hat{x}$ and $\mathsf{T}i\mathsf{T} = -i$.

where Φ_n are the eigenstates corresponding to a non-Hermitian Hamiltonian. We multiply $\exp(-q/2)$ on the left of Eq. (2.41):

$$\begin{aligned} e^{-q/2}He^{q/2}e^{-q/2}\Phi_n &= E_n e^{-q/2}\Phi_n \\ h\Psi_n &= E_n\Psi_n, \end{aligned} \tag{2.42}$$

where $\Psi_n = e^{-q/2}\Phi_n$.

More recently, Mostafazadeh [30] has studied the path integrals of non-Hermitian quantum mechanics. He considers the generating functions (partition function)

$$Z[\vec{J}] = \mathrm{Tr}\left(T\exp\left\{\frac{i}{\hbar}\int_0^t \left(H - \vec{J}\cdot\vec{X}\right)dt\right\}\right), \tag{2.43}$$

where $\vec{J}\cdot\vec{X}$ represents the source term in quantum mechanics, $\vec{X} = (X_1, X_2, ..., X_m)$ are the pseudo-Hermitian dynamical variables and

$$X_m = e^{-q/2}x_m e^{q/2}, \tag{2.44}$$

where x_m are the dynamical configuration variable. By means of Eq. (2.44), he established the identity

$$T\exp\left\{\frac{i}{\hbar}\int_0^t \left(h - \vec{J}\cdot\vec{x}\right)dt\right\} = e^{-q/2}T\exp\left\{\frac{i}{\hbar}\int_0^t \left(H - \vec{J}\cdot\vec{X}\right)dt\right\}e^{q/2}. \tag{2.45}$$

In the discrete-time version, we can write the relation [31, 32]

$$\Phi(X_{n+1}, t_{n+1}) = \int dx_n K(X_{n+1}, t_{n+1}; X_n, t_n)\Phi(X_n, t_n), \tag{2.46}$$

where Φ is the wave function corresponding to a non-Hermitian Hamiltonian and K is identical to the kernel of the quantum mechanical time-evolution operator (propagator)

$$K = \left\langle X_{n+1}\left|\exp\left(-i\Delta t H/\hbar\right)\right| X_n\right\rangle, \tag{2.47}$$

with $\Delta t = t_{n+1} - t_n$. Using Eq. (2.44), we obtain

$$\begin{aligned} e^{-q/2}\Phi(X_{n+1}, t_{n+1}) &= \int dX_n e^{-q/2}K(X_{n+1}, t_{n+1}; X_n, t_n)e^{q/2}e^{-q/2}\Phi(X_n, t_n) \\ \Psi(x_{n+1}, t_{n+1}) &= \int dx_n k(x_{n+1}, t_{n+1}; x_n, t_n)\Psi(x_n, t_n), \end{aligned} \tag{2.48}$$

where

$$\Psi(x_{n+1}, t_{n+1}) = e^{-q/2}\Phi(X_{n+1}, t_{n+1}), \tag{2.49}$$

and

$$k(x_{n+1}, t_{n+1}; x_n, t_n) = e^{-q/2}K(X_{n+1}, t_{n+1}; X_n, t_n)e^{q/2} \tag{2.50}$$

is the propagator corresponding to a Hermitian Hamiltonian

$$k = \langle x_{n+1} \left| \exp\left(-i\Delta t h/\hbar\right) \right| x_n \rangle . \tag{2.51}$$

The quantum harmonic oscillator has been studied in discrete-time version by Field [2]. In this paper, the unitary time-evolution operator of the harmonic oscillator can be expressed in the form

$$\hat{U} = e^{-\frac{i}{\hbar}\tau\frac{\hat{p}^2}{2}}e^{-\frac{i}{\hbar}\tau\omega^2\frac{\hat{x}^2}{2}}. \tag{2.52}$$

The evolution to proceed in integer time steps is defined

$$...., x \equiv x(n)\ ,\quad \bar{x} \equiv x(n+1)\ ,\quad \bar{\bar{x}} \equiv x(n+2), ... \tag{2.53}$$

Then the Hamiltonian for the discrete-time harmonic oscillator is

$$H(\hat{x}, \hat{\bar{p}}) = T(\hat{\bar{p}}) + V(\hat{x}) = \frac{1}{2}\hat{\bar{p}}^2 + \frac{1}{2}\omega^2\hat{x}^2. \tag{2.54}$$

The quantum operator equations of motion have been carried out by using the discrete-time Hamilton's equation. He also studied the one-time step propagator in the hyperbolic regime.

In this section, the propagator of a particle in the complex potential has been calculated in both continuous-time and discrete-time. The Hamiltonian under consideration is

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2}(x^2 + 2i\varepsilon x), \tag{2.55}$$

where ε is a real constant. It is obvious to see that $H \neq H^\dagger$ but $H = H^{\mathsf{PT}}$.

In the next section, the continuous-time path integral has been considered and the propagator has been exactly obtained. Lastly, the discrete-time path integral has been studied and one-step in time propagator is also calculated.

2.3.1 The continuous-time propagator

The propagator of the system Eq. (2.55) can be expressed in the form

$$K(x(t), t; x(0), 0) = \int_{x(0)}^{x(t)} D(x(\tau)) \exp\left(-\frac{i}{\hbar}\int_0^t d\tau L\right), \tag{2.56}$$

where L is the Lagrangian of the system given by

$$L(\dot{x}, x) = \frac{m\dot{x}^2}{2} - \frac{m\omega^2}{2}(x^2 + 2i\varepsilon x). \tag{2.57}$$

The equation of motion of a particle in the PT-symmetric harmonic oscillator is given by

$$\ddot{x} + \omega^2 x = -i\varepsilon\omega^2/m = f, \tag{2.58}$$

which describes the harmonic motion with a constant driving force f. The exact form of the propagator Eq. (2.56) can be found[32] as

$$K(x(t), t; x(0), 0) = \left(\frac{m\omega}{2\pi i\hbar \sin\omega t}\right)^{\frac{1}{2}} \exp\left(\frac{i}{\hbar} S_c\right), \tag{2.59}$$

where S_c is the classical action given by

$$\begin{aligned} S_c \;=\; & -\frac{m\omega^2\varepsilon^2 t}{2} + \frac{m\omega}{2\sin\omega t}\Big[\big(x^2(t) + x^2(0)\big)\cos\omega t - 2x(t)x(0) \\ & +2\big(i\varepsilon\left(x(t) + x(0)\right) - \varepsilon^2\big)\left(\cos\omega t - 1\right)\Big]. \end{aligned} \tag{2.60}$$

Next, we would like to calculate the energy spectrum of the Hamiltonian Eq. (2.55). We use the formula [32]

$$TrK(x, t; x, 0) = \sum_{n=0}^{\infty} e^{-iE_n t/\hbar}, \tag{2.61}$$

where E_n are an energy spectrum and from Eqs. (2.59) and (2.60), we find that

$$TrK(x(t), t; x(0), 0) = \sum_{n=0}^{\infty} e^{-i(n+1/2)\omega t - im\omega^2\varepsilon^2 t/2\hbar^2}. \tag{2.62}$$

Then the energy spectrum for the system can be expressed

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega + \frac{m\omega^2\varepsilon^2}{2}. \tag{2.63}$$

We see that the energy of the system is real and has been shifted by the factor $+m\omega^2\varepsilon^2/2$. The eigenstates of the system can be evaluated by using an expression

$$K(x(t), t; x(0), 0) = \sum_n \Psi_n(x(t))\Psi_n(x(0))e^{-iE_n t/\hbar}, \tag{2.64}$$

where Ψ_n is the wave function of the system and we have

$$\Psi_n(x) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{2\pi\hbar}\right)^{\frac{1}{4}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}(x + i\varepsilon)\right) e^{-m\omega(x+i\varepsilon)^2/\hbar}, \tag{2.65}$$

where H_n are the Hermite polynomial.

Using the transformation between a non-Hermitian and Hermitian Hamiltonian that Bender have found [26], we can show that

$$e^{\varepsilon p} H e^{-\varepsilon p} = \frac{p^2}{2m} + \frac{m\omega^2}{2} x^2 + \frac{m\omega^2}{2} \varepsilon^2 = h, \tag{2.66}$$

since $e^{\varepsilon p} x e^{-\varepsilon p} = x - i\varepsilon$. h corresponds to a Hermitian Hamiltonian after transforming. By using the same technique with Eq. (2.66), we can establish the relation

$$e^{\varepsilon p} K(x(t), t; x(0), 0) e^{-\varepsilon p} = k(x(t), t; x(0), 0), \tag{2.67}$$

where k is the propagator corresponding the Hermitian Hamiltonian h given by

$$\begin{aligned} k(x(t), t; x(0), 0) &= \left(\frac{m\omega}{2\pi i\hbar \sin \omega t}\right)^{\frac{1}{2}} e^{\frac{im\omega^2 \varepsilon^2 t}{2\hbar^2}} \\ &\times \exp\left(\frac{im\omega}{2\hbar \sin \omega t} \left(x^2(t) + x^2(0)\right) \cos \omega t - 2x(t)x(0)\right). \end{aligned} \tag{2.68}$$

It is easy to observe that the energy spectrum of Eq. (2.68) is exactly the same with Eq. (2.59).

From Eq. (2.37), we see that the Hermitian Hamiltonian and non-Hermitian Hamiltonian are related to each other through the similarity transformation. This relation also holds in the context of the path integral, see Eqs. (2.45) and (2.67).

2.3.2 The one step in time propagator

In this section, we would like to calculate the one-step in time propagator of the system Eq. (2.55). We now write the Hamiltonian in the discrete-time version

$$H(\hat{x}, \hat{\bar{p}}) = \frac{\hat{\bar{p}}^2}{2} + \frac{\omega^2}{2}(\hat{x}^2 + 2i\varepsilon \hat{x}). \tag{2.69}$$

The Hamilton's equations are given by

$$\frac{\hat{\bar{p}} - \hat{p}}{\tau} = -\frac{\partial H}{\partial \hat{x}} = -\omega^2(\hat{x} + i\varepsilon), \tag{2.70}$$

$$\frac{\hat{\bar{x}} - \hat{x}}{\tau} = \frac{\partial H}{\partial \hat{\bar{p}}} = \hat{\bar{p}}. \tag{2.71}$$

Combining Eqs. (2.70) with (2.71), we obtain

$$\hat{\bar{x}} + \underline{\hat{x}} = 2\left(1 - \frac{\omega^2 \tau^2}{2}\right) \hat{x} - i\varepsilon\omega^2\tau^2. \tag{2.72}$$

We now introduce $C = \left(1 - \frac{\omega^2\tau^2}{2}\right)$, then we can find the solution of Eq. (2.72)

$$\hat{x}(t) = A\sin(\delta t + \theta) - i\varepsilon \ , \quad C = \cos\delta t \ . \tag{2.73}$$

Making the ansatz, the solution can be found for quantum operators

$$\hat{x}(t) = \hat{x}(0)\cos\delta t + \hat{C}_1\sin\delta t + ai\varepsilon, \tag{2.74}$$

where $a = \cos\delta t - 1$ and using

$$\hat{p}(0) = \frac{\hat{x}(0) - \hat{x}(t)}{\tau},$$

we find that

$$\hat{C}_1 = \frac{1}{\sqrt{4 - \omega^2\tau^2}}\left(\frac{2\hat{p}(0)}{\omega} - \tau\omega\hat{x}(0) + \frac{2ai\varepsilon}{\tau\omega}\right). \tag{2.75}$$

Next, we would like to calculate the discrete-time unitary operator of the Hamiltonian Eq. (2.68) which is given in the form[2]

$$\hat{U} = e^{-\frac{i}{\hbar}\tau T(\hat{p})}e^{-\frac{i}{\hbar}\tau V(\hat{x})} = e^{-\frac{i}{\hbar}\tau\hat{H}_{\text{mod}}(\hat{p},\hat{x})}, \tag{2.76}$$

where H_{mod} is the modified Hamiltonian given by

$$\hat{H}_{\text{mod}}(\hat{p},\hat{x}) = T(\hat{p}) + V(\hat{x}) + \left(\frac{-i\tau}{2\hbar}\right)[T(\hat{p}), V(\hat{x})] + ... \tag{2.77}$$

We now calculate $[T, V]$ yielding

$$[T(\hat{p}), V(\hat{x})] = -\frac{i\omega^2\hbar}{2}(\hat{x}\hat{p} + \hat{p}\hat{x}) + \varepsilon\omega^2\hbar\hat{p}, \tag{2.78}$$

then the modified Hamiltonian becomes

$$\hat{H}_{\text{mod}}(\hat{p},\hat{x}) = \frac{\hat{p}^2}{2} + \frac{\omega^2}{2}(\hat{x}^2 + 2i\varepsilon\hat{x}) + \frac{\tau\omega^2}{4}(\hat{x}\hat{p} + \hat{p}\hat{x}) - \frac{i\varepsilon\omega^2\tau}{2}\hat{p} + ... \tag{2.79}$$

We define

$$\hat{I}(\hat{p},\hat{x}) = \frac{\hat{p}^2}{2} + \frac{\omega^2}{2}(\hat{x}^2 + 2i\varepsilon\hat{x}) + \frac{\tau\omega^2}{4}(\hat{x}\hat{p} + \hat{p}\hat{x}) - \frac{i\varepsilon\omega^2\tau}{2}\hat{p}, \tag{2.80}$$

which is invariant, $\hat{\tilde{I}} = \hat{I}$, under mapping Eqs. (2.70) and (2.71). The modified Hamiltonian for the discrete-time PT-symmetric harmonic oscillator is directly referred to Eq. (2.80).

Next, we would like to consider the Schroedinger's equation for the quantum discrete-time $\hat{I}$

$$-\hbar^2\frac{d^2\Psi_n(x)}{dx^2} + \tau\omega^2\hbar\,(ix - \varepsilon)\,\frac{d\Psi_n(x)}{dx} + \omega^2\left(x^2 + \frac{i\hbar\tau}{2}\right)\Psi_n(x) = E_n\Psi_n(x). \tag{2.81}$$

Solving Eq. (2.81), we obtain the wave function

$$\Psi_n(x) = N \exp\left(\frac{i\tau\omega^2 x^2}{4\hbar^2} + \frac{(\tilde{\Omega}_\varepsilon - \Omega(\Omega_\varepsilon + x\Omega))x}{2\hbar\Omega}\right) \times H\left(\frac{\tilde{\Omega}_\varepsilon^2 - \Omega_\varepsilon^2\Omega^2 - 4\hbar\Omega^3}{8\hbar\Omega^3}, -\frac{\tilde{\Omega}_\varepsilon}{2\sqrt{\hbar}\Omega^{3/2}} + x\sqrt{\frac{\Omega}{\hbar}}\right), \tag{2.82}$$

where $H(n,x) = H_n(x)$ is the Hermite polynomial, N is a normalization constant and

$$\Omega_\varepsilon = \tau\omega^2\varepsilon \ , \quad \tilde{\Omega}_\varepsilon = \tfrac{i\tau^2\omega^4\varepsilon}{2} - 2i\varepsilon \ , \quad \Omega^2 = \omega^2\left(1 - \frac{\tau^2\omega^2}{4}\right). \tag{2.83}$$

The corresponding eigenvalues of Eq. (2.81) are

$$E_n = \frac{\Omega_\varepsilon^2}{4} + \frac{\tilde{\Omega}_\varepsilon^2}{4\Omega^2} + 2\hbar\Omega\left(n + \frac{1}{2}\right). \tag{2.84}$$

Note: in the limit $\varepsilon \to 0$, we find

$$\lim_{\varepsilon\to 0} \Psi_n(x) = N \exp\left(\frac{i\tau\omega^2 x^2}{4\hbar} - \frac{\Omega x^2}{2\hbar}\right) H_n\left(x\sqrt{\frac{\Omega}{\hbar}}\right), \tag{2.85}$$

$$\lim_{\varepsilon\to 0} E_n = 2\hbar\Omega\left(n + \frac{1}{2}\right), \tag{2.86}$$

which are identical with C. M. Field's results[2].

From Eq. (2.75), we find that

$$\hat{p}(0) = \frac{\Omega}{\sin\delta t}\left(\hat{x}(t) - \hat{x}(0)\cos\delta t\right) + \frac{\tau\omega^2}{2}\hat{x}(0) - 2ai\varepsilon\left(\frac{\Omega}{\sin\delta t}\right). \tag{2.87}$$

By using

$$\hat{p}(t) = \frac{\hat{x}(t) - \hat{x}(t-\tau)}{\tau}, \tag{2.88}$$

we also obtain

$$\hat{p}(t) = -\frac{\Omega}{\sin\delta t}\left(\hat{x}(0) - \hat{x}(t)\cos\delta t\right) + \frac{\tau\omega^2}{2}\hat{x}(0) - 2ai\varepsilon\left(\frac{\Omega}{\sin\delta t}\right). \tag{2.89}$$

Hence, acting with $\langle x(t), t|$ and $|x(0), 0\rangle$

$$i\hbar\frac{\partial}{\partial x(0)}\left\langle x(t), t|\, x(0), 0\right\rangle = p(0)\left\langle x(t), t|\, x(0), 0\right\rangle, \tag{2.90}$$

$$-i\hbar\frac{\partial}{\partial x(t)}\left\langle x(t),t\right|x(0),0\rangle=p(t)\left\langle x(t),t\right|x(0),0\rangle\,, \qquad (2.91)$$

we have

$$\begin{aligned}\left\langle x(t),t\right|x(0),0\rangle &= N\exp\left\{\frac{i}{\hbar}\left[\frac{\Omega}{2\sin\delta t}\left(\left(x^2(t)+x^2(0)\right)\cos\delta t-2x(0)x(t)\right)\right.\right.\\ &\left.\left.+\frac{\tau\omega^2}{4}\left(x^2(t)-x^2(0)\right)-\frac{2ai\varepsilon\Omega}{\sin\delta t}\left(x(t)-x(0)\right)\right]\right\}, \qquad (2.92)\end{aligned}$$

where N is the prefactor given by

$$N=\left(\frac{1}{2\pi i\hbar}\right)^{\frac{1}{2}}\left(\frac{\Omega}{\sin\delta t}\right)^{\frac{1}{2}}. \qquad (2.93)$$

The one step in time propagator for a particle in PT-symmetric harmonic oscillator has been calculated in Eq. (2.92). Obviously, if we take $\varepsilon\to 0$, we will obtain the one step in time propagator for a particle in harmonic oscillator which was obtained by Field [2].

2.4 Discrete-time path integral of a quantum harmonic oscillator

Field [2] has studied the path integral approach for integrable discrete-time quantum harmonic oscillator. Degasperis and Ruijsenaars [33] have shown that there are many classes of Hamiltonian producing the same equation of motion of a harmonic oscillator. Then it might be a good idea if we can develop further mathematical details for this system especially both continuous and discrete-time path integrals. Therefore, We have found the corresponding Lagrangian of the Degasperis and Ruijsenaars' Hamiltonian. The details of a calculation have been given in the appendix B.

3 Research plan

3.1 The extended Yang-Baxter for the Calogero-Moser model

In the literature, the classical Yang-Baxter structure for the discrete-time Calogero-Moser systems has not been discovered yet. At first stage of the research, we would like to investigate the Yang-Baxter in which we will start from the rational Calogero-Moser system. At the next stage, the quantum Yang-Baxter of this model will be studied as well. To prepare the ground, we would like to review the papers by F. W. Nijhoff and et al., [15, 16, 23] where the extended quantum

R-matrix of KdV type mapping was established. As a consequence of these papers, not only the Poisson structure of L operators but also the Poisson structure of M operators have to take into account.

In summary, the first stage of this research project comprise:

- Firstly, establish extended classical Yang-Baxter structure for the rational Calogero-Moser model.
- Establish a connection between the classical action and the time-M matrix for the Calogero-Moser model.
- Formulate a time-sliced approach to the discrete Calogero-Moser system.
- Study the (extended) quantum Yang-Baxter structure for this model.

3.2 The connection between integrable systems and quantum information science

More recently, there are many authors which studied entanglement of the Calogero-Moser model. Ghikas et al.,[17] investigated the rate of change of bi-partite entanglement under the variation of a critical parameter and the level-curvature. Later, Katsura [18] calculated the entanglement entropy between two subsets of particles in the ground state of the Calogero-Sutherland model.

We know that an entanglement is purely quantum property and it cannot be encountered in classical physics. However, in the discrete-time (classical) Calogero-Moser model, which was studied by Nijhoff [14], the dynamics is really given by an algebraic correspondence (i.e. a multivalued map) which can be explicitly solved. In other words, the solution of discrete-time equation of motion of the system shows that "each discrete-time value the positions of the particles is uniquely determined up to a permutation of the particles"[14]. Thus, for this classical system we have an essential indistinguishability of particle which never happens in classically physical systems. Then we think that the discrete-time operation may produce quantum behavior of classical systems and we may find the connection between entanglement systems and the Calogero-Moser model. This part of work is in progress.

4 Appendix A

In this section, we would to show how to obtain the relation $M_2^{(n)} = -nTr_2(L_2^{n-1}R)$. Let us consider the family of the Hamiltonians $H_n = Tr(L^n)$ which have the corresponding equations of motion

$$\dot{q} = \{H_i, q\} = \frac{\partial q}{\partial t_i}, \tag{4.94}$$

$$\dot{p} = \{H_i, p\} = \frac{\partial p}{\partial t_i}, \tag{4.95}$$

where t_i indicate the ith time flow. We have the requirement that

$$\begin{aligned}
\frac{\partial}{\partial t_i}\frac{\partial q}{\partial t_j} &= \frac{\partial}{\partial t_j}\frac{\partial q}{\partial t_i}, \\
\frac{\partial}{\partial t_i}\{H_j, q\} &= \frac{\partial}{\partial t_j}\{H_i, q\}, \\
\{H_j, \{H_i, q\}\} - \{H_j, \{H_i, q\}\} &= 0. \qquad (4.96)
\end{aligned}$$

Using Jacobi identity

$$\{F, \{G, H\}\} + \{H, \{F, G\}\} + \{G, \{H, F\}\} = 0.$$

Eq. (4.96) becomes

$$= -\{\{H_i, H_j\}, q\} = 0, \tag{4.97}$$

since $\{H_i, H_j\} = 0$ which furnishes the involution property.

Next we would like to consider the expression $Tr_1\{L_1^n, L_2^m\}$ where $L_1 = L \otimes I$ and $L_2 = I \otimes L$. We can show that

$$Tr_1\{L_1^n, L_2^m\} = \{H_n, L_2^m\} = \partial_{t_n} L_2^m. \tag{4.98}$$

Using the fact that

$$\{L_1^n, L_2^m\} = \sum_{k=0}^{n-1} L_1^k \{L_1, L_2^m\} L_1^{n-1-k}.$$

We now have

$$\begin{aligned}
Tr_1\{L_1^n, L_2^m\} &= Tr_1\left(\sum_{k=0}^{n-1} L_1^k \{L_1, L_2^m\} L_1^{n-1-k}\right) \\
&= Tr_1\left(\sum_{k=0}^{n-1} L_1^k L_1^{n-1-k} \{L_1, L_2^m\}\right) \\
&= Tr_1\left(nL_1^{n-1}\{L_1, L_2^m\}\right) \\
&= nTr_1\left(L_1^{n-1}[R, L_1]\right) - nTr_1\left(L_1^{n-1}[R^*, L_2^m]\right). \qquad (4.99)
\end{aligned}$$

The first term is zero because of cyclicity of the trace. Then Eq. (4.100) becomes

$$\begin{aligned}
Tr_1\{L_1^n, L_2^m\} &= -nTr_1\left(L_1^{n-1}[R^*, L_2^m]\right) \\
&= -n\left[Tr_1\left(L_1^{n-1}R^*\right), L_2^m\right]. \qquad (4.100)
\end{aligned}$$

We now define

$$M_2^{(n)} \equiv -nTr_1\left(L_1^{n-1}R^*\right), \tag{4.101}$$

and equating Eq. (4.100) with Eq. (4.98), we have the Lax equation

$$\partial_{t_n} L_2^m = [M_2^n, L_2^m]. \tag{4.102}$$

Analogy, if we start to perform trace over the second space in Eq. (4.98), we will obtain

$$M_1^{(m)} \equiv mTr_2\left(L_2^{m-1}R\right), \tag{4.103}$$

and

$$\partial_{t_m} L_1^n = [M_1^m, L_1^n]. \tag{4.104}$$

5 Appendix B

Degasperis and Ruijsenaars [33] have considered a family of Hamiltonian satisfying the Newton equation,

$$\frac{\partial^2 H}{\partial x \partial p}\frac{\partial H}{\partial p} - \frac{\partial^2 H}{\partial p^2}\frac{\partial H}{\partial x} + \frac{1}{m}\frac{\partial V}{\partial x} = 0. \tag{5.105}$$

They introduced the multiplicative form of Hamiltonian

$$H(x,p) = F(p)G(x), \tag{5.106}$$

where $F(p)$ is of the form $Ap^2 + Bp + C$, with B and C are arbitrary. Inserting H to Eq. (5.105) with particular choice of arbitrary constants, they obtained

$$H_c(x,p) = 4mc^2 \cosh\left(\frac{p}{2mc}\right)\sqrt{1 + \frac{V(x)}{2mc^2}}. \tag{5.107}$$

This Hamiltonian gives the same equation of motion with the Newton equation. In the limit $c \to \infty$, they recovered an ordinary Hamiltonian

$$\lim_{c\to\infty}\left(H_c - 4mc^2\right) = \frac{p^2}{2m} + V(x). \tag{5.108}$$

Next, we would like to calculate the Lagrangian cooresponding to H_c. Consider the Hamilton's equations

$$\dot{x} = \frac{\partial H}{\partial p} = 2c\sinh\left(\frac{p}{2mc}\right)\sqrt{1 + \frac{V(x)}{2mc^2}}, \tag{5.109}$$

$$\dot{p} = -\frac{\partial H}{\partial x} = -\cosh\left(\frac{p}{2mc}\right)\frac{\frac{\partial V}{\partial x}}{\sqrt{1+\frac{V(x)}{2mc^2}}}. \tag{5.110}$$

From Eq. (5.109), we can write the momenta in the form

$$p = 2mc\sinh^{-1}\left(\frac{\dot{x}}{2c\sqrt{1+\frac{V(x)}{2mc^2}}}\right) = \frac{\partial L}{\partial \dot{x}}. \tag{5.111}$$

We also use the fact that

$$\int \sinh^{-1}\alpha d\alpha = -\sqrt{1+\alpha^2} + \alpha\sinh^{-1}\alpha. \tag{5.112}$$

We obtain the Lagrangian in the form

$$L_c(x,\dot{x}) = 4mc^2\mu(x)\left(-\sqrt{1+\left(\frac{\dot{x}}{2c\mu(x)}\right)^2} + \frac{\dot{x}}{2c\mu(x)}\sinh^{-1}\left(\frac{\dot{x}}{2c\mu(x)}\right) + C(x)\right), \tag{5.113}$$

where $C(x)$ is an arbitrary function of x and

$$\mu(x) = \sqrt{1+\frac{V(x)}{2mc^2}}.$$

Consider in the limit $c \to \infty$

$$\lim_{c\to\infty}\left(L_c + 4mc^2\right) = \frac{m\dot{x}^2}{2} - V(x) + \left(V(x) + 4mc^2\right)C(x). \tag{5.114}$$

If we choose $C(x) = 0$, we obtain

$$\lim_{c\to\infty}\left(L_c + 4mc^2\right) = \frac{m\dot{x}^2}{2} - V(x) = L(x,\dot{x}). \tag{5.115}$$

Then we have the Lagrangian in the form

$$L_c(x,\dot{x}) = -4mc^2\sqrt{1+\left(\frac{\dot{x}}{2c\mu(x)}\right)^2} + 2mc\dot{x}\sinh^{-1}\left(\frac{\dot{x}}{2c\mu(x)}\right). \tag{5.116}$$

However, we can obtain the Lagrangian in Eq. (5.116) by using the Legendre transformation

$$L_c(x,\dot{x}) = p\dot{x} - H_c(x,p). \tag{5.117}$$

Inserting Eqs. (5.107) and (5.111), we have

$$L_c(x,\dot{x}) = 2mc\dot{x}\sinh^{-1}\left(\frac{\dot{x}}{2c\mu(x)}\right) - 4mc^2\cosh\left(\frac{p}{2mc}\right)\sqrt{1+\frac{V(x)}{2mc^2}}, \quad (5.118)$$

and we find that

$$\cosh\left(\frac{p}{2mc}\right) = \sqrt{1+\sinh^2\left(\frac{p}{2mc}\right)} = \sqrt{1+\left(\frac{\dot{x}}{2c\mu(x)}\right)^2}. \quad (5.119)$$

Finally, we obtain the Lagrangian in the same form with Eq. (5.116)

$$L_c(x,\dot{x}) = -4mc^2\sqrt{1+\left(\frac{\dot{x}}{2c\mu(x)}\right)^2} + 2mc\dot{x}\sinh^{-1}\left(\frac{\dot{x}}{2c\mu(x)}\right). \quad (5.120)$$

Next, we consider the Euler-Lagrange equation of motion

$$\frac{d}{dt}\left(\frac{\partial L_c}{\partial \dot{x}}\right) - \frac{\partial L_c}{\partial x} = 0. \quad (5.121)$$

We find that

$$\frac{\partial L_c}{\partial x} = -4mc^2\sqrt{1+\left(\frac{\dot{x}}{2c\mu(x)}\right)^2}\frac{d\mu(x)}{dx}, \quad (5.122)$$

$$\frac{d}{dt}\left(\frac{\partial L_c}{\partial \dot{x}}\right) = \frac{m}{\sqrt{1+\left(\frac{\dot{x}}{2c\mu(x)}\right)^2}}\left(\frac{\ddot{x}}{\mu(x)} - \frac{\dot{x}^2}{\mu^2(x)}\frac{d\mu(x)}{dx}\right). \quad (5.123)$$

Inserting Eqs. (5.122) and (5.123) into Eq. (5.121), we obtain

$$\frac{m}{\sqrt{1+\left(\frac{\dot{x}}{2c\mu(x)}\right)^2}}\frac{\ddot{x}}{\mu(x)} - \left(\frac{m}{\sqrt{1+\left(\frac{\dot{x}}{2c\mu(x)}\right)^2}}\frac{\dot{x}^2}{\mu^2(x)} - 4mc^2\sqrt{1+\left(\frac{\dot{x}}{2c\mu(x)}\right)^2}\right)\frac{d\mu(x)}{dx} = 0. \quad (5.124)$$

or

$$m\ddot{x} + \frac{dV}{dx} = 0. \quad (5.125)$$

Here, we can recover the Newton equation of motion.

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