Towards an entanglement measure based on the partial transpose

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Abstract

We propose an alternative approach to measure the amount of entanglement for a given state. Our method is based on the PPT criterion: first we form a convex combination of a state ρ_{AB} with its partial transpose ρ_{AB}^{PT}: $\rho(\lambda) = (1-\lambda)\rho_{AB} + \lambda\rho_{AB}^{PT}$; then we search for the largest allowed value of λ such that $\rho(\lambda)$ remains a density matrix. Finally, we define the entanglement measure as $E_{PT}(\rho_{AB}) = 1 - \lambda_c$. We provide some examples of our measure for Werner and Bell diagonal states. For pure states, this measure can distinguish between product states and entangled states. For mixed states, we can show that this measure is zero, if and only if the state is separable. The amount of entanglement is a function of the eigenvalues of a given state and its partial transpose for entangled states. This measure has been shown to be invariant under local unitary operations. We show you the examples that the amount of entanglement does not increase under particular LOCC operations. However, it is still elusive in general whether the amount of entanglement does increase or not.

Contents

1 Introduction

Quantum theory is one of the most successful theories of the 20th century. Physicists use it to explore the behavior of matter such as atoms and molecules on a microscopic scale. It also shows that such behaviors are totally different from macroscopic scale or classical world which is described by Newton's laws. Nowadays, quantum physics is relevant in various fields, such as biology, chemistry, computation and information. Especially in computational and information areas, quantum features offer more efficient operations than classical one [1]. The important quantum feature is *entanglement* which we will be discussed throughout this work.

We will focus on how to measure the amount of entanglement for a given state. There are many criterions to distinguish between classical and quantum correlations [14]. PPT is the best criterion among them in lower dimensional quantum systems. Partial transpose of a density matrix can be used to identify entangled states from separable. Unfortunately, PPT-criterion cannot differentiate the amount of entanglement among entangled states. However, we realize that it is possible to measure the entanglement for a given state by the use of partial transpose of the density matrix. Our entanglement measure is based on the PTT-criterion [2] which holds only in $2 \otimes 2$ and $2 \otimes 3$ dimensional quantum system. This method is simpler than those methods in literature [3].

We would like to give you a map to follow this article. In the second section, we will discuss some aspects of quantum correlations and Bell's theorem which are fundamental to the idea of quantum entanglement. In the third section, Quantum states are also discussed. A density matrix is the most general to express quantum states. Next, we would like to discuss quantum states change under certain of operations. The idea of PTT-criterion is briefly discussed in the fifth section. We also provide you with some examples of PPT-criterion in action in this section. A number of entanglement measures, which were proposed by various researchers, will be reviewed in Section 6. We then summarize some properties that all entanglement measures should satisfy. In the Section 7, we introduce our entanglement measure which we call PT-entanglement measure. We apply this measure to the Werner states and Bell decomposition states and derive some of its properties. Later, we represent that PT-entanglement measure satisfy three basic properties of entanglement measure. The first one is that the entanglement is zero if and only if a give state is separable. Secondly, the entanglement is invariant under local unitary operations. The last one is that the entanglement does not increase under LOCC operations. For the third condition, we demonstrate that the amount of entanglement does not increase under LOCC operations for a particular set of operators. The lower and upper limits of PT-entanglement measure have been derived by using Weyl's and Ostrowski's theorems. In the last section, we summarize our results, draw conclusion and point to some open questions.

2 Quantum correlations and Bell's theorem

In 1935, Einstein, Podolsky and Rosen [4] formulated a quantum mechanical thought experiment to claim that quantum theory is an incomplete theory which is called EPR-paradox. Later Schrödinger published two papers to discuss EPR-argument [5]. Here the word *entanglement* was introduced. Quantum entanglement continuous to play an important role in quantum information, quantum teleportation, quantum cryptography and quantum computation [1].

2.1 Einstein's argument

Originally [4], a EPR-pair is created from a source in a certain quantum state. Later they are separated from each other one of which is given to Alice and another is obtained by Bob. According to Einstein assumption on locality and separability, there is no interaction between EPR-particles. Alice and Bob consider the measurement results of the positions and momenta of a EPR-pair. There are perfect correlations between both the positions and their momenta. A measurement of either positions or momenta on one particle will allow us to certainly predict the results of the position or momentum measurements on the other particle. Because the position and momentum operators do *not* commute, Alice and Bob cannot obtain the position and momenta of their particles simultaneously. However, Alice and Bob are allowed to communicate via classical channels. Then Alice and Bob may possibly obtain the position and momenta of their particles in the same time. Let us consider Bob's measurement in the first place. Bob can choose to measure the momentum of his particle. On the other hand, Alice can choose to measure the position of her particle. At this point, Bob can asks Alice's result of the measurement and he immediately know the position of his particle . By means of this process, Bob perfectly knows the position and the momentum of his particle simultaneously. But Bob cannot certainly know the position and the momentum of his particle in the same time because of the commutation between position and momentum operators. The position measurement will disrupt the correlation between momentum values. Then EPR-argument was claimed that quantum state is incomplete.

Later, Bohm [6] proposed another model of EPR-correlation which involves a spin system. The advantage of this model is that it can be performed experimentally [7, 8]. Let us now consider the system with total angular momentum $\vec{J}$ zero. Imagine, it explodes into two parts carrying opposite angular momenta $\vec{j}_A$ and $\vec{j}_B$ in which $\vec{j}_A + \vec{j}_B = 0$.

Suppose now Alice receives part one with angular momentum $\vec{j}_A$. Bob gets the other one with opposite angular momentum $\vec{j}_B$. Classically, Alice can predict the the sign (direction) of Bob's angular momentum precisely via the measurement on her system. If Alice chooses to measure the sign of the angular momentum along the x-axis and she gets $+j_A^x$, say spin-up along the x-axis, she can conclude immediately that Bob's angular momentum is -j_B^x , spin-down along the x-axis. Bob just asks Alice about her result and he will predict the sign of his angular momentum in advance without measuring on his system. In the same fashion, Bob can perform a measurement of angular momentum along the y-axis. If he gets $+j_B^y$, immediately, he can conclude that Alice's sign of the angular momentum is

-j_A^y along the y-axis.

Actually, Alice and Bob can measure the spin between the x and z-axes or y and z-axes but they cannot measure the sign of the angular momentum along the x, y, z-axeses simultaneously. The results of measurements are based on Einstein's *local realism* which is the combination of the principle of locality with the realistic assumption: meaning that all objects must objectively have their properties already before these properties are measured, and distant object cannot have direct influence on one another (relativity). From this statement, we can conclude that if Alice measures angular momentum along the x-axis and gets $+j_A^x$, the sign of Bob's angular momentum -j_B^x is a real physical property whether Bob measures or not and it is already there.

Above situation can be found in daily life (classically). Actually, from a quantum point of view, the EPR argument goes one step in advance. We can measure Alice's angular momentum along the x-axis and the y-axis with certainly. The problem is that the observable of angular momentum along the x-axis and the y-axis do not commute. Then these spin operators satisfy uncertainty principle. What does not agree with the postulate of quantum mechanics about the measurement and uncertainty principle. The quantum mechanical interpretation of the above situation is that the measurement is a selection process. We can write the quantum-state, a singlet state, of the system as follows [1]:

$$\left|\Psi_{AB}^1\right\rangle = \frac{1}{\sqrt{2}}\left(|+j_A^x\rangle \otimes |-j_B^x\rangle - |-j_A^x\rangle \otimes |+j_B^x\rangle\right), \qquad (1)$$

or

$$\left|\Psi_{AB}^2\right\rangle = \frac{1}{\sqrt{2}}\left(|+j_A^y\rangle \otimes |-j_B^y\rangle - |-j_A^y\rangle \otimes |+j_B^y\rangle\right). \qquad (2)$$

If angular momentum along the x-axis of Alice's system is measured and found to be $+j_1^x$, the state $|+j_A^x\rangle \otimes |-j_B^x\rangle$ is selected from the quantum state $|\Psi_{AB}^1\rangle$ which is an entangled state. A subsequent measurement of Bob's angular momentum along x-axis merely ascertains that the system is in the state $|+j_A^x\rangle \otimes |-j_B^x\rangle$ not in $|+j_A^y\rangle \otimes |-j_B^y\rangle$ or $|+j_A^z\rangle \otimes |-j_B^z\rangle$ or any other states. In the same way, Bob can choose to obtain an angular momentum along the y-axis. If his measure result is $+j_B^y$, the state $|+j_A^y\rangle \otimes |-j_B^y\rangle$ is selected from $|\Psi_{AB}^2\rangle$. Later, Alice's measure along y-axis will be produced from $|+j_A^y\rangle \otimes |-j_B^y\rangle$ not any other states.

But from the EPR-argument, we can predict Alice's angular momentum along the x-axis and the y-axis simultaneously. Let say, if Alice decides to measure her angular momentum along x-axis, Alice quantum state will be described by $|\Psi_{AB}^1\rangle$. We know that Alice cannot measure the angular momentum along the y-axis directly. But she can conclude from Bob's measurement along the y-axis for instance. From this situation, Einstein claimed that quantum mechanics is an incomplete theory because Alice can determine the sign of the angular momentum

[1] The composite system will be discussed later.

along the y-axis and it is an element of reality but it is not contained in Alice's wave function $|\Psi^1_{AB}\rangle$. Then the wave function is not a complete description of the physical system.

2.2 Bell's inequality

From the statement of the EPR argument as we mentioned before, it seems that quantum mechanics needs something which has not yet been discovered to describe the EPR behavior. Something that quantum mechanics needs must contain variables, which are called *hidden variables* λ [9], corresponding to all the *elements of reality*. It should be the answer to the problem of *non-commuting quantum observables*.

John Bell [10] showed that the predictions of quantum mechanics in the EPR argument are actually different from the predictions of hidden variable theory. There are two key assumptions in Bell's analysis which is based on Einstein's local realism principle.

A1. Each measurement shows an objective physical property of the system. This is sometimes known as the assumption of *realism*.

A2. A measurement taken by one observer has no effect on the measurement taken by the other. This is sometimes known as the assumption of *locality*.

We now consider a two-component system and a pair of instruments that can measure a two-valued variable on each of the components. The possible results of a measurement are taken ± 1. Therefore, we assume that there is a function $A(\hat{a}, \lambda) = \pm 1$ that determines that results of the measurement on the first particle, saying on Alice side. Similarly, we assume that there is a function $B(\hat{b}, \lambda) = \pm 1$ that determines Bob's results. Here $\hat{a}$ and $\hat{b}$ are referred to measuring angles of Alice's and Bob's angular momenta, respectively. According to Einstein's principle of locality, we assume that the result of a measurement on the first particle does not effect to the measurement on the second particle and vice versa.

We wish to construct the classical correlation between the results of the measurement on two particles. The uncontrollable parameters λ are subject to some probability distribution $\delta(\lambda)$ which it has been called local hidden variables. The classically correlation function is taken a form

$$C_{cl}(\hat{a}, \hat{b}) = \int A\,(\hat{a}, \lambda) B(\hat{b}, \lambda) \delta(\lambda) d\lambda. \tag{3}$$

where $A\,(\hat{a}, \lambda) = B(\hat{b}, \lambda) = \pm 1$ and $\int \delta(\lambda) d\lambda = 1$. The absolute values of the

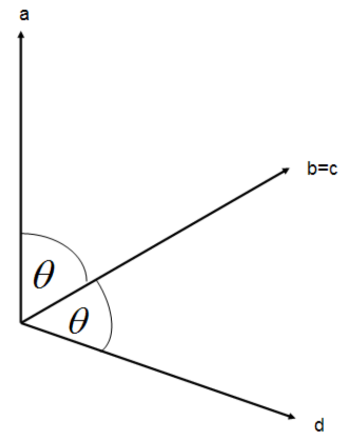

Figure 1: Choice of directions leading to a violation of the Bell' inequality.

expectation values cannot exceed unity [10],

$$|A(\hat{a},\lambda)| \leq 1, \left|B(\hat{b},\lambda)\right| \leq 1$$

Using above statement, we can show that [11, 12]

$$\left|C_{cl}(\hat{a},\hat{b}) - C_{cl}(\hat{a},\hat{d})\right| + \left|C_{cl}(\hat{c},\hat{b}) - C_{cl}(\hat{c},\hat{d})\right| \leq 2. \tag{4}$$

This expression is known as *Bell's inequality*. Now we consider a quantum correlation which can be written as [11, 12]

$$C_{qm}(\hat{a},\hat{b}) = \langle\Psi_A|\,\vec{j}_1\cdot\hat{a}\otimes\vec{j}_2\cdot\hat{b}\,|\Psi_A\rangle = -\cos\theta_{\hat{a}\hat{b}}. \tag{5}$$

where $\theta_{\hat{a}\hat{b}}$ is the angle between $\hat{a}$ and $\hat{b}$. If we set the directions which are shown in Fig. 1, we have

$$\begin{aligned} |C_{qm}(\theta) - C_{qm}(2\theta)| + |C_{qm}(\theta) - C_{qm}(0)| &\leq 2 \\ 2\cos\theta - \cos 2\theta &\leq 1. \end{aligned} \tag{6}$$

The inequality is violated for a wide reange of θ. The maximum violation occurs for $\theta = \pi/3$, we obtain a contradiction,

$$\frac{3}{2} \leq 1.$$

Quantum mechanical prediction are not compatible with Bell's inequality (probabilistically). This shows that difference between quantum mechanics and the theory satisfying Einstein's locality principle. We may ask a question what does this mean from the result of calculations. To answer this question, we have to reconsider the assumption A1 and A2 that used to derive the Bell inequality. The combination of A1 and A2 is known as *local realism.* They work very well on every experiment (classical world). Therefore Bell shows that one or both of these assumptions must be wrong.

How do we interpret from Bell's inequality. Some physicists say the locality must be dropped because the quantum world is not locally realistic. On the other hand, some say realistic point of view must be ignored. But what we can say from Bell's results of calculations is that either or both of locality and realism must be dropped from our point of view of classical world if we would like to deeply understand quantum world. Then EPR-correlation is not the contradiction of quantum theory but it is a fundamental feature of quantum world.

3 Density Matrix

In real situations in quantum mechanics, we often need to consider systems consisting of a huge number of atoms or molecules and they can be in different quantum states (except for very special conditions when the various atoms or molecules are prepared in the same state such as in a laser tube). To handle such a system in quantum mechanics, Von Neumann and Weyl [13] introduced the concept of density matrix.

Consider a quantum system which occupies of a number of states $|\Psi_i\rangle$, with respective classical probabilities p_i. We shall call $\{p_i, |\Psi_i\rangle\}$ or $\{\rho_i\}$ an *ensemble of pure states.* The density operator of the system is defined as

$$\rho \equiv \sum_i p_i \left|\Psi_i\right\rangle \left\langle\Psi_i\right| . \tag{7}$$

The classical distribution satisfies

$$\sum_i p_i = 1 \ , \quad p_i = p_i^* \ , \quad p_i \geq 0 \quad . \tag{8}$$

3.1 Pure and mixed states

Consider the set of all states which it is called pure states

$$\rho = \left|\Psi\right\rangle \left\langle\Psi\right| , \tag{9}$$

where $|\Psi\rangle$ is a state vector. The average value of an observable $\hat{\mathrm{Q}}$ in a pure state $\langle\Psi|$ is

$$\left\langle \hat{Q} \right\rangle = \mathrm{Tr}\left(\hat{Q}\left|\Psi\right\rangle\left\langle\Psi\right|\right) = \left\langle\Psi\right|\hat{Q}\left|\Psi\right\rangle . \tag{10}$$

The condition on the density matrix to describe pure state is

$$\rho^2 = (|\Psi\rangle\langle\Psi|)(|\Psi\rangle\langle\Psi|) = |\Psi\rangle\langle\Psi| = \rho. \tag{11}$$

From Eq. (7), we can show that in general

$$\begin{aligned}
\rho^2 &= \left(\sum_i p_i |\Psi_i\rangle\langle\Psi_i|\right)\left(\sum_k p_k |\Psi_k\rangle\langle\Psi_k|\right) \\
&= \sum_{ik} p_i p_k |\Psi_i\rangle\langle\Psi_i |\Psi_k\rangle\langle\Psi_k| \\
&= \sum_i p_i^2 |\Psi_i\rangle\langle\Psi_i|. \qquad (12)
\end{aligned}$$

Then we have

$$\begin{aligned}
\mathrm{Tr}(\rho^2) &= \mathrm{Tr}\left(\sum_i p_i^2 |\Psi_i\rangle\langle\Psi_i|\right) \\
&= \sum_i p_i^2. \qquad (13)
\end{aligned}$$

Using Eq. (11), we can show that $\mathrm{Tr}(\rho^2) = \mathrm{Tr}(\rho)$. It means that $p_i^2 - p_i = 0$, the solutions are $p_i = 1, 0$. According to Eq. (8), it must be the case that exactly one of them has the value 1 and all others are 0. Thus Eq. (7) consists of a single term and it must be pure state, Eq. (9).

For a mixed state, we can write $\rho = \sum_i p_i |\Psi_i\rangle\langle\Psi_i|$. The expectation value of the observable Q for a mixed state take the form

$$\left\langle \hat{Q} \right\rangle = \mathrm{Tr}\left(\hat{Q}\sum_i p_i |\Psi_i\rangle\langle\Psi_i|\right) = \sum_i p_i \langle\Psi_i|\hat{Q}|\Psi_i\rangle. \tag{14}$$

Obviously, we see that $\rho^2 \neq \rho$. It means that there are more than one classical distribution p_i and we also find that

$$\mathrm{Tr}(\rho^2) = \sum_i p_i^2 \leq 1. \tag{15}$$

3.2 The density matrix of a qubit

In the context of quantum information and quantum computation, we use *quantum bit*, or *qubit* for short instead of the classical bits. The state of the classical

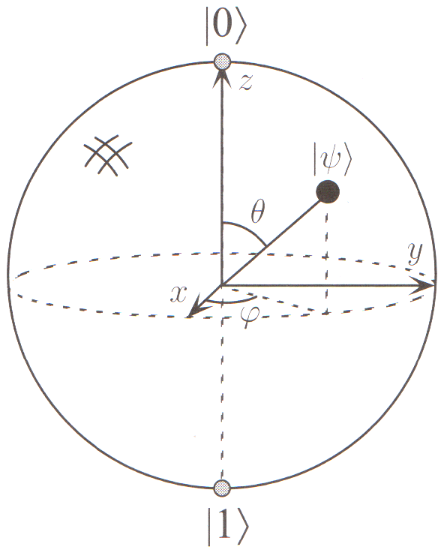

Figure 2: The visualization of a qubit [1].

bit is defined by either 0 or 1. Analogously, qubits can be represented by two possible states $|0\rangle$ and $|1\rangle$. The difference between bits and qubits is that a qubit can be decomposed in *linear combinations* of states $|0\rangle$ and $|1\rangle$, called *superposition*

$$|\Psi\rangle = \alpha |0\rangle + \beta |1\rangle , \tag{16}$$

where α and β are complex numbers and $|\alpha|^2 + |\beta|^2 = 1$. The useful visualization of a qubit is the *Bloch sphere* (see Fig. 2) [1]. We can write Eq. (16) in form

$$|\Psi(\theta, \varphi)\rangle = \cos\frac{\theta}{2} |0\rangle + e^{i\varphi} \sin\frac{\theta}{2} |1\rangle , \tag{17}$$

where θ and φ are real numbers. We now write the corresponding density matrix

$$\rho(\theta, \varphi) = |\Psi(\theta, \varphi)\rangle \langle \Psi(\theta, \varphi)| , \tag{18}$$

and its matrix representation in $\{|0\rangle , |1\rangle\}$ basis is

$$\rho(\theta, \varphi) = \begin{bmatrix} \cos^2\frac{\theta}{2} & \cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{-i\varphi} \\ \cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{i\varphi} & \sin^2\frac{\theta}{2} \end{bmatrix} . \tag{19}$$

It is easy to show that $\rho^2(\theta, \varphi) = \rho(\theta, \varphi)$, as it must be a pure state.

Next we consider the density matrix for the mixed state of a qubit which takes a form

$$\rho(x, y, z) = \frac{1}{2} \begin{bmatrix} 1+z & x-iy \\ x+iy & 1-z \end{bmatrix} . \tag{20}$$

We have seen that the density matrix is positive and therefore its eigenvalues are positive. Thus, we obtain the condition $\det \rho(x, y, z) \geq 0$. From Eq. (20), we have

$$\det \rho = \frac{1}{4}\left(1 - |\vec{r}|^2\right), \tag{21}$$

where $\vec{r} = (x, y, z)$ is called as the *Bloch vector*. We see that $\det \rho \geq 0$ if and only if $0 \leq |\vec{r}| \leq 1$. For a pure state, the density matrix ρ has eigenvalues 1 and 0. Then $\det \rho = 0$, which implies that $|\vec{r}| = 1$. We can conclude that pure states are located on the surface of the Bloch ball. On the other hand, mixed states are located inside space of the Bloch sphere.

3.3 Composite systems

Suppose we have n subsystems in pure states $|\Psi_i\rangle \in \mathbb{H}_i$, then we can represent the state of the entire quantum system by the tensor product

$$|\Psi\rangle = |\Psi_1\rangle \otimes |\Psi_2\rangle \otimes ... \otimes |\Psi_n\rangle. \tag{22}$$

If $\{|j_1\rangle, |j_2\rangle, ..., |j_n\rangle\}$ are basis of $\mathbb{H}_1, \mathbb{H}_2, ..., \mathbb{H}_n$ respectively, then

$$\{|j_1\rangle \otimes |j_2\rangle \otimes ... \otimes |j_n\rangle\}$$

is basis of $\mathbb{H}_1 \otimes \mathbb{H}_2 \otimes ... \otimes \mathbb{H}_n$. Eq. (22) can be expanded in terms of the basis

$$|\Psi\rangle = \sum_{j_1, j_1, ..., j_n} c_{j_1, j_1, ..., j_n} |j_1\rangle \otimes |j_2\rangle \otimes ... \otimes |j_n\rangle, \tag{23}$$

where $c_{j_1, j_1, ..., j_n}$ are complex numbers and

$$\sum_{j_1, j_1, ..., j_n} c_{j_1, j_1, ..., j_n} c^*_{j_1, j_1, ..., j_n} = 1$$

In this paper, we will be mainly concerned with two subsystems A and B. Then we can write

$$|\Psi_{AB}\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle, \tag{24}$$

or

$$|\Psi_{AB}\rangle = \sum_{j_A, j_B} c_{j_A, j_B} |j_A\rangle \otimes |j_B\rangle. \tag{25}$$

The density matrix of a pure state can be written

$$\rho_{AB} = |\Psi_{AB}\rangle \langle\Psi_{AB}| = \sum_{j_A, j_B, \bar{j}_A, \bar{j}_B} c_{j_A, j_B} c^*_{\bar{j}_A, \bar{j}_B} |j_A\rangle \langle\bar{j}_A| \otimes |j_B\rangle \langle\bar{j}_B|. \tag{26}$$

The state in $\mathbb{H}_A \otimes \mathbb{H}_B$ which are not product states are called *entangled*. They can be written only as a superposition of product states. Consider the state

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \tag{27}$$

Let us express $|\Phi^+\rangle$ as a combination of two qubits

$$|\Psi_A\rangle = \alpha|0\rangle + \beta|1\rangle,$$
$$|\Psi_B\rangle = \gamma|0\rangle + \delta|1\rangle.$$

We would like to find the values of α, β, γ and δ:

$$\begin{aligned}\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) &= |\Psi_A\rangle \otimes |\Psi_B\rangle \\ &= \left(\alpha|0\rangle + \beta|1\rangle\right)\left(\gamma|0\rangle + \delta|1\rangle\right) \\ &= \alpha\gamma|00\rangle + \beta\gamma|10\rangle + \alpha\delta|01\rangle + \beta\delta|11\rangle. \end{aligned} \tag{28}$$

Equating the appropriate parts, we obtain $\alpha\gamma = \beta\delta = 1/\sqrt{2}$ and $\beta\gamma = \alpha\delta = 0$. The result $\beta\gamma = 0$ shows that either β or γ is zero. However, this cannot be the case, since the condition $\alpha\gamma = \beta\delta = 1/\sqrt{2}$. Then we say that the state $|\Phi^+\rangle$ cannot be written in product state of $|\Psi_A\rangle \otimes |\Psi_B\rangle$ and we conclude that $|\Phi^+\rangle$ is entangled.

A mixed state of the composite system takes the form

$$\rho_{AB} = \sum_k p_k \left|\Psi_{AB}^k\right\rangle\left\langle\Psi_{AB}^k\right|, \tag{29}$$

where p_k define a classical probability distributions with $p_k \geq 0$, $\sum_k p_k = 1$. We will use this notation in later sections. Using Eq. (25), Eq. (29) becomes

$$\rho_{AB} = \sum_k p_k \sum_{j_A, j_B, \bar{j}_A, \bar{j}_B} c^k_{j_A, j_B}\left(c^k_{\bar{j}_A, \bar{j}_B}\right)^* \left|j_A^k\right\rangle\left\langle\bar{j}_A^k\right| \otimes \left|j_B^k\right\rangle\left\langle\bar{j}_B^k\right|. \tag{30}$$

If the density matrix of a mixed state can be written in the form

$$\rho_{AB} = \sum_k p_k \rho_A^k \otimes \rho_B^k, \tag{31}$$

the state ρ_{AB} is separable. Otherwise it is an entangled state. We will discuss in more details on this stuff in section 5.

3.4 The reduced density matrices

In the context of quantum entanglement, we can reduce density matrix operator which is a useful tool to analysis properties of composite quantum systems [1]. The reduced density matrix for system A is defined by

$$\rho_A \equiv \mathrm{Tr}_B\left(\rho_{AB}\right), \tag{32}$$

where Tr_B is a partial trace over system B and ρ_A is called a reduced density matrix for subsystem A. Suppose that the product state was defined in Eq. (25)

and the density matrix is defined in Eq. (59). Tracing out the system B now give us

$$\begin{aligned}
\mathrm{Tr}_B(\rho_{AB}) &= \sum_{j_A,j_B,\bar{j}_A,\bar{j}_B} c_{j_A,j_B}\left(c_{\bar{j}_A,\bar{j}_B}\right)^* |j_A\rangle\langle\bar{j}_A| \otimes \mathrm{Tr}\left(|j_B\rangle\langle\bar{j}_B|\right) \\
&= \sum_{j_A,j_B,\bar{j}_A,\bar{j}_B} c_{j_A,j_B}\left(c_{\bar{j}_A,\bar{j}_B}\right)^* \left(\langle\bar{j}_B|\,|j_B\rangle\right)|j_A\rangle\langle\bar{j}_A| \\
&= \sum_{j_A,\bar{j}_A} C_{j_A,\bar{j}_A}|j_A\rangle\langle\bar{j}_A|, \qquad (33)
\end{aligned}$$

where

$$C_{j_A,\bar{j}_A} = \sum_{j_B} c_{j_A,j_B}\left(c_{\bar{j}_A,j_B}\right)^*. \qquad (34)$$

Using Eq. (30), we find a reduced density matrix for subsystem A of a mixed state

$$\begin{aligned}
\mathrm{Tr}_B(\rho_{AB}) &= \sum_k p_k \sum_{j_A,j_B,\bar{j}_A,\bar{j}_B} c^k_{j_A,j_B}\left(c^k_{\bar{j}_A,\bar{j}_B}\right)^* \left|j^k_A\right\rangle\left\langle\bar{j}^k_A\right| \otimes \mathrm{Tr}\left(\left|j^k_B\right\rangle\left\langle\bar{j}^k_B\right|\right) \\
&= \sum_k p_k \sum_{j_A,j_B,\bar{j}_A,\bar{j}_B} c^k_{j_A,j_B}\left(c^k_{\bar{j}_A,\bar{j}_B}\right)^* C^k_{j_A,\bar{j}_A}\left(\left\langle\bar{j}^k_B\right|\left|j^k_B\right\rangle\right)\left|j^k_A\right\rangle\left\langle\bar{j}^k_A\right| \\
&= \sum_k p_k \sum_{j_A,\bar{j}_A} C^k_{j_A,\bar{j}_A}\left|j^k_A\right\rangle\left\langle\bar{j}^k_A\right|. \qquad (35)
\end{aligned}$$

where

$$C^k_{j_A,\bar{j}_A} = \sum_{j_B} c^k_{j_A,j_B}\left(c^k_{\bar{j}_A,j_B}\right)^*. \qquad (36)$$

It is important to point out that, even ρ_{AB} is a pure state of the composite system, it is not assured that the reduced density matrix ρ_A and ρ_B are a pure state. Let us consider the state in Eq. (27). Tracing out the second system gives us

$$\begin{aligned}
\rho_A \equiv \mathrm{Tr}_B\left(\left|\Phi^+\right\rangle\left\langle\Phi^+\right|\right) &= \mathrm{Tr}_B\left(\frac{1}{2}\left(|00\rangle + |11\rangle\right)\left(\langle 00| + \langle 11|\right)\right) \\
&= \frac{1}{2}\left(|0\rangle\langle 0|\langle 0|\,|0\rangle + |1\rangle\langle 0|\langle 0|\,|1\rangle\right. \\
&\quad \left. + |0\rangle\langle 1|\langle 1|\,|0\rangle + |1\rangle\langle 1|\langle 1|\,|1\rangle\right) \\
&= \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) = \frac{I}{2}. \qquad (37)
\end{aligned}$$

We also find that

$$\rho_B \equiv \mathrm{Tr}_A\left(\left|\Phi^+\right\rangle\left\langle\Phi^+\right|\right) = \frac{I}{2}. \qquad (38)$$

It is easy to show that ρ_A and ρ_B are mixed states: we have $\rho_A^2 = \rho_B^2 = I/4$ and $\mathrm{Tr}(\rho_A^2) = \mathrm{Tr}(\rho_A^2) = 1/2 < 1$. This result shows that any mixture can be consider as part of a pure state.

3.5 The Schmidt decomposition

Consider a pure bipartite state Eq. (25), we can write the coefficient $\mathbf{c} = \mathbf{udv}$, where $\mathbf{d}$ is a diagonal matrix with non-negative elements, and $\mathbf{u}$ and $\mathbf{v}$ are unitary matrices. Then Eq. (25) can be written

$$|\Psi_{AB}\rangle = \sum_{j_A, j_B, l} u_{j_A,l} d_{ll} v_{l,j_B} |j_A\rangle \otimes |j_B\rangle . \tag{39}$$

We now define

$$|l_A\rangle = \sum_{j_A} u_{j_A,l} |j_A\rangle ,$$

$$|l_B\rangle = \sum_{j_B} v_{l,j_B} |j_B\rangle ,$$

and $\xi_l \equiv d_{ll}$, we obtain

$$|\Psi_{AB}\rangle = \sum_l \xi_l |l_A\rangle \otimes |l_B\rangle , \tag{40}$$

where ξ_l are called *Schmidt coefficients* satisfying $\sum_l \xi^2 = 1$. The bases $|l_A\rangle$ and $|l_B\rangle$ are called the *Schmidt bases* for A and B, respectively.

We find that the reduced density matrices for A and B can be written

$$\rho_A = \sum_l \xi_l^2 |l_A\rangle \langle l_A| , \tag{41}$$

$$\rho_B = \sum_l \xi_l^2 |l_B\rangle \langle l_B| . \tag{42}$$

Both reduced states have the same non-negative eigenvalues ξ_l^2. We know that the eigenvalues of the states ρ_A and ρ_B are invariant under a unitary transformation

$$U_i \rho_i U_i^\dagger = \sum_l \xi_l^2 U_i |l_i\rangle \langle l_i| U_i^\dagger = \sum_l \xi_l^2 |n_i\rangle \langle n_i| , \tag{43}$$

where $i = A, B$ and $U_i |l_i\rangle = |n_i\rangle$. Then the Schmidt coefficients are unchanged under a unitary transformation on one of the subsystems

$$U_A \otimes I |\Psi_{AB}\rangle = \sum_l \xi_l U_A |l_A\rangle \otimes |l_B\rangle , \tag{44}$$

or

$$I \otimes U_B |\Psi_{AB}\rangle = \sum_l \xi_l |l_A\rangle \otimes U_B |l_B\rangle . \tag{45}$$

4 Quantum operations

4.1 Quantum processes

In order to determine the properties of a quantum system, quantum measurements must be performed. In this section, we will show the effect of the quantum measurements on a quantum system which is described by the density matrix ρ in a Hilbert space of dimension N. All quantum operations can be constructed by composting four elementary transformations [14].

i) **Extend the system**: We can define a new state after adding an auxiliary system σ to the original one,

$$\rho \rightarrow \rho' = \rho \otimes \sigma, \tag{46}$$

where

$$\sigma = \sum_{\nu=1}^{R} q_\nu |\nu\rangle \langle\nu| \tag{47}$$

The auxiliary system is often refereed to as the ancilla or an environment. This process will expand the size of the Hilbert space.

ii) **Unitary transformations**: Unitary operations can be perform on the system ρ', resulting in a new state

$$\rho' \rightarrow \rho'' = U\rho \otimes \sigma U^\dagger. \tag{48}$$

iii) **Partial trace**: This process leads to a reduction of the size of the Hilbert space. After tracing out the ancilla, we find [1, 15] that the state ρ'' is given by

$$\begin{aligned}
\rho''' &= \mathrm{Tr}_{ancilla}\left[U\left(\rho \otimes \sum_{\nu=1}^{R} q_\nu |\nu\rangle\langle\nu|\right)U^\dagger\right] \\
&= \sum_{\nu=1}^{R}\sum_{\mu=1}^{K} q_\nu \langle\mu| U |\nu\rangle \rho \langle\nu| U^\dagger |\mu\rangle \\
&= \sum_{l=1}^{RK} A_l \rho A_l^\dagger,
\end{aligned} \tag{49}$$

where $\{|\mu\rangle\}_{\mu=1}^{K}$ is a basis in the ancilla's Hilbert space and

$$A_l = \sqrt{q_\nu} \langle\mu| U |\nu\rangle \ \ ; \ \ l = \mu + \nu(K-1)\ . \tag{50}$$

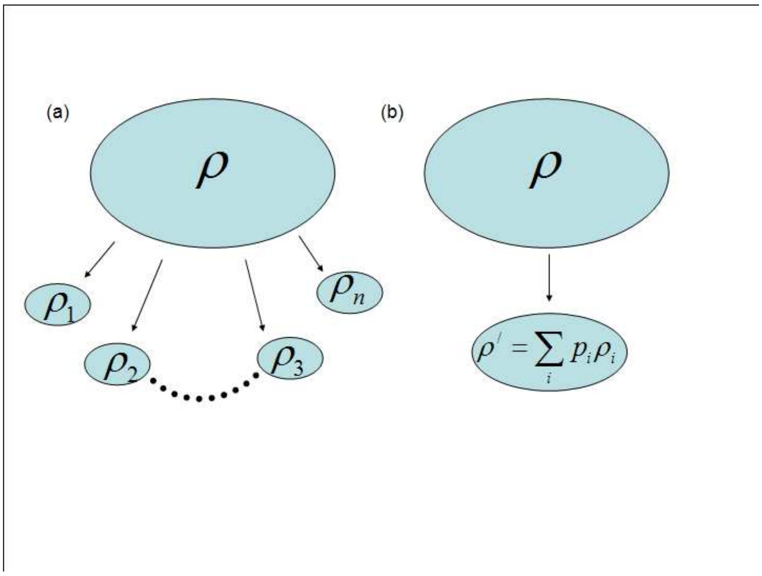

Figure 3: Schematic picture of the quantum operations with and without subselection show in part(a) and part(b), respectively [16]

We see that

$$\begin{aligned}\sum_{l=1}^{RK} A_l^\dagger A_l &= \sum_{\nu=1}^{R}\sum_{\mu=1}^{K} q_\nu \langle\mu| U |\nu\rangle \langle\nu| U^\dagger |\mu\rangle \\ &= \sum_{\nu=1}^{R} q_\nu \langle\nu| UU^\dagger |\nu\rangle = \sum_{\nu=1}^{R} q_\nu = I_N. \end{aligned} \tag{51}$$

iv) **Selective measurement**: In this process, a probabilistic of the outcome is well defined which will be discussed in the next sections. This is called a probabilistic quantum operation.

In conclusion, if we consider the time evolution of the quantum system through a unitary transformation, we have

$$\rho \to \rho' = U\rho U^\dagger \ , \quad UU^\dagger = I_N$$

Furthermore, if an ancilla is added into the system, later removed by a partial trace, we will end up with operations of the form

$$\rho \to \rho' = \sum_{l}^{RK} A_l \rho A_l^\dagger \ ; \quad \sum_{l}^{RK} A_l^\dagger A_l = I_N \tag{52}$$

Generally, this is the operator sum representation of a complete positive map [14]

4.2 Quantum measurement postulate

Let us define measurement operators A_i, with k possible measurement outcomes, which satisfy the completeness relation $\sum_i^k A_i^\dagger A_i = I_N$. If the initial state is ρ, after the measurement, the state is ρ_i with probability p_i,

$$\rho \rightarrow \rho_i = \frac{A_i \rho A_i^\dagger}{\mathrm{Tr}\left(A_i \rho A_i^\dagger\right)} \ ; \ \ p_i = \mathrm{Tr}\left(A_i \rho A_i^\dagger\right) \tag{53}$$

Thesse measurements are called selective because the outcomes labeled by i are recorded. Note that the collapse of the wave function happens in the measurement process that we are performing. Fig. 3(a) shows that the state ρ collapses into new states ρ_i with probabilities p_i. If we perform the measurements without recording the results, the state ρ will transform into a convex combination of all possible outcomes, Eq. (52), as illustrated in Fig. 3(b).

In quantum mechanics, there are two different types of quantum measurements called repeatable and non-repeatable. In the next two sections, we will defined these types of quantum measurements.

4.3 Projective measurements

We can represent any observable Q in the spectral decomposition as

$$Q = \sum_{i=1}^{N} \lambda_i P_i, \tag{54}$$

where λ_i are the eigenvalues of Q. The measurement operators are orthogonal projectors. Let us define $A_i = A_i^\dagger = P_i$ and $P_i P_j = \delta_{ij} P_i$. The orthogonal measurement operators are $P_i = |e_i\rangle\langle e_i|$, where $|e_i\rangle$ are the eigenstates of Q. In a non-selective projective measurement, the initial state ρ is transformed into the mixture

$$\rho \rightarrow \rho' = \sum_{i=1}^{N} P_i \rho P_i. \tag{55}$$

On the other hand, if we record the outcomes labeled by λ_i with probabilistic p_i, the initial state becomes

$$\rho \rightarrow \rho_i = \frac{P_i \rho P_i}{\mathrm{Tr}\left(P_i \rho P_i\right)} \ , \ \ p_i = \mathrm{Tr}\left(P_i \rho P_i\right) \ . \tag{56}$$

The expectation value of the observable Q can be found as

$$\langle Q \rangle = \sum_{i=1}^{N} p_i \lambda_i = \mathrm{Tr}\,(Q\rho). \tag{57}$$

We observe that the state in Eq. (56) remains the same and will give the same outcomes, if we repeat the projective measurement.

4.4 Positive Operator Valued Measures

By relaxing the orthogonality constraint on the measurement operators, we are led to a new set of operators defining a positive operator valued measures as POVM. Consider the set of *positive* operators F_i, where $i = 1....k$, acting on an N-dimensional Hilbert space, which also satisfy the completeness condition

$$\sum_{i=1}^{k} F_i = I_N, \tag{58}$$

and $F_i = F_i^\dagger$ and $F_i \geq 0$. A POVM measurement acting on the initial state ρ produces the ith outcome with probability $p_i = \text{Tr}(F_i\rho)$. POVMs are competible with the general framework of the quantum measurement postulate (see also section 4.2). If we choose

$$A_i = U_i\sqrt{F_i}, \tag{59}$$

we find that $F_i = A_i^\dagger A_i$. Note that a POVM does not uniquely determine the measurement operators A_i since the unitary transformation U_i dropout. Therefore, the unitaries U_i have no influence on the probabilities p_i but they contribute to determining the outcome state ρ_i. Because we do not restrict with the orthogonal property of the quantum operators. Then the outcomes state ρ_i change if the POVM measurements are preformed repeatable.

In this section, we would like to give you an example of a POVM measurement. Consider a system of a qubit which is described by the density matrix ρ in term of the Bloch vector $\vec{r}$

$$\rho = \frac{1}{2}\left(I + \vec{r}\cdot\vec{\sigma}\right), \tag{60}$$

where $\vec{\sigma}$ is the vector of the Pauli spin matrices. Then we now set the POVM operators as

$$F_i = a_i I + b_i \vec{n}_i \cdot \vec{\sigma}, \tag{61}$$

where $\vec{n}_i$ is a unit vector in R^3 and a_i and b_i are supposed to be non-negative real numbers. So we can show that F_i and $1 - F_i$ are positive operators. The completeness relation Eq. (58) leads to the conditions

$$\sum_i a_i = 1, \tag{62}$$

$$\sum_i b_i \vec{n}_i = 0. \tag{63}$$

The probabilities for the measurement outcomes of the POVM measurement are

$$p_i = \mathrm{Tr}(\rho F_i) = a_i + b_i \vec{n}_i \cdot \vec{r}. \tag{64}$$

From above result, in order to determine the value of $\vec{r}$, the non-vanishing vectors $b_i\vec{n}_i$ must span in R^3 and with the condition Eq. (63), there must be at least four vectors $\vec{n}_i$ $(i = 1, 2, 3, 4)$. We give an explicit example which fulfills Eqs. (62) and (63). We now set $a_i = b_i = 1/4$ and

$$\begin{aligned}
\vec{n}_1 &= (0, 0, 1)\,, \\
\vec{n}_2 &= \left(\frac{2\sqrt{2}}{3}, 0, -\frac{1}{3}\right), \\
\vec{n}_3 &= \left(-\frac{\sqrt{2}}{3}, \sqrt{\frac{2}{3}}, -\frac{1}{3}\right), \\
\vec{n}_4 &= \left(-\frac{\sqrt{2}}{3}, -\sqrt{\frac{2}{3}}, -\frac{1}{3}\right).
\end{aligned} \tag{65}$$

Inserting into Eq. (61) leads to the *informationally-complete* POVM for a qubit. The other examples of POVMs can be found in the standard text books [1, 17]

4.5 LOCC operations

In the context of quantum entanglement and quantum information theory, there is usually a number of parties sharing quantum states. In this paper, we will restrict ourselves to two parties, say Alice and Bob. They can perform quantum operations locally (LO) on their states and they are also allowed to contact each other via classical communication (CC). Combining these two processes, we obtain LOCC or LQCC [3, 18, 19]. Notice that LOs correspond to the general quantum operation such as POVM, whereas CC leads to classical correlations between the subsystems.

4.5.1 One-way LOCC operations: Forward and Backward

This type of operations performed by Bob depend on Alice's operations, but not conversely. Let us define a map $\Lambda^{A\to B}_{LOCC}$ such that

$$\Lambda^{A\to B}_{LOCC}(\rho_{AB}) = \sum_{i,j=1}^{K,L} (I_2^A \otimes B_{ij})(A_i \otimes I_1^B)\rho_{AB}(A_i^\dagger \otimes I_1^B)(I_2^A \otimes B_{ij}^\dagger), \tag{66}$$

where I_2^A and I_1^B are the unit operators acting on the Hilbert spaces such that $A_i : H_1^A \to H_2^A$ and $B_{ij} : H_1^B \to H_2^B$, respectively. The operations A_i and B_{ij} also satisfy

$$\sum\nolimits_{i=1}^{K} A_i^\dagger A_i = I_1^A$$

and

$$\sum\nolimits_{j=1}^{L} B_{ij}^{\dagger} B_{ij} = I_1^B$$

By means of the Choi-Kraus [20, 21, 22, 23] representation for any operation Λ_{LOCC}, we can write

$$\Lambda_{LOCC} = \Lambda_{LOCC}^{A} \otimes I_B$$

where Λ_{LOCC}^{A} is a completely positive map on Alice's system and I_B is the identity operator on Bob's system. Then this process is called a one-way LOCC operation from Alice to Bob.

On the opposite way, backward, the type of operations performed by Bob depend on Alice's operations, but not conversely. Let us define a map $\Lambda_{LOCC}^{A\leftarrow B}$ such that

$$\Lambda_{LOCC}^{A\leftarrow B}(\rho_{AB}) = \sum_{i,j=1}^{K,L} (A_{ij} \otimes I_2^B)(I_1^A \otimes B_i)\rho_{AB}(I_1^A \otimes B_i^{\dagger})(A_{ij}^{\dagger} \otimes I_2^B). \tag{67}$$

where I_1^A and I_2^B are the unit operations acting on the Hilbert spaces such that $A_i : H_1^A \to H_2^A$ and $B_{ij} : H_1^B \to H_2^B$, respectively. The operations B_i and A_{ij} also satisfy

$$\sum\nolimits_{i=1}^{K} B_i^{\dagger} B_i = I_1^B$$

and

$$\sum\nolimits_{j=1}^{L} A_{ij}^{\dagger} A_{ij} = I_1^A$$

4.5.2 Two-way LOCC operations

In this situation, bidirectional classical communication is allowed. Let us define a two-way LOCC map $\Phi_{LOCC}^{A\rightleftarrows B}$ such that [18]

$$\Phi_{LOCC}^{A\rightleftarrows B}(\rho_{AB}) = \sum_{i_1,\ldots,i_{2n}=1}^{K_1,\ldots,K_{2n}} M_{i_1,\ldots,i_{2n}}^{AB} \rho_{AB} \left(M_{i_1,\ldots,i_{2n}}^{AB}\right)^{\dagger}, \tag{68}$$

where $M_{i_1,\ldots,i_{2n}}^{AB}$ is given by

$$M_{i_1,\ldots,i_{2n}}^{AB} = (I_{n+1}^A \otimes B_{i_{2n},\ldots,i_1}^{2n})(A_{i_{2n-1},\ldots,i_1}^{2n-1} \otimes I_n^B)(I_n^B \otimes B_{i_{2n-1},\ldots,i_1}^{2n-2})\ldots(I_2^A \otimes B_{i_2,i_1}^{2})(A_{i_1}^{1} \otimes I_1^B), \tag{69}$$

and we have

$$\sum_{i_{2k+1}=1}^{K_{2k+1}} \left(A_{2k+1}^{i_{2k+1},\ldots,i_1}\right)^{\dagger} A_{2k+1}^{i_{2k+1},\ldots,i_1} = I_{k+1}^A,$$

$$\sum_{i_{2k}=1}^{K_{2k}} \left(B_{2k}^{i_{2k},\ldots,i_1}\right)^{\dagger} B_{2k}^{i_{2k},\ldots,i_1} = I_k^B.$$

We see that a one-way LOCC operation is a subclass of a two-way LOCC operation.

Actually, there is an another important operation: S-LOCC operations which can be written as

$$\Upsilon_{S-LOCC}^{A\rightleftarrows B}(\rho_{AB}) = \sum_{i=1}^{k} (A_i \otimes B_i)\rho_{AB} \left(A_i \otimes B_i\right)^{\dagger}, \tag{70}$$

where $\sum_{i=1} (A_i \otimes B_i)^{\dagger} (A_i \otimes B_i) = I_A \otimes I_B$. The class of S-LOCC is larger than the LOCC class [24].

5 PPT-criterion

5.1 Positive partial transpose

Entanglement is an inseparable state of composite quantum system which represents the quantum correlation. The question *how to distinguish quantum correlations from the classical correlations* plays an important role in quantum entanglement research. Peres [2] proposed a method to distinguish between entangled and separable states. This criterion involves with the positivity of the partial transpose of a given state.

If the density matrix of the composite quantum systems AB can be written as

$$\rho_{AB} = \sum_{i}^{N} p_i \rho_A^i \otimes \rho_B^i, \tag{71}$$

then the system is in a *separable* state, here $p_i \geq 0$ define a classical probability distribution which satisfies $\sum_i p_i = 1$, ρ_A^i and ρ_B^i are density matrices for quantum systems A and B, respectively. A separable state always satisfies Bell's inequality, but the converse is not necessarily true. This means that not all entangled states violate a given Bell inequality.

Eq. (71) holds if the eigenvalues of the partially transposed density matrices

$$\rho_{AB}^{T_A} = \sum_{i} p_i \left(\rho_A^i\right)^T \otimes \rho_B^i \tag{72}$$

and

$$\rho_{AB}^{T_B} = \sum_{i} p_i \rho_A^i \otimes \left(\rho_B^i\right)^T \tag{73}$$

are *not* negative, which is known as PPT-criterion [2]. From a mathematical point of view, we use the operator of transposition to map a separable density matrix[2] $\rho_{AB} \rightarrow \rho_{AB}^{PT}$ (see Fig. 4) both of which live in the region of separable density

[2] ρ_{AB}^{PT} is the partial transpose of ρ_{AB} which refers to either $\rho_{AB}^{T_A}$ or $\rho_{AB}^{T_B}$

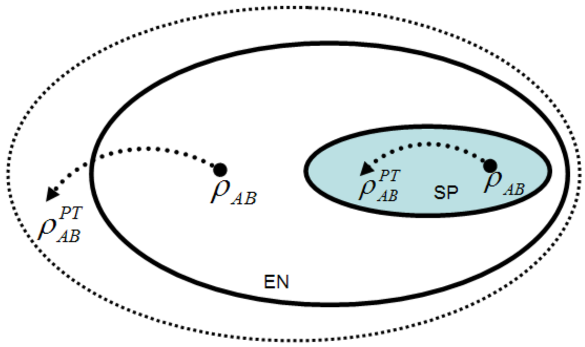

Figure 4: This graph illustrates that the PT-operation maps a separable state to another separable state, while it may map an entangled state to another state that cannot be density matrix. The areas of the separable and entangled states are denoted by SP and EN, respectively. The dash line represent the boundary of all states.

matrices. On the other hand, if the state ρ_{AB}^{PT} has some negative eigenvalues, the transposition operator maps the density ρ_{AB} to another matrix outside the region of density matrix. In this case the density matrix ρ_{AB} cannot be expressed in the form given in Eq. (71). The PPT-criterion is exact only for $2 \otimes 2$ and $2 \otimes 3$ quantum systems [25]. For product spaces involving higher dimensions, this criterion was shown not be necessary but sufficient.

Consider the states $|\Psi_{AB}^i\rangle = |\Psi_A^i\rangle \otimes |\Psi_B^i\rangle$, we can write the density matrix Eq. (71) as

$$\rho_{AB} = \sum_i^N p_i \left|\Psi_{AB}^i\right\rangle \left\langle\Psi_{AB}^i\right| . \tag{74}$$

The density matrix ρ_{AB} can be represented in the computational basis, namely $|0\rangle$ and $|1\rangle$, by introducing

$$\left|\Psi_{AB}^i\right\rangle = \sum_{ab} c_{ab}^i \left|a\right\rangle \otimes \left|b\right\rangle , \tag{75}$$

where c_{ab}^i are complex numbers satisfying the completeness condition $\sum_i c_{ab}^i = 1$. Inserting Eq. (75) into Eq. (74), we obtain

$$\rho_{AB} = \sum_{aa'} \sum_{bb'} w_{aa'bb'} |a\rangle \langle a'| \otimes |b\rangle \langle b'| , \tag{76}$$

where

$$w_{aa'bb'} = \sum_{i}^{N} p_i c^i_{ab} \left(c^i_{a'b'}\right)^* . \tag{77}$$

Then we can find the partial transpose of state ρ_{AB} with respect to A and B as follows:

$$\rho_{AB}^{T_A} = \sum_{aa'} \sum_{bb'} w_{a'abb'} \left(|a\rangle \langle a'|\right)^T \otimes |b\rangle \langle b'| = \sum_{aa'} \sum_{bb'} w_{a'abb'} |a'\rangle \langle a| \otimes |b\rangle \langle b'| , \tag{78}$$

and

$$\rho_{AB}^{T_B} = \sum_{aa'} \sum_{bb'} w_{aa'b'b} |a\rangle \langle a'| \otimes \left(|b\rangle \langle b'|\right)^T = \sum_{aa'} \sum_{bb'} w_{a'ab'b} |a\rangle \langle a'| \otimes |b'\rangle \langle b| . \tag{79}$$

Let us have a look the action of partial transpositions ,defined in Eqs. (78) and (79), in a matrix representation. We now consider a $M \times N$-block matrix

$$W = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{80}$$

where A, B, C and D have the size $m \times n$ and $M = 2m$ and $N = 2n$. The partial transpositions of this matrix are given by [14]

$$W^{T_A} = \begin{pmatrix} A^T & B^T \\ C^T & D^T \end{pmatrix}, \quad W^{T_B} = \begin{pmatrix} A & C \\ B & D \end{pmatrix} , \tag{81}$$

which is consistent with $W^{T_B} = (W^{T_A})^T$.

According to *Fano* [26, 14], any density matrix ρ_{AB}, in d-dimensional Hilbert space with $d = NK$, can be represented by means of Pauli's spin matrices σ_i in the following form

$$\rho_{AB} = \frac{1}{NK} \left(I_N \otimes I_K + \sum_{i=1}^{N^2-1} \tau_i^A \sigma_i \otimes I_K + \sum_{j=1}^{K^2-1} \tau_j^B I_N \otimes \sigma_j + \sum_{i=1}^{N^2-1} \sum_{j=1}^{K^2-1} t_{ij} \sigma_i \otimes \sigma_j \right), \tag{82}$$

where τ_i^A and τ_j^B are real Bloch vectors of the partially reduced states. The real matrix t describes the correlation between the subsystems. If $t = 0$, the state ρ_{AB} is separable, but the reverse is not true [27]. From the expression in Eq. (82), a partial transposition flips one of the Bloch vectors,

$$\rho_{AB}^{T_A} = \frac{1}{NK} \left(I_N \otimes I_K - \sum_{i=1}^{N^2-1} \tau_i^A \sigma_i \otimes I_K + \sum_{j=1}^{K^2-1} \tau_j^B I_N \otimes \sigma_j - \sum_{i=1}^{N^2-1} \sum_{j=1}^{K^2-1} t_{ij} \sigma_i \otimes \sigma_j \right), \tag{83}$$

and

$$\rho_{AB}^{T_B} = \frac{1}{NK}\left(I_N \otimes I_K + \sum_{i=1}^{N^2-1} \tau_i^A \sigma_i \otimes I_K - \sum_{j=1}^{K^2-1} \tau_j^B I_N \otimes \sigma_j - \sum_{i=1}^{N^2-1}\sum_{j=1}^{K^2-1} t_{ij}\sigma_i \otimes \sigma_j\right). \tag{84}$$

In the two-qubit case, reflection of all three components of the Bloch vector, $\vec{\tau}^B \to -\vec{\tau}^B$, is equivalent to changing the sign of its single component τ_y^B, followed by a π-rotation about the y-axis.

5.2 PPT-criterion examples

To watch the PPT-criterion in action, we will consider two families of states, one in $\mathbb{C}^2 \otimes \mathbb{C}^2$, and one in $\mathbb{C}^2 \otimes \mathbb{C}^3$.

Example 1.1 Consider the Werner state [28] which is a convex combination of the identity $I \in \mathbb{C}^2 \otimes \mathbb{C}^2$, a separable state, and a maximally entangled state

$$\rho_W(x) = x\left|\Phi^+\right\rangle\left\langle\Phi^+\right| + (1-x)\frac{I}{4} \equiv \rho_{\Phi^+}(x), \quad x \in [0,1] \tag{85}$$

where $|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ is a Bell state. Its matrix representation is given by

$$\rho_{\Phi^+}(x) = \frac{1}{4}\begin{pmatrix} 1+x & 0 & 0 & 2x \\ 0 & 1-x & 0 & 0 \\ 0 & 0 & 1-x & 0 \\ 2x & 0 & 0 & 1+x \end{pmatrix}, \tag{86}$$

with eigenvalues

$$\nu_{\Phi^+}^1 = \frac{1+3x}{4},$$

$$\nu_{\Phi^+}^2 = \nu_{\Phi^+}^3 = \nu_{\Phi^+}^4 = \frac{1-x}{4}.$$

The partial transpositions of the ρ_{Φ^+} with respect to A or B are identical, reading

$$\rho_{\Phi^+}^{T_A}(x) = \rho_{\Phi^+}^{T_B}(x) = \frac{1}{4}\begin{pmatrix} 1+x & 0 & 0 & 0 \\ 0 & 1-x & 2x & 0 \\ 0 & 2x & 1-x & 0 \\ 0 & 0 & 0 & 1+x \end{pmatrix}. \tag{87}$$

The eigenvalues of $\rho_{\Phi^+}^{PT}$ are

$$\mu_{\Phi^+}^1(x) = \mu_{\Phi^+}^2(x) = \mu_{\Phi^+}^3(x) = \frac{(1+x)}{4},$$

$$\mu_{\Phi^+}^4(x) = \frac{(1-3x)}{4}.$$

The fourth eigenvalue is negative for $x > 1/3$, and the state ρ_{Φ^+} is entangled state. On the other hand, the state is separable if $x \leq 1/3$.

Example 1.2 Consider the Werner state which consists of a separable state $I \in \mathbb{C}^2 \otimes \mathbb{C}^3$ and an entangled state of $2 \otimes 3$ quantum system as

$$\rho_W(x) = x\left(|\Phi_1\rangle\langle\Phi_1| + |\Phi_2\rangle\langle\Phi_2|\right) + \frac{(1-x)}{6} I \equiv \rho_{2\otimes3}(x) \qquad (88)$$

where $x \in [0,1]$ and $|\Phi_1\rangle = (|11\rangle + |22\rangle)/\sqrt{2}$ and $|\Phi_2\rangle = (|12\rangle - |23\rangle)/\sqrt{2}$. We now define the computational basis in $\mathbb{C}^3$ as

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad . \qquad (89)$$

It is easy to show that

$$\rho_{2\otimes3}(x) = \begin{pmatrix} \frac{1}{6}+\frac{x}{12} & 0 & 0 & 0 & \frac{x}{4} & 0 \\ 0 & \frac{1}{6}+\frac{x}{12} & 0 & 0 & 0 & -\frac{x}{4} \\ 0 & 0 & \frac{1-x}{6} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-x}{6} & 0 & 0 \\ \frac{x}{4} & 0 & 0 & 0 & \frac{1}{6}+\frac{x}{12} & 0 \\ 0 & -\frac{x}{4} & 0 & 0 & 0 & \frac{1}{6}+\frac{x}{12} \end{pmatrix}, \qquad (90)$$

and the eigenvalues of $\rho_{2\otimes3}(x)$ are

$$\nu_{2\otimes3}^1 = \nu_{2\otimes3}^2 = \nu_{2\otimes3}^3 = \frac{1-x}{6}$$

$$\nu_{2\otimes3}^4 = \frac{1+2x}{6}, \quad \nu_{2\otimes3}^5 = \frac{2-\sqrt{10}x}{12}, \quad \nu_{2\otimes3}^6 = \frac{2+\sqrt{10}x}{12}.$$

Again, partial transpositions of $\rho_{2\otimes3}$ with respect to A and B are identical,

$$\rho_{2\otimes3}^{T_A}(x) = \rho_{2\otimes3}^{T_B}(x) = \begin{pmatrix} \frac{1}{6}+\frac{x}{12} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{6}+\frac{x}{12} & 0 & \frac{x}{4} & 0 & 0 \\ 0 & 0 & \frac{1-x}{6} & 0 & -\frac{x}{4} & 0 \\ 0 & \frac{x}{4} & 0 & \frac{1-x}{6} & 0 & 0 \\ 0 & 0 & -\frac{x}{4} & 0 & \frac{1}{6}+\frac{x}{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{6}+\frac{x}{12} \end{pmatrix}. \qquad (91)$$

The eigenvalues are

$$\mu_{2\otimes3}^1(x) = \mu_{2\otimes3}^2(x) = \frac{2-x}{12},$$

$$\mu_{2\otimes3}^{3}\left(x\right)=\mu_{2\otimes3}^{4}\left(x\right)=\frac{4-\left(1-3\sqrt{5}\right)x}{24},$$

$$\mu_{2\otimes3}^{5}\left(x\right)=\mu_{2\otimes3}^{6}\left(x\right)=\frac{4-\left(1+3\sqrt{5}\right)x}{24}.$$

The third and fourth eigenvalues are non-negative if $x\leq 4/(1+3\sqrt{5})$ so that the state $\rho_{2\otimes3}(x)$ is separable, otherwise it is not. From these examples, we see that the PPT-criterion is easy to distinguish separable states from entangled states.

6 A brief review of entanglement measures

There are many approaches to measure the amount of entanglement present in a pure or mixed states [3]. Many of them are based on physical intuition and they are often difficult to calculate for a given state. In this section, we briefly review some of the important entanglement measures.

Let us first collect some properties of an entanglement measure $E(\rho)$ should possess. Vedral et al. [29] list a set of requirement that every measure of entanglement should satisfy. They are given by

E1. The entanglement measure $E(\rho)$ vanishes, $E(\rho)=0$, if and only if ρ is a separable state.
E2. The entanglement measure $E(\rho)$ is invariant under local unitary operations, $U_{LO}=U_A\otimes U_B$:

$$E(U_{LO}\rho U_{LO}^{\dagger})=E(\rho). \tag{92}$$

E3.1. The entanglement measure $E(\rho)$ should decrease, on average, under probabilistic LOCC,

$$E(\rho)\geq\sum_i p_iE\left(\rho_i\right), \tag{93}$$

where

$$\rho_i=\frac{A_i\otimes B_i\rho A_i^{\dagger}\otimes B_i^{\dagger}}{\mathrm{Tr}\left(A_i\otimes B_i\rho A_i^{\dagger}\otimes B_i^{\dagger}\right)}, \tag{94}$$

and

$$p_i=\mathrm{Tr}\left(A_i\otimes B_i\rho A_i^{\dagger}\otimes B_i^{\dagger}\right). \tag{95}$$

This is called monotonicity under *probabilistic* LOCC operation. The operators A_i and B_i represent the LOCC operations performed by Alice and Bob, respectively.
E3.2. The entanglement measure $E(\rho)$ should, on average, decrease under deterministic LOCC

$$E(\rho)\geq E(\Phi_{LOCC}\rho), \tag{96}$$

where Φ_{LOCC} is defined the LOCC map [14]. This is called monotonicity under *deterministic* LOCC

The first condition **E1** tells us that every entanglement measure must distinguish between entangled and separable states. The second condition **E2** states that if a local unitary transformation is appllied to ρ,

$$\rho' = U_{LO} \rho U_{LO}^{\dagger},$$

the amount of entanglement is invariant under this operation, $E(\rho') = E(\rho)$. The condition **E3.1** requires that after the measurement the entanglement average over the possible output states ρ_i is less than or equal to the original entanglement. Eq. (93) expresses the fact that it is impossible to create or increase entanglement by performing procedures composed of loach quantum operations and classical communication(LOCC) alone. Many authors referred the condition **E3.1** is more strong requirement than **E3.2**. The difference between **E3.1** and **E3.2** is that **E3.1** stipulates that entanglement cannot increase on average under LOCC, while **E3.2** states that entanglement cannot increase for any operation which acts on individual systems and is composed of LOCC [29, 18].

Actually, there are other conditions [3, 14] which an entanglement measure should satisfy. An interesting condition is the convexity of any entanglement measure:

E4: The entanglement measure $E(\rho)$ should be convex in the respect to convex combinations of states,

$$E\left(p\rho + (1-p)\sigma\right) \leq pE\left(\rho\right) + (1-p)E\left(\sigma\right)$$

with $p \in [0, 1]$. This condition guarantees that we cannot increase entanglement by mixing different states. According to Vidal [31], we will call any entanglement measure satisfying **E3.1**, **E3.2**, and **E4** an *entanglement monotone*. In this paper we focus on the three conditions **E1**, **E2**, **E3.2**.

6.1 Entanglement measures for pure states

Let us consider a pure state $\rho_{AB} = \left|\Psi_{AB}\right\rangle\left\langle\Psi_{AB}\right|$. The partial trace $\rho_{A/B} = \mathrm{Tr}_{A/B}\left|\Psi_{AB}\right\rangle\left\langle\Psi_{AB}\right|$[3] is mixed if ρ_{AB} is a entangled state. In addition, if it is a maximally entangled state, its partial trace is also maximally mixed. For a given pure state $\left|\Psi_{AB}\right\rangle$, we can write using the Schmidt decomposition [1];

$$\left|\Psi_{AB}\right\rangle = \sum_i \eta_i \left|a_i b_i\right\rangle, \tag{97}$$

[3] $\rho_{A/B}$ represents the state after performing partial trace either A or B.

where $\eta_i \in \mathbb{R}$ are the *Schmidt coefficients*. You can also use the number of non-zero Schmidt coefficients as a measure. In order to use the η_i's to quantify entanglement, it suggests to use the von Neumann entropy (Shannon entropy) of ρ_{AB}, which measures how much a state is mixed, as an entanglement measure for pure states. We define [24, 30]

$$E_{vN}(\rho_{AB}) = -\mathrm{Tr}\left[\mathrm{Tr}_{A/B}\rho_{AB}\ln\left(\mathrm{Tr}_{A/B}\rho_{AB}\right)\right] = -\sum_i \eta_i \ln \eta_i. \tag{98}$$

It is easy to show that the von Neumann entropy satisfies the following conditions

E2). E_{vN} is invariant under local unitary operation because it is a function of the η_i's only
E5). E_{vN} is additive: $E_{vN}(\rho_{AB} \otimes \sigma_{AB}) = E_{vN}(\rho_{AB}) + E_{vN}(\sigma_{AB})$.
E3.2). Monotonicity under LOCC: From a majorization theorem of Nielsen [32] which relates LOCC operations (for pure states). Consider two states Schmidt decomposition

$$|\Psi_{AB}\rangle = \sum_i \eta_i \, |a_i b_i\rangle \,, \tag{99}$$

and

$$|\Phi_{AB}\rangle = \sum_i \mu_i \, |a_i' b_i'\rangle \,, \tag{100}$$

respectively. The pure state $|\Psi_{AB}\rangle$ can be transformed into the pure state $|\Phi_{AB}\rangle$ if and only if the Schmidt coefficients of $|\Psi_{AB}\rangle$ are majorized by those of $|\Phi_{AB}\rangle$, which we denote by $\eta \prec \mu$. Suppose the Schmidt coefficients are labeled in decreasing order, $\eta_1 \geq \eta_2 \geq ... \geq \eta_N$ and $\mu_1 \geq \mu_2 \geq ... \geq \mu_N$. Then we have

$$\eta \prec \mu \Leftrightarrow \sum_{j=1}^{k} \eta_j \leq \sum_{j=1}^{k} \mu_j \;\;, \quad k = 1, ..., N \;\;. \tag{101}$$

By means of Eq. (101), one can show that the von Neumann entropy does not increase under LOCC operations:

$$E_{vN}\left(|\Psi_{AB}\rangle\langle\Psi_{AB}|\right) = -\sum_{j=1}^{k} \eta_j \,\mathrm{ln}\eta_j \leq E_{vN}\left(|\Phi_{AB}\rangle\langle\Phi_{AB}|\right) = -\sum_{j=1}^{k} \mu_j \ln \mu_j. \tag{102}$$

6.2 Entanglement measures for mixed states

In the previous section, we have shown that the entanglement of pure states can be quantified by using the von Neumann. Unfortunately, this method cannot be applied to mixed states as follows from considering two specific mixed states

$$\rho_E = \left|\Psi^+\right\rangle\left\langle\Psi^+\right| = \frac{1}{4}\left(|01\rangle\langle 01| + |01\rangle\langle 10| + |10\rangle\langle 01| + |10\rangle\langle 10|\right), \tag{103}$$

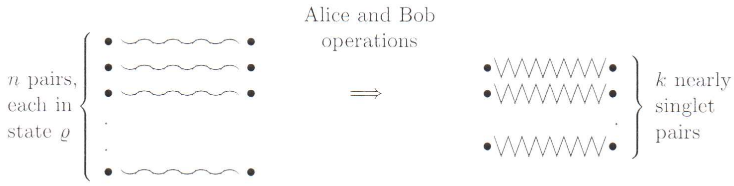

Figure 5: This figure shows the formation of entanglement states with the a certain number of maximally entangled pairs m which is manipulated by local operation and classical communication, LOCC, and converted into non-maximally entangled pairs n. The converse of this process is the entanglement of distillation.

$$\rho_S = \left|\Psi^+\right\rangle \left\langle\Psi^+\right| + \left|\Psi^-\right\rangle \left\langle\Psi^-\right| = \frac{1}{4}\left(|01\rangle \langle 01| + |10\rangle \langle 10|\right). \tag{104}$$

We see that ρ_E is a maximally entanglement state, while ρ_S which is a mixture of two maximally entangled states which is completely separable. Obviously, both have the same von Neumann entropy for the reduced density matrices. Thus, the von Neumann entropy cannot be used to quantify entanglement for mixed states. Many authors have tried to propose entanglement measures for mixed states which we will briefly review in the following sections.

6.2.1 Entanglement of Formation (or Creation) and entanglement cost

The entanglement of formation of the mixed state ρ_{AB} is defined as the average entanglement of the pure states of the decomposition:

$$E_F\left(\rho_{AB}\right) = \min \sum_i p_i S\left(\rho_A^i\right). \tag{105}$$

where $S\left(\rho_A\right) = -\mathrm{Tr}\rho_A \log \rho_A$ is the von Neumann entropy, and the minimum is taken over all the possible realizations of the state

$$\rho_{AB} = \sum_j p_i \left|\Psi_{AB}^i\right\rangle \left\langle\Psi_{AB}^i\right|, \tag{106}$$

and

$$\rho_A^i = \mathrm{Tr}_B(\left|\Psi_{AB}^i\right\rangle \left\langle\Psi_{AB}^i\right|). \tag{107}$$

Eq. (105) tells us about the physical interconvertibility of a collection of pairs in an arbitrary pure state $|\Psi_{AB}\rangle$ and a collection of pairs in the standard singlet states [24]. In addition, we can interpret the physical meaning of this measure as the minimal pure entangled states required to build up the mixed state.

The entanglement of formation is related to the *entanglement cost*, E_C. The physical meaning of this measure is that Alice and Bob would like to create an ensemble of n copies of the nonmaximally entangled states $\rho_{AB}^{\otimes n}$, from a number m of the maximally entangled states[4] $P\left(\left|\Psi^+\right\rangle\right)^{\otimes m}$ by LOCC operations $P\left(\left|\Psi^+\right\rangle\right)^{\otimes m} \rightarrow \rho_{AB}^{\otimes n}$, in the limit as the number of shared pairs goes to infinity,

$$E_C\left(\rho_{AB}\right) = \lim_{n\rightarrow\infty} \frac{E_F\left(\rho_{AB}^{\otimes n}\right)}{n}. \tag{108}$$

Wooters [33] has shown that, for bipartite qubit states, Eq. (105) can be expressed as an explicit function of ρ_{AB} as

$$E_F\left(\rho_{AB}\right) = e\left(C\left(\rho_{AB}\right)\right), \tag{109}$$

where

$$e\left(C\right) = h\left(\frac{1+\sqrt{1-C^2}}{2}\right), \tag{110}$$

$$h\left(x\right) = -x\log x - \left(1-x\right)\log(1-x). \tag{111}$$

The *concurrence* C is defined as

$$C\left(\rho_{AB}\right) = \max\left\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\right\}, \tag{112}$$

where the λ_i are the eigenvalues, in decreasing order, of the Hermitian matrix $R \equiv \sqrt{\sqrt{\rho_{AB}}\tilde{\rho}_{AB}\sqrt{\rho_{AB}}}$ and $\tilde{\rho}_{AB} = \left(\sigma_y \otimes \sigma_y\right)\rho_{AB}^*\left(\sigma_y \otimes \sigma_y\right)$. ρ_{AB}^* being the complex conjugate of ρ_{AB} and σ_y is the y-component of the Pauli's spin matrices.

The variational problem of the entanglement of formation is extramely difficult to solve in general[34, 35, 36, 37]. However, it has been shown that this entanglement measure satisfies the requirements **E1**, **E2**, **E3**[24].

The explicit form of the concurrence can be found if, for example, we consider the Werner states,

$$\rho_{\Phi^\pm}(x) = x\left|\Phi^\pm\right\rangle\left\langle\Phi^\pm\right| + (1-x)\frac{I}{4}, \tag{113}$$

$$\rho_{\Psi^\pm}(x) = x\left|\Psi^\pm\right\rangle\left\langle\Psi^\pm\right| + (1-x)\frac{I}{4}, \tag{114}$$

where the maximally entangled states are defined

$$\left|\Phi^\pm\right\rangle = \frac{1}{\sqrt{2}}\left(\left|00\right\rangle \pm \left|11\right\rangle\right), \tag{115}$$

$$\left|\Psi^\pm\right\rangle = \frac{1}{\sqrt{2}}\left(\left|01\right\rangle \pm \left|01\right\rangle\right). \tag{116}$$

The concurrence is calculated to be given by [14]

$$C\left(\rho_{\Phi^\pm}(x)\right) = C\left(\rho_{\Psi^\pm}(x)\right) = \begin{cases} (3x-1)/2 & , \quad x \geq 1/3 \\ 0 & , \quad x < 1/3 \end{cases}. \tag{117}$$

Only the maximally entangled states, $x = 1$, is the concurrence equal to one.

[4] $P\left(\left|\Psi^+\right\rangle\right) \equiv \left|\Psi^+\right\rangle\left\langle\Psi^+\right|$ is the projection operator of the Bell state $\left|\Psi^+\right\rangle$[33].

6.2.2 Entanglement of distillation (purification)

The amount of enatanglement of a state ρ_{AB} has been defined by the asymtotic proportion of singlets that can be distilled using a LOCC-operation. This is the opposite process of the entanglement formation (see Fig. 5).

The functional associated with this entanglement measure is defined as the *maximum* fraction of the maximally entangled states that can be extracted from n copies of ρ_{AB} by the LOCC operations $\rho_{AB}^{\otimes n} \to P\left(|\Psi^{+}\rangle\right)^{\otimes m}$, in the asmytotic limit as $n \to \infty$,

$$E_D\left(\rho_{AB}\right) = \lim_{n\to\infty}\left[\sup\left(\frac{m}{n}\right)\right]. \tag{118}$$

Plenio et al., [3] have proposed another expression of the entanglement of distillation,

$$E_D\left(\rho_{AB}\right) = \sup\left\{r : \lim_{n\to\infty}\left[\inf_{\Psi} \mathrm{Tr}\left|\Psi\left(\rho_{AB}^{\otimes n}\right) - \Phi\left(2^{rn}\right)\right|\right] = 0\right\}. \tag{119}$$

This quantity is sometimes called the *free entanglement* because it can be viewed as analogous to the free energy in thermodynamics [15]. Distillable entanglement is a measure of a fundamental importance, since it tells us how much entanglement one may extract out of a given state. Nevertheless, the value obtained is generally small. This means that the formation of states is irreversible, in the sense that more pure entanglement state cannot be distilled from PPT states than may be have been used to assist in thier creation [38]. These states are called *bound entangled states*.

According to condition **E3.2**, it must be the case that [16, 39]

$$E_D\left(\rho_{AB}\right) \leq E_F\left(\rho_{AB}\right). \tag{120}$$

which reflects the irreversible character of state mixing. It is also natural to consider the difference $B\left(\rho_{AB}\right) = E_F\left(\rho_{AB}\right) - E_D\left(\rho_{AB}\right)$, between the entanglement of formation and distillation, known as the bound entanglement.

The computation of the entanglement of distillation is difficult; it is known however, that for pure states $E_D\left(\rho_{AB}\right)$ is equivalent to the entropy of entanglement [41, 40].

6.2.3 Relative entropy of entanglement

This measure is based on the geometric measure. The measure shows us that the amount of entanglement in ρ_{AB} is its distance from the closet disentangled states σ_{AB} [41, 42] (see Fig. 6),

$$E_R\left(\rho_{AB}\right) = \min_{\sigma_{AB}\in SP} S\left(\rho_{AB}\left\|\sigma_{AB}\right.\right), \tag{121}$$

where $S\left(\rho_{AB}\left\|\sigma_{AB}\right.\right)$ is the quantum relative entropy which is defined by

$$S\left(\rho\left\|\sigma\right.\right) = \mathrm{Tr}\left\{\rho\log\rho - \rho\log\sigma\right\}. \tag{122}$$

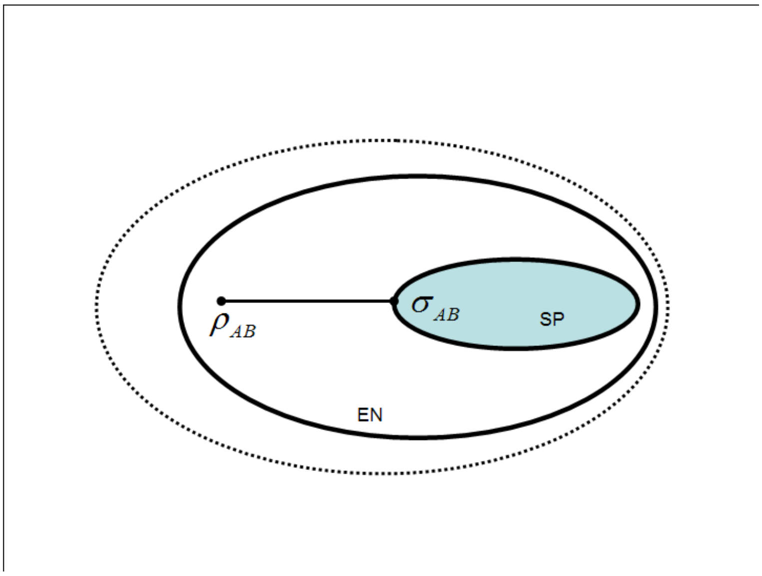

Figure 6: This diagram shows the structure of the relative entropy of entanglement which is based on distance geometry. The distance represents the amount of entanglement for a given state.

Audenaert et al., [43] have shown that the relative entropy can be obtained by using ideas of semi-definite programming and optimization theory [44], resulting in

$$E_R\left(\rho_{AB}\left(x\right)\right) = \begin{cases} 1-h(x) & , \quad x < \frac{d+2}{2d} \\ x\log\frac{d+2}{d} + (1-x)\log\frac{d-2}{d+2} & , \quad x > \frac{d+2}{2d} \end{cases}. \tag{123}$$

where $h(x)$ is defined in Eq. (111) and $\rho_{AB}(x)$ is the Werner states,

$$\rho_{AB}\left(x\right) = x\sigma_a + (1-x)\sigma_s, \tag{124}$$

and $\sigma_a(\sigma_s)$ are states proportional to the projectors onto anti-symmetric subspace. Analytical results for $E_R\left(\rho_{AB}\right)$ has been found in certain cases [29, 35]. Furthermore, it has been shown that $E_R\left(\rho_{AB}\left(x\right)\right)$ satisfies the conditions **E1-E3.2** [45, 41].

6.2.4 Robustness of entanglement

This entanglement measure was proposed by Vidal and Tarrach [46]. It shows that the endurance of entanglement by quantifying the minimal amount of mixing with separable states needs to wipe out the entanglement. For a entangled state ρ_{AB}, there always exist separable states ρ_{sep} for which the robustness $R(\rho_{AB} \|$

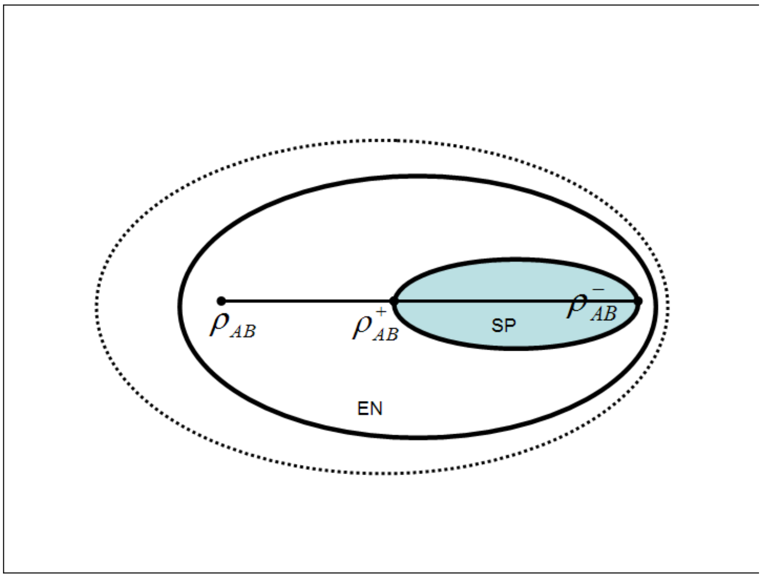

Figure 7: This diagram shows the structure of the robustness of entanglement. The entanglement state ρ_{AB} can be composed of two separable states ρ^{+}_{AB} and ρ^{-}_{AB}.

ρ_{sep}) is given as the minimal amount of t such that

$$\frac{1}{1+t}\left(\rho_{AB}+t\rho_{sep}\right) \tag{125}$$

is separable. According to a varity of states ρ_{sep}, robustness of entanglement is defined

$$R\left(\rho_{AB}\right)=\inf_{\rho_{sep}} R\left(\rho_{AB}\left|\rho_{sep}\right.\right). \tag{126}$$

Note that $R\left(\rho_{AB}\right)$ is zero if and only if ρ_{AB} is separable. Therefore, by using local pseudomixture theorem [47], we can also rewrite the state ρ_{AB} as

$$\rho_{AB}=(1+s)\rho^{+}_{AB}-s\rho^{-}_{AB}\ ,\quad 0\leq s\leq\infty\ , \tag{127}$$

or

$$\rho_{AB}=\sum_{k=1}^{l<\infty} r_k\left|\Psi_k\right\rangle\left\langle\Psi_k\right|. \tag{128}$$

where $\sum_{k=1}^{l<\infty} r_k=1$ and $r_k\in\mathbb{R}$. Then robustness can be expressed as

$$R\left(\rho_{AB}\right)=\min_{\rho^{+}_{AB}\in SP}\left\{\min_{a}:\rho^{+}_{AB}=a\rho_{AB}+(1-a)\rho^{-}_{AB}\in SP\right\}. \tag{129}$$

The meaning of Eq. (129) is that $R\left(\rho_{AB}\right)$ measures the minimal amount of the state ρ^{+}_{AB} that must be mixed with the state ρ^{-}_{AB} to make ρ_{AB} separable. As shown in Fig. 7, we can also interpret this measure as the minimal ratio of the distance $1-a$ of ρ_{AB} from the set SP of separable states with the width a of this set,

$$R = \frac{1-a}{a}. \tag{130}$$

It has been proved that the robustness is be convex and monotone [46].

The explicit form of robustness can be found for pure states [46]. Consider a pure state with Hilbert space $\mathbb{C}^3 \otimes \mathbb{C}^3$,

$$|\Psi_i\rangle = \sum_{i=1}^{m} a_i |i\rangle \otimes |i\rangle \;\; ; \;\; a_i \geq a_{i+1} \geq 0 \;,\;\; \sum_{i=1}^{m} a_i^2 \quad = 1, \tag{131}$$

where a_i are Schmidt coefficients. its robustness is then given by

$$R\left(|\Psi_i\rangle\langle\Psi_i|\right) = \left(\sum_{i=1}^{m} a_i\right)^2 - 1. \tag{132}$$

For a two qubit system, the robustness can be expressed as [46]

$$\frac{|\lambda|}{\cos^2\theta} \leq R\left(\rho_{AB}\right) \leq 2\left|\lambda\right|, \tag{133}$$

where λ is the negative eigenvalue of ρ_{AB}^{PT} and $|\theta\rangle$ is the associated eigenvector,

$$|\theta\rangle = \cos\theta\,|1\rangle \otimes |1\rangle + \sin\theta\,|2\rangle \otimes |2\rangle \;\; ; \;\; \theta \in [0, \pi/4] \;. \tag{134}$$

In reference [46], the state $\rho = p\rho_D + (1-p)\,|\theta\rangle\langle\theta|$ is considered where

$$\rho_D = \begin{pmatrix} q_1 & 0 & 0 & 0 \\ 0 & q_2/2 & 0 & 0 \\ 0 & 0 & q_2/2 & 0 \\ 0 & 0 & 0 & q_3 \end{pmatrix} \;,\;\; q_i \geq 0 \;,\;\; \sum_{i}^{3} q_i \quad = 1. \tag{135}$$

The robustness of ρ_D is found to be

$$R(\rho_D) = \left\{ \begin{array}{cc} 0 & ,\rho^{T_B} \geq 0 \\ (1-p)\sin 2\theta - pq_2 & ,\text{otherwise} \end{array} \right. . \tag{136}$$

If we set $q_1 = q_3 + q_2/2 = 1/4$, $\theta = \pi/4$ and $p = 4(1-F)/3$, we find that

$$R(\rho) = 2F - 1, \tag{137}$$

where F is the fidelity [48] of a Werner state.

For a two-qubit state diagonal in the Bell basis, the explicit from of the robustness has been calculated by Akhtarshenas et al. [49].

6.2.5 Logarithmic Negativity

We know that the density matrix ρ_{AB} of a bipartite system can be represented conveniently by using the computational basis, Eq. (76). Its partial transposition with respect to party A and B of a state ρ_{AB} are defined in Eqs. (78) and (79). The positivity of the partial transpose of a state is a necessary condition for separability, and is sufficient to prove that $N(\rho_{AB}) = 0$. Negativity essentially measures the degree to which ρ_{AB}^{PT} fails to be positive. From Eqs. (78) and (79), we can construct two useful quantities. The first one is the *Negativity*

$$N\left(\rho_{AB}\right) = \frac{\left\|\rho_{AB}^{PT}\right\| - 1}{2}, \tag{138}$$

where $\|X\| = \mathrm{Tr}\sqrt{X^{\dagger}X}$ is the trace norm. The negativity sufficient the deficiency that it is not additive while being a conserve entanglement monotone [50, 51]. A more suitable quantity for an entanglement monotone may therefore be the *logarithmic negativity* which is defined as

$$E_N\left(\rho_{AB}\right) = \log_2\left\|\rho_{AB}^{PT}\right\|. \tag{139}$$

E_N is an entanglement monotone that cannot increase under LOCC-operations because of the monotone property of the negativity.

The explicit form of E_N can be found if we consider the Werner states [14]. Recalling the states in Eqs. (113) and (114), we find that the negativities are

$$N\left(\rho_{\Phi^{\pm}}(x)\right) = N\left(\rho_{\Psi^{\pm}}(x)\right) = \left\{ \begin{array}{ccc} (3x-1)/2 & , & x \geq 1/3 \\ 0 & , & x < 1/3 \end{array} \right. . \tag{140}$$

Vidal and Werner [51] have shown the monotonicity of $N(\rho_{AB})$ under LOCC. They consider a family $\mathbb{M}_i$ of completely positive linear maps such that $\mathbb{M}_i(\rho) = p_i\rho_{AB}^i$ and these maps also satisfy the normalization condition $\sum_i \mathrm{Tr}\left[\mathbb{M}_i(\rho)\right] = \mathrm{Tr}(\rho)$. According to the Choi-Kraus representation, we can write the map $\mathbb{M}_i(\rho)$ which is taking into account a local measurement by Bob as

$$\mathbb{M}_i(\rho) = (I_A \otimes M_i)\rho(I_A \otimes M_i^{\dagger}). \tag{141}$$

where M_i are the Kraus operators which must satisfy the normalization condition $\sum_i M_i^{\dagger}M_i \leq I_B$. Consider the combination

$$\rho_{AB}^{PT} = (1+N)\rho^{+} - N\rho^{-},$$

where $\rho^{\pm}$ are density matrices with $\mathrm{Tr}(\rho^{\pm}) = 1$ and $N = N(\rho_{AB})$. Then we can show that

$$p_i(\rho_{AB}^i)^{PT} = \mathbb{M}_i(\rho_{AB}^i)^{PT} = \mathbb{M}_i(\rho_{AB}^{iPT}) = (1+N)\mathbb{M}_i(\rho^{+}) - N\mathbb{M}_i(\rho^{-}).$$

Dividing by p_i, we have a decomposition of the form

$$(\rho^i_{AB})^{PT} = \frac{(1+N)}{p_i}\mathbb{M}_i(\rho^+) - \frac{N}{p_i}\mathbb{M}_i(\rho^-).$$

According to the variational form of the negativity [51], we can define the negativity of (ρ^i_{AB}) as follows

$$N(\rho^i_{AB}) = \inf \left\{ a_- \left| (\rho^i_{AB})^{PT} = a_+\mathbb{M}_i(\rho^+) - a_-\mathbb{M}_i(\rho^-) \right. \right\}. \tag{142}$$

where $a_+ = (1+N)/p_i$ and $a_- = N/p_i$. This variation shows that the codfficient $a_- = N/p_i$ must be larger than the infimum. i.e., $N(\rho^i_{AB}) \leq N/p_i$. Multiplying by p_i and summing, we obtain

$$\sum_i p_i N(\rho^i_{AB}) \leq N(\rho_{AB}). \tag{143}$$

This inequality shows that the negativity is indeed an entanglement monotone.

6.2.6 Best separable approximation: BSA

Lewenstein and Sunpera [52, 53] have proposed the decomposition of a statistical operator ρ_{AB} (in $\mathbb{C}^2 \otimes \mathbb{C}^2$)

$$\rho_{AB} = \Lambda\rho_{sep} + (1-\Lambda)\, P\left(|\Psi_{ent}\rangle\right), \tag{144}$$

with $\Lambda \in [0,1]$ is maximal, where ρ_{sep} is separable and $P\left(|\Psi_{ent}\rangle\right) \equiv |\Psi_{ent}\rangle\langle\Psi_{ent}|$ is the projector for a fully entangled state. This expression exists for any two-qubit state; however, the decomposition is not unique. We have to search for an optimal value Λ_{max} which is sometimes referred to as the degree of *separability* and also can be viewed as the degree of *classically* of the state [54].

The idea of BSA is that, because of the fact that the set of separable states is compact, for any density matrix ρ_{AB} there exist an *optimal* separable matrix ρ_{sep} and *optimal* $\Lambda \geq 0$ such that $\Lambda\rho_{sep}$ can be subtracted from ρ_{AB} maintaining the positivity of the difference, $\delta\rho_{AB} = \rho_{AB} - \Lambda\rho_{sep} \geq 0$. According to the decomposition Eq. (144), this leads straightforwardly to an unambiguous measure of the entanglement for any mixed state ρ_{AB}

$$E_{BSA}(\rho_{AB}) = (1-\Lambda)\, E_{vN}\left(|\Psi_{ent}\rangle\right), \tag{145}$$

where $E_{vN}\left(|\Psi_{ent}\rangle\right)$ is the von Neumann entropy of the reduced density matrix [55] as defined in Eq. (98). For a particular state [52] ρ_W is defined as

$$\rho_W\left(x\right) = \frac{1}{4}\begin{pmatrix} 1-x & 0 & 0 & 0 \\ 0 & 1+x & -2x & 0 \\ 0 & -2x & 1+x & 0 \\ 0 & 0 & 0 & 1-x \end{pmatrix}, \tag{146}$$

with $x \in [0,1]$. The above state can always be decomposed as

$$\rho_W(x) = \Lambda(x) I + (1 - \Lambda(x)) \left|\Psi^-\right\rangle \left\langle \Psi^-\right|, \tag{147}$$

where $|\Psi^-\rangle = 1/2(|01\rangle - |10\rangle)$. For $x \leq 1/3$, $\Lambda = 1$, and the state $\rho_W = I$ is separable. For $x > 1/3$, $0 \leq \Lambda < 1$, and a measure of the entanglement of ρ_W is thus provided by the value of the corresponding Λ,

$$E_{BSA}(\rho_W(x)) = 1 - \Lambda(x). \tag{148}$$

Karnas and Lewenstein [53] have shown that the BSA-entanglement measure does not increase under LOCC operations,

$$E_{BSA}(\rho_{AB}) \geq \sum_i p_i E_{BSA}(\rho^i_{AB}), \tag{149}$$

where $V_i = A_i \otimes B_i$ are local POVM operators satisfying $\sum\limits_i V_i V_i^\dagger = 1$ and $p_i = \mathrm{Tr}\left(V_i \rho_{AB} V_i^\dagger\right)$. This inequality shows that BSA is entanglement monotone.

7 The alternative method of entanglement measure: PT-entanglement measure

7.1 Introducing criterion

In this contribution, we propose a new entanglement measure which is based on PPT-criterion and simple to define.

Consider composite systems of dimensions $2\otimes 2$ or $2\otimes 3$. Let us form a convex combination of the given density matrix ρ_{AB} with its partial transpose ρ_{AB}^{PT},

$$\rho_{AB}(\lambda) = (1 - \lambda)\rho_{AB} + \lambda \rho_{AB}^{PT} \ ; \qquad \lambda \in [0,1] \ , \tag{150}$$

which interpolates between a density matrix and a matrix with at least one negative eigenvalue if ρ_{AB} is entangled.

Definition: *We propose a candidate of a measure of entanglement given by*

$$E_{PT}(\rho_{AB}) = 1 - \lambda_c \tag{151}$$

where

$$\lambda_c = \max_{\lambda \in [0,1]} \left\{ E_k\left((1-\lambda)\rho_{AB} + \lambda \rho_{AB}^{PT}\right) \geq 0 \ , \quad k = 1...4(6) \ \right\}, \tag{152}$$

is the ***critical value*** *of λ: for $\lambda_c + \varepsilon$ with any positive ε at least one of the eigenvalues $E_k(\rho_{AB}(\lambda))$ becomes negative.*

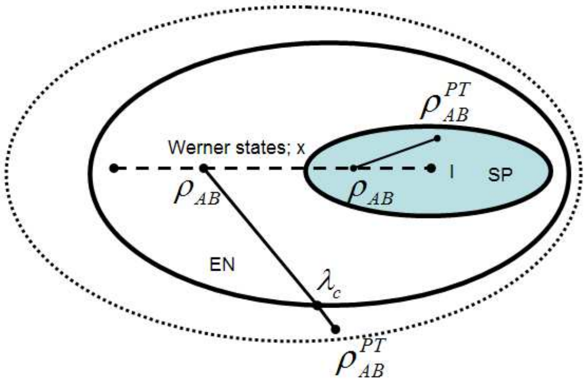

Figure 8: This diagram shows the structure of PT-measure. See the details in the text.

The idea of this approach is to quantify how much of the state ρ_{AB}^{PT} need to be mixed with the state ρ_{AB} in order to have $\rho_{AB}(\lambda)$ acquire a negative eigenvalue. The value of λ_c then relates to the amount of entanglement for a give state ρ_{AB}.

As shown in Fig. 8, if the initial state ρ_{AB} is separable, its partial transpose is also separable. Then the new state $\rho_{AB}(\lambda)$ is the convex combination of the separable states and also lives in the region of separable states (SP). On the other hand, if the initial state ρ_{AB} is entangled, its partial transpose is not a density matrix hence it lives both outside EN and SP-region. In this case, the operator $\rho_{AB}(\lambda)$ will have both positive and negative eigenvalues. Our quest is to search the critical value λ_c, beyond which at least one of the eigenvalues of $\rho_{AB}(\lambda)$ becomes negative.

We know that the maximally entangled states lie on the boundary of the area of entangle states. We will always find one maximally entangled state from our entanglement measure (Fig. 8). If we set $\lambda = \lambda_c$, we obtain

$$\rho_{MAX}(\lambda_c) \equiv (1-\lambda_c)\,\rho_{AB} + \lambda_c \rho_{AB}^{PT}. \tag{153}$$

By using above statement, our measure quantifies the amount of entanglement for a given state by using the variable λ as a measure, saying how far of the state ρ_{AB} from the maximally entangled state which has been created by composting it and its partial transpose.

In order to calculate the amount of entanglement, we have to find the critical value of λ for which $\rho_{AB}(\lambda)$ acquires a negative eigenvalue. We provide some examples of this approach in this section and we will investigate in how far this candidate for a measure of entanglement also satisfies some conditions **E1**, **E2**, **E3** in later sections.

7.2 Pure States

Consider a pure state

$$|\Psi_{AB}\rangle = \alpha\,|1_A\rangle \otimes |1_B\rangle + \beta\,|2_A\rangle \otimes |2_B\rangle\,, \tag{154}$$

where α and β are real numbers which is entangled if both α and β are not zero. We can express the density matrix as

$$\rho_{AB} = \alpha^2\,|1_A 1_B\rangle\,\langle 1_A 1_B| + \beta^2\,|2_A 2_B\rangle\,\langle 2_A 2_B| + \alpha\beta\,(|1_A 1_B\rangle\,\langle 2_A 2_B| + |2_A 2_B\rangle\,\langle 1_A 1_B|)\,. \tag{155}$$

The eigenvalues of the state ρ_{AB} are

$$\nu_1 = \nu_2 = \nu_3 = 0\ ,\quad \nu_4 = 1\ ,$$

since $\rho_{AB} \equiv |\Psi_{AB}\rangle\,\langle\Psi_{AB}|$ is a projective on a pure state. The partial transpose can be found as

$$\rho_{AB}^{PT} = \alpha^2\,|1_A 1_B\rangle\,\langle 1_A 1_B| + \beta^2\,|2_A 2_B\rangle\,\langle 2_A 2_B| + \alpha\beta\,(|1_A 2_B\rangle\,\langle 2_A 1_B| + |2_A 1_B\rangle\,\langle 1_A 2_B|)\,. \tag{156}$$

Its eigenvalues are

$$\mu_1 = \alpha^2\ ,\quad \mu_2 = \beta^2\quad ,\quad \mu_3 = \alpha\beta\quad ,\quad \mu_4 = -\alpha\beta\ .$$

We find that the operator ρ_{AB}^{PT} always have negative eigenvalues μ_3 or μ_4 as long as α and β are not zero. Next, we form a convex combination between the original density matrix and its partial transpose

$$\rho_{AB}\,(\lambda) = (1-\lambda)\rho_{AB} + \lambda\rho_{AB}^{PT}\ ,\quad 0 \le \lambda \le 1\ . \tag{157}$$

In order to find the eigenvalues of this state, we have to obtain the characteristic equation as

$$\det\left(\rho_{AB}\,(\lambda) - EI\right) = \left(E^2 - E + \alpha^2\beta^2\left(2\lambda - \lambda^2\right)\right)(E + \lambda\alpha\beta)\,(E - \lambda\alpha\beta) = 0$$

where E is a real parameter. We see that there is a pair of eigenvalues $\pm\lambda\alpha\beta$ always producing a the negative sign for λ as small as desired. Thus, the critical value of λ is zero and we end up with $E_{PT}(\rho_{AB}^{ent}) = 1 - \lambda_c = 1 - 0 = 1$, otherwise, the entanglement measure is zero.

Obviously, the measure introduced have does not distinguish between degree of entanglement for pure states as does the von Neumann entropy, for example. We just know that if the pure state is separable, the entanglement measure is zero. On the other hand, if the pure state is entangled, the entanglement measure is one.

7.3 Mixed States

7.3.1 Werner states

In this section, we provide two examples for our entanglement measure in space $\mathbb{C}^2 \otimes \mathbb{C}^2$ and $\mathbb{C}^2 \otimes \mathbb{C}^3$, respectively.

Example 3.1. From Example 1.1, we can show that the state $\rho_{\Phi^+}(\lambda; x)$ is

$$\begin{aligned} \rho_{\Phi^+}(\lambda; x) &= (1-\lambda)\rho_{\Phi^+}(x) + \lambda \rho_{\Phi^+}^{PT}(x) \\ &= \frac{1}{4}\begin{pmatrix} 1+x & 0 & 0 & 2x\,(1-\lambda) \\ 0 & 1-x & 2x\lambda & 0 \\ 0 & 2x\lambda & 1-x & 0 \\ 2x\,(1-\lambda) & 0 & 0 & 1+x \end{pmatrix}. \end{aligned} \tag{158}$$

The eigenvalues are

$$\begin{gathered} E^1_{\Phi^+}(\lambda; x) = E^2_{\Phi^+}(\lambda; x) = (1 - x + 2x\lambda)/4, \\ E^3_{\Phi^+}(\lambda; x) = (1 - x - 2x\lambda)/4, \\ E^4_{\Phi^+}(\lambda; x) = (1 + 3x - 2x\lambda)/4. \end{gathered}$$

If we set $x = 1/2$, the state $\rho_{\Phi^+}(\lambda; x = 1/2)$ is entangled. We can plot the graph of the eigenvalues as in Fig. 9 and the third eigenvalue gives a negative sign if $\lambda > 1/2$. For $x > x_c$, we can see that the third eigenvalue is positive as long as λ satisfies the inequality

$$\lambda \leq \frac{1-x}{2x}$$

illustrated in Fig. 10. We find that

1. If $0 \leq x \leq 1/3$, the Werner state $\rho_{\Phi^+}(x)$ is separable. Then the state $\rho_{\Phi^+}(\lambda; x)$ always gives positive eigenvalues. We see that the quantity $(1 - x)/2x > 1$, except for $x = 0$, but we know that $0 \leq \lambda \leq 1$. Then we may say $\lambda_c = 1$ for separable states.

2. If $1/3 < x \leq 1$, the Werner state $\rho_{\Phi^+}(x)$ is entangled. The values of λ satisfying the inequality

$$\lambda > \frac{1-x}{2x}$$

produce a negative sign for the eigenvalue $E^3_{\Phi^+}(\lambda; x)$. We see that the upper part of curve, see Fig. 10, will give the negative sign while the lower part

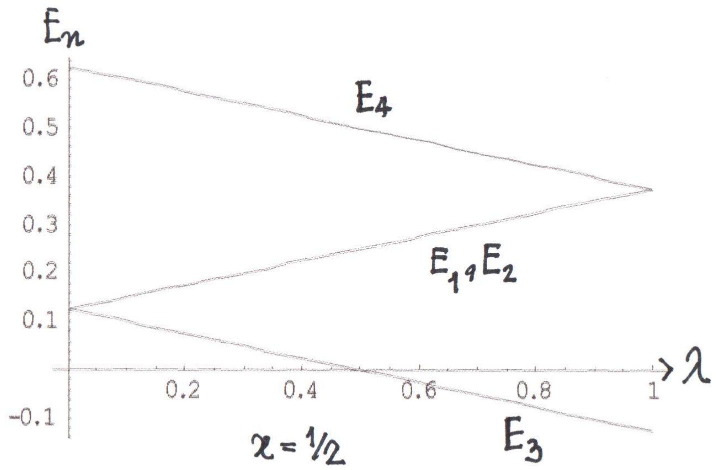

Figure 9: This diagram shows the characteristic of the eigenvalues of the state $\rho_{\Phi^+}(\lambda; x)$.

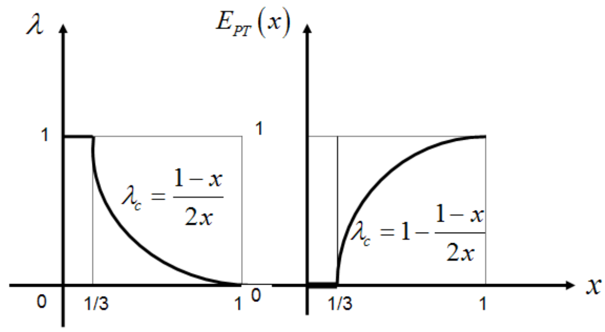

Figure 10: The diagrams show the characteristic of the $\lambda - x$ and $E_{PT}(x)$ curves of the state $\rho_{\Phi^+}(\lambda; x)$.

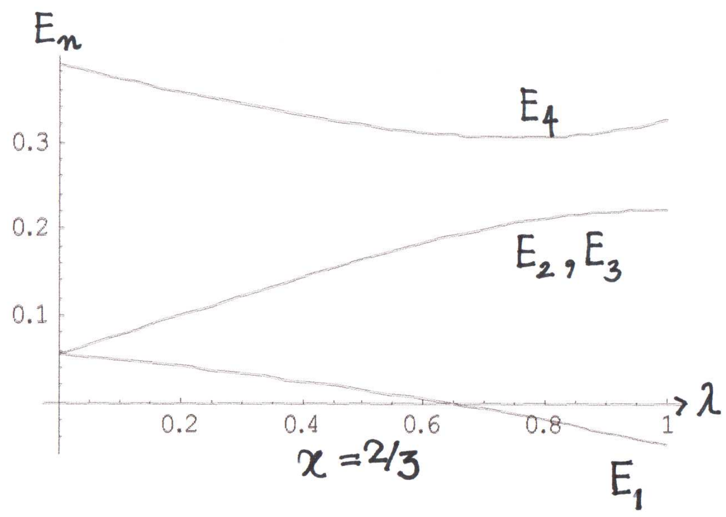

Figure 11: This diagram shows the characteristic of the eigenvalues of the state $\rho_{2\otimes 3}(\lambda; x)$.

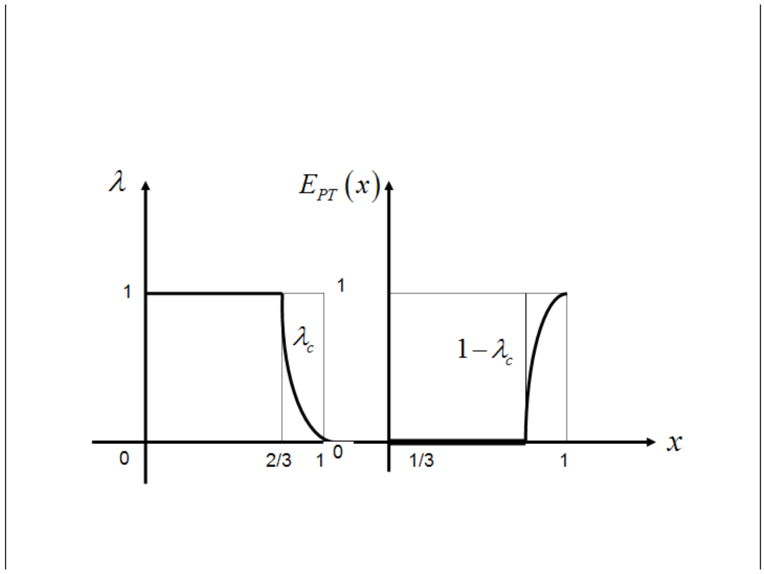

Figure 12: The diagrams show the characteristic of the $\lambda - x$ and $E_{PT}(x)$ curves of the state $\rho_{2\otimes 3}(\lambda; x)$ where $\lambda_c(x) = -\frac{2}{3}\left(\frac{3-3x+(x-1)\sqrt{8+14x+5x^2}}{x-4}\right)$.

will produce the positive sign. Thus, we can define the critical value of λ as

$$\lambda_c(x) = \frac{1-x}{2x}.$$

Example 3.2. For the state of Eq. (88), we have

$$\rho_{2\otimes 3}(\lambda; x) = \begin{pmatrix} \frac{1}{6}+\frac{x}{12} & 0 & 0 & 0 & \frac{x}{4}(1-\lambda) & 0 \\ 0 & \frac{1}{6}+\frac{x}{12} & 0 & \frac{x}{4}\lambda & 0 & -\frac{x}{4}(1-\lambda) \\ 0 & 0 & \frac{1-x}{6} & 0 & -\frac{x}{4}\lambda & 0 \\ 0 & \frac{x}{4}\lambda & 0 & \frac{1-x}{6} & 0 & 0 \\ \frac{x}{4}(1-\lambda) & 0 & -\frac{x}{4}\lambda & 0 & \frac{1}{6}+\frac{x}{12} & 0 \\ 0 & -\frac{x}{4}(1-\lambda) & 0 & 0 & 0 & \frac{1}{6}+\frac{x}{12} \end{pmatrix}. \tag{159}$$

For above matrix, we cannot find the analytical expression for the eigenvalues. we consider the value $x = 2/3$ and we can plot the graph of the eigenvalues as in Fig. 11 and the first eigenvalue produces the negative sign. We can conclude that, for $\frac{4}{1+3\sqrt{5}} < x \leq 1$, the first eigenvalue is positive as long as satisfies the inequality

$$\lambda < -\frac{2}{3}\left(\frac{3-3x+(x-1)\sqrt{8+14x+5x^2}}{x-4}\right).$$

We can show the relation between x and λ as Fig. 12 and we can defined the critical value of λ as

$$\lambda_c(x) = -\frac{2}{3}\left(\frac{3-3x+(x-1)\sqrt{8+14x+5x^2}}{x-4}\right).$$

7.3.2 The characteristics of $\lambda - x$ curve of the Werner state.

From above examples, we can conclude that there is a curve, denoted by $\lambda_c = F(x)$, that separates the area between positive eigenvalues only and a negative eigenvalue. The behavior of the curve can be described in two situations (see Fig. 13).

1. If $0 \leq x \leq x_c$, A1, the Werner state $\rho_W(x)$ is separable. The partial transpose operator maps $\rho_W(x)$ to another separable state $\rho_W^{PT}(x)$ which also lives inside the separable area (see Fig. 5). Then the state $\rho_W(\lambda; x) = (1-\lambda)\rho_W(x) + \lambda\rho_W^{PT}(x)$ has positive eigenvalues only and is a separable state.

2. If $x_c < x \leq 1$, A2 and A3, the Werner state $\rho_W(x)$ is entangled and its partial transpose matrix lives in the area of the non-density matrices (see

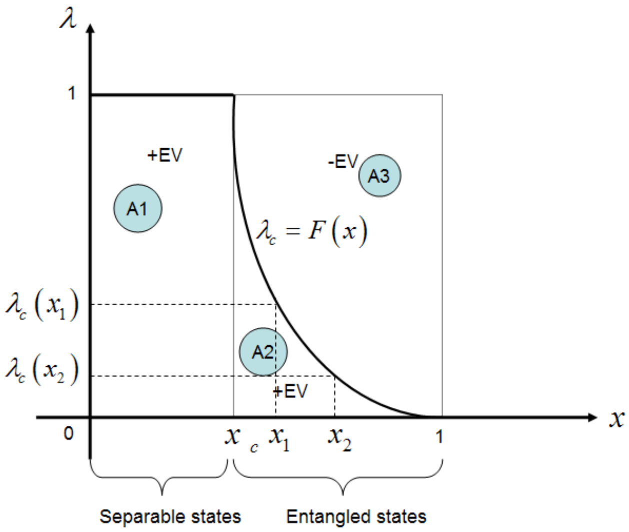

Figure 13: This figure shows the generalize characteristics of $\lambda - x$ curve of the Werner states.

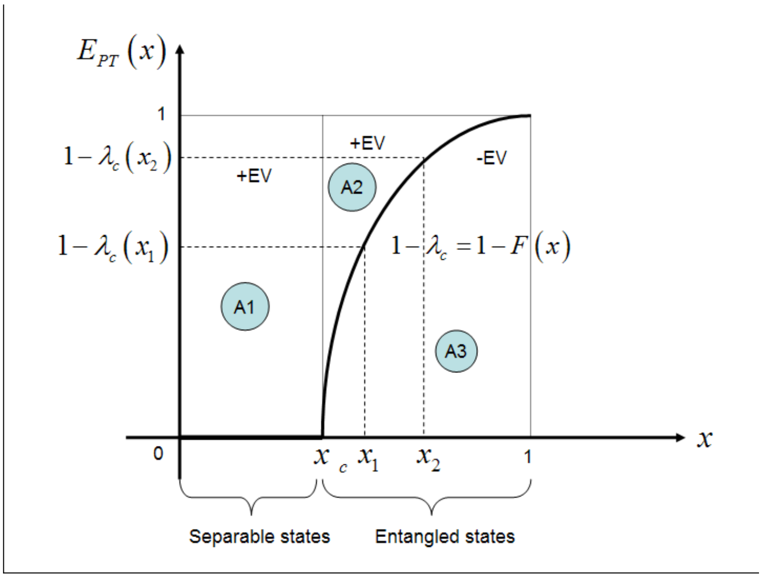

Figure 14: This figure shows the generalize characteristics of $E_{PT}(x)$ curve of the Werner states.

Fig. 1). The state $\rho_W(\lambda; x) = (1-\lambda)\rho_W(x) + \lambda\rho_W^{PT}(x)$ is a density matrix as long as λ satisfying the inequality

$$\lambda(x) \leq F(x) \ .$$

On the other hand, it is not a density matrix. Then we can define the critical value of λ as

$$\lambda_c(x) = F(x).$$

In this case, the states that lie on the curve is the maximally entangled states $\rho_{MAX}(\lambda_c; x) = (1-\lambda_c)\rho_{AB}(x)+\lambda\rho_{AB}(x)$. These maximally entangled states are probably not Bell states. That means this curve represents the boundary between the entangled states and all other states which are not density matrices (see also Fig. 8).

Consider Werner states $\rho_{AB}(x_1)$ and $\rho_{AB}(x_2)$ with x_1 and x_2 satisfying the inequality

$$x_c < x_1 < x_2 < 1.$$

Then we find that

$$1 = \lambda_c(x_c) > \lambda_c(x_1) > \lambda_c(x_2) > \lambda_c(1) = 0.$$

or

$$0 = 1 - \lambda_c(x_c) < 1 - \lambda_c(x_1) < 1 - \lambda_c(x_2) < 1 - \lambda_c(1) = 1.$$

These inequalities say that the state $\rho_{AB}(x_2)$ contains more entanglement than the state $\rho_{AB}(x_1)$ because $1 - \lambda_c(x_1) < 1 - \lambda_c(x_2)$ (see Fig. 14). This implies that the state $\rho_{AB}(x_2)$ is located nearer the boundary of maximally entangled states than the state $\rho_{AB}(x_1)$. In addition, we can say that the eigenvalues of the state $\rho_{AB}(x_2)$ fail to be positive *easier* than those of the state $\rho_{AB}(x_1)$ when mixing the negative eigenvalues of their corresponding partial transpose matrices. For the maximally entangled state $|\Phi^+\rangle\langle\Phi^+|$, $x = 1$, the critical value of λ is 0 because it lives on the boundary. Then at least one of its eigenvalues is always negative.

For any entangled states ρ_{AB}, we can use λ_c to classify the family of the entangled states (see Fig. 15). The states that produce the same values of λ_c can be grouped represented by the dash line, in one class because they attributed the same amount of entanglement. In the next section, we will give you a particular set of the states that produce the same critical value of λ.

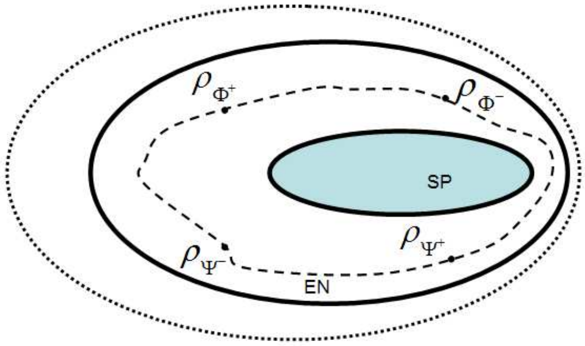

Figure 15: This figure shows the group of the states that contain the same amount of entanglement.

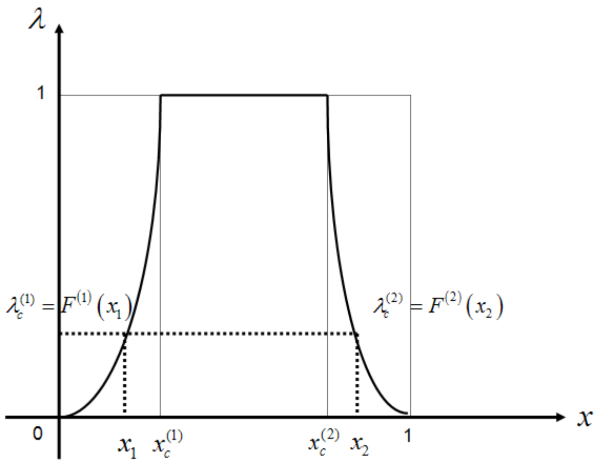

Figure 16: This figure shows over all trend of the critical value of λ for Bell diagonal states.

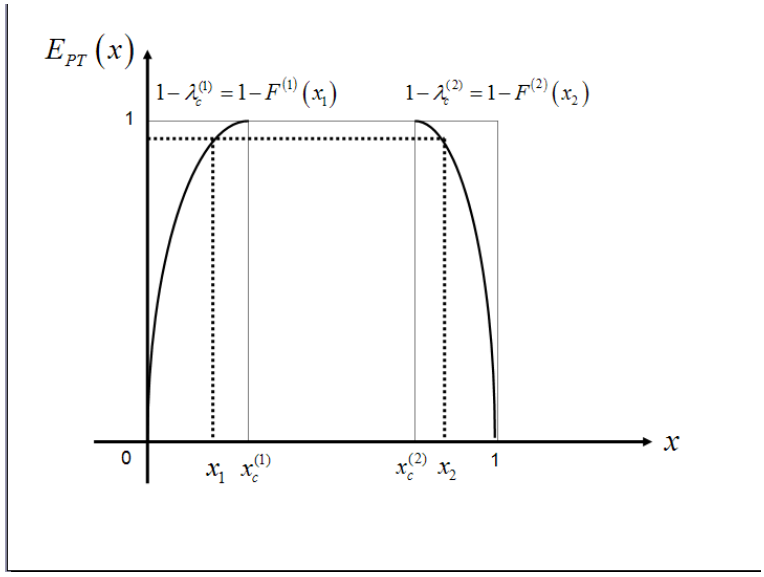

Figure 17: This figure shows over all trend of E_{PT} for Bell diagonal states.

7.3.3 The characteristics of the $\lambda - x$ curve for Bell diagonal states

From the examples studied in Appendix A, we can extract the following features of the $\lambda - x$ curve:

1. Symmetric case: Consider two Bell states δ_{BD} and σ_{BD} which may be transformed into each other by a local unitary transformation. When we form the convex combination $\rho_{BD}(x) = x\delta_{BD} + (1-x)\sigma_{BD}$, we find that the typically has the form shown in Fig. 16. For some values of λ there are three regions which are separated by two critical values of x, say $x_c^{(1)}$ and $x_c^{(2)}$.

   I. For $0 \leq x < x_c^{(1)}$, we find the critical value of λ as
   $$\lambda_c^{(1)} = F^{(1)}(x).$$
   The operator $\rho_{BD}(\lambda; x)$ is a density matrix if $\lambda \leq \lambda_c^{(1)}$, otherwise it is not.

   II. For $x_c^{(1)} \leq x \leq x_c^{(2)}$, all states are always density matrices and separable.

   III. For $x_c^{(2)} < x \leq 1$, we find the critical value of λ as
   $$\lambda_c^{(2)} = F^{(2)}(x).$$
   Similar to case I, the operator $\rho_{BD}(\lambda; x)$ is a density matrix only if $\lambda \leq \lambda_c^{(2)}$. In this particular example, we observe that if we consider the amount of

entanglement of the states which $0 \leq x_1 < x_c^{(1)}$ and $x_c^{(2)} < x_2 \leq 1$, we find

$$\lambda_c^{(1)} = F^{(1)}(x_1) = \lambda_c^{(2)} = F^{(2)}(x_2).$$

That means there is a symmetry of entanglement distribution (see also Fig. 17), $1 - \lambda_c$, of the convex combination between $\rho_1(p)$ and $\rho_2(q)$.

2. Asymmetric case: Consider two Bell states δ_{BD} and σ_{BD} which we cannot find a local unitary transformation mapping them to each other. When we form the convex combination $\rho_{BD}(x) = x\delta_{BD} + (1-x)\sigma_{BD}$, we find that the characteristic curve is separated by two critical values of x, say $x_c^{(1)}$ and $x_c^{(2)}$. In this case, we found one zero eigenvalue, at least, of the state $\rho_{BD}(x)$. For some appropriate values of p, q, x, we find that the critical value of λ is zero. We can conclude that the PT-entanglement measure fails to distinguish the amount of entanglement, if the states considered have zero eigenvalues (see Figs. 30 and 31).

7.4 Classification of entangled states

In this section, we will show a particular set of entangled states which can be grouped via the PT-entanglement measure. Consider the set of the Werner states

$$\rho_{\Phi^\pm}(x) = x \left|\Phi^\pm\right\rangle \left\langle\Phi^\pm\right| + (1-x)\frac{I}{4}, \tag{160}$$

and

$$\rho_{\Psi^\pm}(x) = x \left|\Psi^\pm\right\rangle \left\langle\Psi^\pm\right| + (1-x)\frac{I}{4}, \tag{161}$$

where $|\Phi^\pm\rangle$ and $|\Psi^\pm\rangle$ are the maximally entangled states which are defined in Eqs. (115) and (115). We can show that the matrix representations of Eqs. (160) and (161) are given by

$$\rho_{\Phi^\pm}(x) = \frac{1}{4}\begin{pmatrix} 1+x & 0 & 0 & \pm 2x \\ 0 & 1-x & 0 & 0 \\ 0 & 0 & 1-x & 0 \\ \pm 2x & 0 & 0 & 1+x \end{pmatrix}, \tag{162}$$

and

$$\rho_{\Psi^\pm}(x) = \frac{1}{4}\begin{pmatrix} 1-x & 0 & 0 & 0 \\ 0 & 1+x & \pm 2x & 0 \\ 0 & \pm 2x & 1+x & 0 \\ 0 & 0 & 0 & 1-x \end{pmatrix}, \tag{163}$$

their eigenvalues are

$$\nu^1_{\Phi^\pm/\Psi^\pm} = \frac{1+3x}{4},$$

$$\nu^2_{\Phi^\pm/\Psi^\pm} = \nu^3_{\Phi^\pm/\Psi^\pm} = \nu^4_{\Phi^\pm/\Psi^\pm} = \frac{1-x}{4}.$$

The partial transpose of the states above can be found as

$$\rho^{PT}_{\Phi^\pm}(x) = \frac{1}{4}\begin{pmatrix} 1+x & 0 & 0 & 0 \\ 0 & 1-x & \pm 2x & 0 \\ 0 & \pm 2x & 1-x & 0 \\ 0 & 0 & 0 & 1+x \end{pmatrix}, \tag{164}$$

and

$$\rho^{PT}_{\Psi^\pm}(x) = \frac{1}{4}\begin{pmatrix} 1-x & 0 & 0 & \pm 2x \\ 0 & 1+x & 0 & 0 \\ 0 & 0 & 1+x & 0 \\ \pm 2x & 0 & 0 & 1-x \end{pmatrix}, \tag{165}$$

and their eigenvalues are

$$\mu^1_{\Phi^\pm/\Psi^\pm}(x) = \mu^2_{\Phi^\pm/\Psi^\pm}(x) = \mu^3_{\Phi^\pm/\Psi^\pm}(x) = \frac{(1+x)}{4},$$

$$\mu^4_{\Phi^\pm/\Psi^\pm}(x) = \frac{(1-3x)}{4}.$$

We find that these states are separable as long as $x \le 1/3$, otherwise they are entangled states. We now construct the convex combination to determine the amount of entanglement for those states

$$\rho_{\Phi^\pm}(\lambda; x) = \frac{1}{4}\begin{pmatrix} 1+x & 0 & 0 & \pm 2x(1-\lambda) \\ 0 & 1-x & \pm 2x\lambda & 0 \\ 0 & \pm 2x\lambda & 1-x & 0 \\ \pm 2x(1-\lambda) & 0 & 0 & 1+x \end{pmatrix}, \tag{166}$$

and

$$\rho_{\Psi^\pm}(\lambda; x) = \frac{1}{4}\begin{pmatrix} 1-x & 0 & 0 & \pm 2x\lambda \\ 0 & 1+x & \pm 2x(1-\lambda) & 0 \\ 0 & \pm 2x(1-\lambda) & 1+x & 0 \\ \pm 2x\lambda & 0 & 0 & 1-x \end{pmatrix}. \tag{167}$$

The eigenvalues of these states can be found as

$$E^1_{\Phi^\pm/\Psi^\pm}(\lambda; x) = E^2_{\Phi^\pm/\Psi^\pm}(\lambda; x) = (1 - x + 2x\lambda)/4,$$

$$E^3_{\Phi^\pm/\Psi^\pm}(\lambda; x) = (1 - x - 2x\lambda)/4,$$

$$E^4_{\Phi^\pm/\Psi^\pm}(\lambda; x) = (1 + 3x - 2x\lambda)/4.$$

The third eigenvalue $E^3_{\Phi^\pm/\Psi^\pm}(\lambda;x)$ is positive as long as λ satisfies the inequality

$$\lambda \leq \frac{1-x}{2x}, \quad x \in (1/3, 1] \ ,$$

and we have

$$\lambda_c(x) = \frac{1-x}{2x}. \tag{168}$$

We see that the set of states $\{\rho_{\Phi^\pm}(x), \rho_{\Psi^\pm}(x)\}$ are the same function of λ_c. If we set $x = 1/2$, $\lambda_c = 1/2$, then the set of the states

$$\{\rho_{\Phi^\pm}(x = 1/2), \rho_{\Psi^\pm}(x = 1/2)\}$$

can be grouped together containing the same amount of entanglement. By using our entanglement measure, the amount of entanglement that these states contain is $1 - \lambda_c = 1/2$. Similarly, if we choose $x = 3/4$, we find $\lambda_c = 1/6$. The set of states

$$\{\rho_{\Phi^\pm}(x = 3/4), \rho_{\Psi^\pm}(x = 3/4)\}$$

can be grouped together as well and the amount of entanglement of this set of these states is $1 - \lambda_c = 5/6$. We see that $1 - \lambda_c(x = 3/4) > 1 - \lambda_c(x = 1/2)$. This implies that the set of states with $x = 3/4$ contain more entanglement than the states for which $x = 1/2$.

We know that there exists the unitary transformation such that

$$\left|\Phi^\pm\right\rangle \rightleftarrows \left|\Psi^\pm\right\rangle$$

Consequently, there also exist the unitrary matrices such that

$$\rho_{\Phi^\pm}(x = 1/2) \rightleftarrows \rho_{\Psi^\pm}(x = 1/2)$$

and

$$\rho_{\Phi^\pm}(x = 3/4) \rightleftarrows \rho_{\Psi^\pm}(x = 3/4)$$

This statement tells us about the amount of entanglement invariant under unitary transformations. We will discuss (local) unitary transformations in more details later.

7.5 A special case of the function $F(x)$

In this section, we will find some conditions of function $F(x)$. We know that the Hermitian density matrix can be written in the spectral decomposition given by

$$\rho_{AB} = \sum_l^M \nu_l \left|\Phi_l\right\rangle \left\langle\Phi_l\right|, \tag{169}$$

where ν_l are the eigenvalues corresponding to the eigenvectors $|\Phi_l\rangle$. Now we assume that the density matrix ρ_{AB} and its partial transpose ρ_{AB}^{PT} commute, satisfying the condition $\left[\rho_{AB}, \rho_{AB}^{PT}\right] = 0$. That means, they can be written in the spectral decomposition with the same eventuates. Then we can introduce

$$\rho_{AB}^{PT} = \sum_{l}^{M} \mu_l \left|\Phi_l\right\rangle \left\langle\Phi_l\right|, \tag{170}$$

where μ_j are eigenvalues of the state ρ_{AB}^{PT} corresponding to the eigenvectors $|\Phi_j\rangle$. Using Eqs. (169) and (170), we can construct the state $\rho_{AB}(\lambda)$ as

$$\rho_{AB}(\lambda) = \sum_{l}^{M} \left[(1-\lambda)\,\nu_l + \lambda\mu_l\right] \left|\Phi_l\right\rangle \left\langle\Phi_l\right|. \tag{171}$$

We know that if $\rho_{AB}(\lambda)$ is the density matrix, its eigenvalues must be positive values. Then we have the condition

$$(1-\lambda)\,\nu_l + \lambda\mu_l \geq 0. \tag{172}$$

If the first eigenvalue μ_1 of ρ_{AB}^{PT} are negative. Then we can find the inequality and the critical value of λ as follows

$$\lambda \leq \frac{\nu_1}{\nu_1 - \mu_1}. \tag{173}$$

and we define the critical value of λ

$$\lambda_c = \frac{\nu_1}{\nu_1 - \mu_1}, \tag{174}$$

that means if $\lambda < \lambda_c$ the state $\rho_{AB}(\lambda)$ is the entangled state, otherwise it is not. We now define PT-entanglement measure for commuting between ρ_{AB} and ρ_{AB}^{PT}

$$E_{PT}^{C}(\rho_{AB}) = 1 - \frac{\nu_l}{\nu_l - \mu_l} = \frac{\mu_l}{\mu_l - \nu_l}. \tag{175}$$

From Section 7.7, it is easily to show that

$$\left[\rho_{\Psi^\pm/\Phi^\pm}, \rho_{\Psi^\pm/\Phi^\pm}^{PT}\right] = 0, \tag{176}$$

and we find that

$$\lambda_c(x) = \frac{\nu_{\Psi^\pm/\Phi^\pm}^{1}}{\nu_{\Psi^\pm/\Phi^\pm}^{1} - \mu_{\Psi^\pm/\Phi^\pm}^{1}} = \frac{(1-x)/4}{(1-x)/4 - (1-3x)/4} = \frac{1-x}{2x}, \tag{177}$$

and

$$E_{PT}^{C}(x) = \frac{3x-1}{2x}. \tag{178}$$

Above equation is identical with Eq. (168). However, this condition holds if and only if $\left[\rho_{AB}, \rho_{AB}^{PT}\right] = 0$.

7.6 Conditions on entanglement measures

In this section, we will show that our entanglement measure satisfies some conditions required of every measure of entanglement.

7.6.1 The condition E1

Pure states: It was shown in section 7.2 that this measure can distinguish between entangled and separable pure states but it does not differentiate between degrees of entanglement of pure states;

$$E_{PT}\left(|\Psi\rangle\langle\Psi|_{sep}\right) = 0 \ , \quad E_{PT}\left(|\Psi\rangle\langle\Psi|_{ent}\right) = 1 \ .$$

Mixed states: Our operation cannot create an entanglement state from an unentangled state. If we pick up a separable state, the convex combination $\rho_{AB}(\lambda)$ will always have positive eigenvalues only. That means $\lambda_c = 1$, then we can say $E_{PT}(\rho_{AB}^{sep}) = 0$. On the other hand, if we consider a entangled state, we found that $\lambda_c = F(x)$, so that $E_{PT}(\rho_{AB}) = 1 - F(x)$. The function $F(x)$ is related to the eigenvalues of a given state and its partial transpose.

However, there are some cases that the critical value of λ is always zero whether the states are more entangled or not. This situation arises when a given state ρ_{AB} has a least one zero eigenvalue. We will show the simple case of this situation. We assume that the state ρ_{AB} and ρ_{AB}^{PT} commute. From Eq. (172), we have the condition for the eigenvalue of $\rho_{AB}(\lambda)$. If the first eigenvalue μ_1 of ρ_{AB}^{PT} is negative and the first eigenvalue ν_1 of ρ_{AB} is zero, we immediately get a new condition for the eigenvalue of $\rho_{AB}(\lambda)$

$$\lambda\mu_1 \geq 0.$$

Obviously, this inequality holds if and only if $\lambda = 0$. Then we can state that the critical value of λ is zero.

7.6.2 The condition E2

In this section we would like to show that the PT-entanglement measure is invariant under the local unitary operations, $U_{LO} = U_A \otimes U_B$

$$E_{PT}\left(\rho'_{AB}\right) = E_{PT}\left(\rho_{AB}\right), \tag{179}$$

where

$$\rho'_{AB}(\lambda) = (1-\lambda)U_{LO}\rho_{AB}U_{LO}^{\dagger} + \lambda(U_{LO}\rho_{AB}U_{LO}^{\dagger})^{PT} \tag{180}$$

Pure states: Consider the density matrix $\rho_{AB} = |\Psi_{AB}\rangle\langle\Psi_{AB}|$, where $|\Psi_{AB}\rangle\langle\Psi_{AB}|$

is either a product or an entangled state. We know that the state ρ_{AB} can be represented in the Schmidt decomposition Eq. (40). From the definition of E_{PT}, it follows that the eigenvalues of the state ρ'_{AB} is

$$\begin{aligned}
&E_k\left[(1-\lambda)U_A\otimes U_B\rho_{AB}U_A^{\dagger}\otimes U_B^{\dagger}+\lambda\left(U_A\otimes U_B\rho_{AB}U_A^{\dagger}\otimes U_B^{\dagger}\right)^{PT}\right]= \\
&E_k\left[U_A\otimes I_B\left((1-\lambda)I_A\otimes U_B\rho_{AB}I_A\otimes U_B^{\dagger}+\lambda I_A\otimes U_B^{*}\rho_{AB}^{PT}I_A\otimes (U_B^{*})^{\dagger}\right)U_A^{\dagger}\otimes I_B\right] \\
&=E_k\left[(1-\lambda)I_A\otimes U_B\rho_{AB}I_A\otimes U_B^{\dagger}+\lambda I_A\otimes U_B^{*}\rho_{AB}^{PT}I_A\otimes (U_B^{*})^{\dagger}\right]. \qquad (181)
\end{aligned}$$

We use the fact that the Schmidt coefficients are unchanged under a unitary operator on one of the subsystems Eq. (44). We now that the state ρ_{AB} can be written in Schmidt decomposition

$$\begin{aligned}
\rho_{AB} &= |\Psi_{AB}\rangle\langle\Psi_{AB}| \\
&= \left(\sum_l \xi_l\,|l_A\rangle\otimes|l_B\rangle\right)\left(\sum_k \xi_k\,\langle k_A|\otimes\langle k_B|\right) \\
&= \sum_{jk}\xi_l\xi_k\,|l_A\rangle\langle k_A|\otimes|l_B\rangle\langle k_B|, \qquad (182)
\end{aligned}$$

and we also write

$$\begin{aligned}
I_A\otimes U_B\rho_{AB}I_A\otimes U_B^{\dagger} &= \sum_{jk}\xi_l\xi_k\,|l_A\rangle\langle k_A|\otimes U_B\,|l_B\rangle\langle k_B|\,U_B^{\dagger} \\
&= \sum_{jk}\xi_l\xi_k\,|l_A\rangle\langle k_A|\otimes|l'_B\rangle\langle k'_B|, \qquad (183)
\end{aligned}$$

where $|l'_B\rangle = U_B\,|l_B\rangle$ and $\langle k_B| = \langle k_B|\,U_B^{\dagger}$. We know that the unitary operation transforms one set of orthonormal states to another set of orthonormal states. We can freely choose a computatinal basis of $I_A\otimes U_B\rho_{AB}I_A\otimes U_B^{\dagger}$ in which ρ_{AB} and $I_A\otimes U_B\rho_{AB}I_A\otimes U_B^{\dagger}$ have the same matrix elements. Then the states ρ_{AB} and $I_A\otimes U_B\rho_{AB}I_A\otimes U_B^{\dagger}$ have the same eigenvalues and their linear combinations with their partial transposes also have the same eigenvalues. So Eq. (181) becomes

$$= E_k\left[(1-\lambda)\rho_{AB}+\lambda\rho_{AB}^{PT}\right]. \qquad (184)$$

Then we can conclude that

$$E_{PT}\left(\rho_{AB}\right) = E_{PT}\left(U_A\otimes U_B\rho_{AB}U_A^{\dagger}\otimes U_B^{\dagger}\right).$$

Mixed states: Consider a given density matrix $\rho_{AB} = \sum_i p_i\,|\Psi^i_{AB}\rangle\langle\Psi^i_{AB}|$, where

p_i are real and $\sum_i p_i = 1$. We know that we can also express the state in Fano form as

$$\rho_{AB} = \frac{1}{4}\left(I_A \otimes I_B + \vec{\tau}_A \cdot \vec{\sigma}_A \otimes I_B + I_A \otimes \vec{\tau}_B \cdot \vec{\sigma}_B + \sum_{kl} t_{kl}\sigma_k \otimes \sigma_l\right).$$

Under local unitary transform, the state becomes

$$\rho_{AB} \to U_A \otimes U_B \rho_{AB} U_A^\dagger \otimes U_B^\dagger = \rho'_{AB}$$

and its partial transpose is

$$\left(U_A \otimes U_B \rho_{AB} U_A^\dagger \otimes U_B^\dagger\right)^{PT}$$

Similarly with pure states, we can show that the eigenvalues of state $\rho'_{AB}(\lambda)$ is

$$\begin{aligned}
&E_k\left[(1-\lambda)U_A \otimes U_B \rho_{AB} U_A^\dagger \otimes U_B^\dagger + \lambda\left(U_A \otimes U_B \rho_{AB} U_A^\dagger \otimes U_B^\dagger\right)^{PT}\right] = \\
&E_k\left[U_A \otimes I_B\left((1-\lambda)I_A \otimes U_B \rho_{AB} I_A \otimes U_B^\dagger + \lambda I_A \otimes U_B^* \rho_{AB} I_A \otimes (U_B^*)^\dagger\right) U_A^\dagger \otimes I_B\right] \\
&= E_k\left[(1-\lambda)I_A \otimes U_B \rho_{AB} I_A \otimes U_B^\dagger + \lambda I_A \otimes U_B^* \rho_{AB}^{PT} I_A \otimes (U_B^*)^\dagger\right]. \qquad (185)
\end{aligned}$$

Now we consider

$$\begin{aligned}
&I_A \otimes U_B \rho_{AB} I_A \otimes U_B^\dagger \\
= \; &\frac{1}{4}\left(I_A \otimes I_B + \vec{\tau}_A \cdot \vec{\sigma}_A \otimes I_B + I_A \otimes \vec{\tau}_B \cdot U_B \vec{\sigma}_B U_B^\dagger + \sum_{kl} t_{kl}\sigma_k \otimes U_B \sigma_l U_B^\dagger\right). \qquad (186)
\end{aligned}$$

If we define $\vec{\sigma}'_B = U_B \vec{\sigma}_B U_B^\dagger$ and $\vec{\sigma}'_l = U_B \vec{\sigma}_l U_B^\dagger$, we find that eigenvalues of the Pauli's spin matrices are invariant under unitary transform. We now that the Puali's spin matices can be represented in the computational basis. Analogously with pure states, we can freely choose the bases of $\vec{\sigma}'_B$ and $\vec{\sigma}'_l$ in which $\vec{\sigma}'_B$, $\vec{\sigma}'_l$, $\vec{\sigma}_B$ and $\vec{\sigma}_l$ have the same matrix elements. Then the state ρ_{AB} and $I_A \otimes U_B \rho_{AB} I_A \otimes U_B^\dagger$ have the same eigenvalues. We know that their linear combinations with their partial transposes also have the same eigenvalues. Using these results, Eq. (185) becomes

$$= E_k\left[(1-\lambda)\rho_{AB} + \lambda\rho_{AB}^{PT}\right]. \qquad (187)$$

Then we can conclude that

$$E_{PT}(\rho_{AB}) = E_{PT}\left(U_A \otimes U_B \rho_{AB} U_A^\dagger \otimes U_B^\dagger\right).$$

7.6.3 The condition E3

In this section, we provide some examples that the measure of entanglement $E_{PT}(\rho_{AB})$ does not increase under LOCC given by Φ, i.e., $E_{PT}(\Phi_{LOCC}\rho_{AB}) \leq E_{PT}(\rho_{AB})$. We expect this result to hold for both pure and mixed states. There are three different ingredients consisted of local operations, classical communications and post-selection that aim to decrease correlations between two quantum subsystems locally.

A. Local Operations: LO

Two parities, A and B, perform measurements separately with two sets of operators satisfying the completeness relations $\sum_i A_i^\dagger A_i = I$ and $\sum_i B_j^\dagger B_j = I$. The joint operation of the two parties is described by $\sum_{ij} A_i \otimes B_j$ which refers to a general local measurement. Next we will show an example and we find that under this local general measurement, the amount of entanglement E_{PT} does not increase.

Example 4.3.1: We propose the following Kraus operators $\{M_1, M_2\}$:

$$M_1 = \begin{pmatrix} \cos\delta & 0 \\ 0 & \sin\delta \end{pmatrix}, \quad M_2 = \begin{pmatrix} \sin\delta & 0 \\ 0 & \cos\delta \end{pmatrix}, \tag{188}$$

while satisfy the condition $\sum\limits_i M_i^\dagger M_i = I_2$ and $0 \leq \delta \leq \pi/4$. Now Alice and Bob choose their Kraus operators as

$$\begin{aligned} A_1 &= \begin{pmatrix} \cos\delta & 0 \\ 0 & \sin\delta \end{pmatrix}, \quad A_2 = \begin{pmatrix} \sin\delta & 0 \\ 0 & \cos\delta \end{pmatrix} \\ B_1 &= \begin{pmatrix} \cos\delta & 0 \\ 0 & \sin\delta \end{pmatrix}, \quad B_2 = \begin{pmatrix} \sin\delta & 0 \\ 0 & \cos\delta \end{pmatrix} \end{aligned} \tag{189}$$

and they also satisfy $\sum\limits_{ij} A_i A_i^\dagger \otimes B_j B_j^\dagger = I_2 \otimes I_2$. Recalling the Werner state in Example 1.1, we have

$$\rho_{\Phi^+}(x) = \frac{1}{4}\begin{pmatrix} 1+x & 0 & 0 & 2x \\ 0 & 1-x & 0 & 0 \\ 0 & 0 & 1-x & 0 \\ 2x & 0 & 0 & 1-x \end{pmatrix}, \tag{190}$$

for which the critical value is $x_c = 1/3$. We now construct

$$\rho_{\Phi^+}(\lambda; x) = (1-\lambda)\,\rho_{\Phi^+}(x) + \lambda \rho_{\Phi^+}^{PT}(x)\,, \tag{191}$$

and we have the critical value of λ as

$$\lambda_c = \frac{1-x}{2x}.$$

Now Alice and Bob perform local operations on their systems, without recording the results and thus not communicating with each other. The Werner state transforms to

$$\rho_{\Phi^+}(x) \longmapsto \sigma_{\Phi^+}(x) = \sum_{ij} A_i \otimes B_j \rho_{\Phi^+}(x) A_i^\dagger \otimes B_j^\dagger, \tag{192}$$

or explicitly,

$$\sigma_{\Phi^+}(x) = \frac{1}{4}\begin{pmatrix} 1+x & 0 & 0 & 8x\cos^2\delta\sin^2\delta \\ 0 & 1-x & 0 & 0 \\ 0 & 0 & 1-x & 0 \\ 8x\cos^2\delta\sin^2\delta & 0 & 0 & 1+x \end{pmatrix}, \tag{193}$$

while it reads in bra-ket notation as

$$\sigma_{\Phi^+}(x) \;=\; x\rho_{\Phi^+}(\delta) + (1-x)\frac{I}{4}, \tag{194}$$

where

$$\rho_{\Phi^+}(\delta) = \frac{1}{2}\left(|00\rangle\langle 00| + |11\rangle\langle 11|\right) + 2\cos^2\delta\sin^2\delta\left(|00\rangle\langle 11| + |11\rangle\langle 00|\right) \tag{195}$$

is a new entangled state. For $\delta = \pi/4$, we recover the Bell state

$$\rho_{\Phi^+}\left(\delta = \frac{\pi}{4}\right) = \frac{1}{2}\left(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\right) = \left|\Phi^+\right\rangle\left\langle\Phi^+\right|. \tag{196}$$

For $\delta = \pi/3$,

$$\rho_{\Phi^+}\left(\delta = \frac{\pi}{3}\right) = \frac{1}{2}\left(|00\rangle\langle 00| + |11\rangle\langle 11|\right) + \frac{3}{8}\left(|00\rangle\langle 11| + |11\rangle\langle 00|\right). \tag{197}$$

and for $\delta = 0$, we obtain

$$\rho_{\Phi^+}(\delta = 0) = \frac{1}{2}\left(|00\rangle\langle 00| + |11\rangle\langle 11|\right). \tag{198}$$

Obviously, the state $\sigma_{\Phi^+}(x)$ is the combination between the separable state I and non-maximally entangled state[5] $\rho_{\Phi^+}(\delta)$. On the other hand, the Werner

[5] According to the entropy, the density matrix gives the maximum entropy when the probability distributions are the same for each state, such as Bell states, which it is called the maximally entangled state.

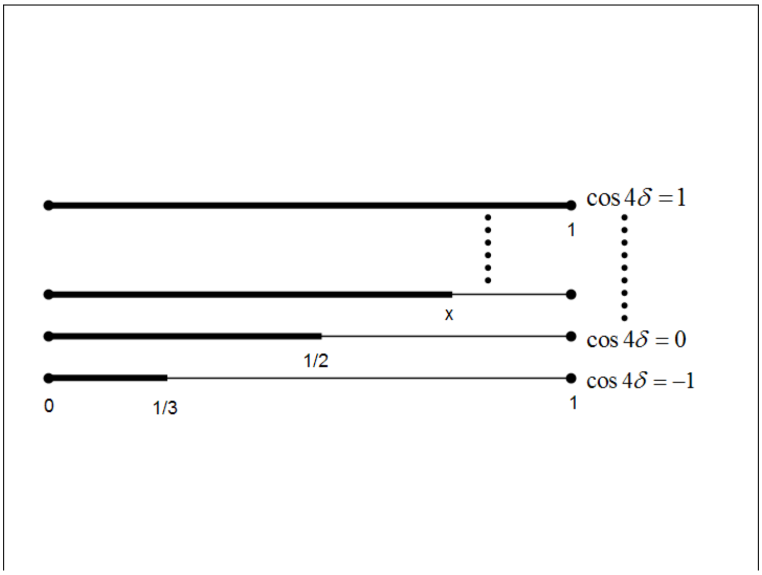

Figure 18: This figure shows that the proportional of entanglement in the Werner state decreases while the angle changes.

state consists of the maximally entangled state $|\Phi^+\rangle\langle\Phi^+|$ and the separable one. We find that the state will contain less entanglement after the performing of local operations. To confirm this statement, we have to calculate the critical value of x of the state $\sigma_{\Phi^+}(x)$. The partial transpose of this state is given by

$$\sigma_{\Phi^+}^{PT}(x) = \frac{1}{4}\begin{pmatrix} 1+x & 0 & 0 & 0 \\ 0 & 1-x & 8x\cos^2\delta\sin^2\delta & 0 \\ 0 & 8x\cos^2\delta\sin^2\delta & 1-x & 0 \\ 0 & 0 & 0 & 1+x \end{pmatrix}. \tag{199}$$

The eigenvalues are

$$E_{\Phi^+}^1 = E_{\Phi^+}^2 = (1+x)/4,$$
$$E_{\Phi^+}^3 = (1 - x\cos 4\delta)/4,$$
$$E_{\Phi^+}^4 = (1 - x(2+\cos 4\delta))/4$$

The eigenvalue $E_{\Phi^+}^4$ is positive as long as $x > 1/(2-\cos 4\delta)$ and we know that $-1 \leq \cos 4\delta \leq 1$ then $1/3 \leq 1/(2-\cos 4\delta) \leq 1$. We see that the critical value now depends on δ, changing from $x_c = 1/3$ to $x_c \in [1/3, 1]$. Only for $\delta = n\pi/4$, where $n = 1, 3, 5, ..$, the density matrix $\rho_{\Phi^+}(\delta)$ becomes $|\Phi^+\rangle\langle\Phi^+|$, the maximally entangled state, and we recover the same critical value $x_c = 1/3$. Form these results, we can say that the proportion of the entangled state in the Werner state decreases while the critical value increases from 1/3 to 1 when $\cos 4\delta$ increases, see Fig. 18, after performing the local operations. Next we would like to find the

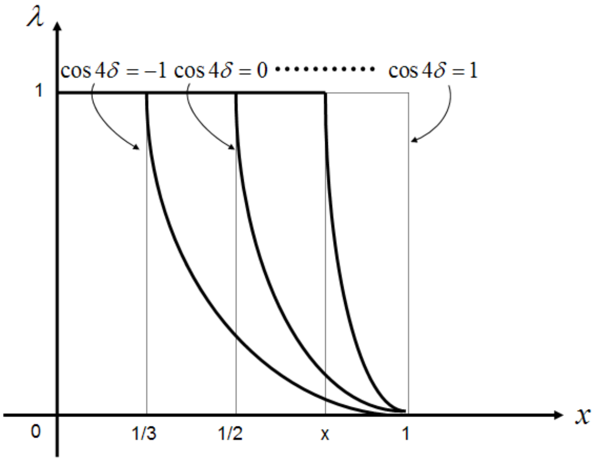

Figure 19: This figure shows that the quantitative dependence of λ remain unchanged.

critical value of λ as a function of x. The state $\sigma_{\Phi^+}(\lambda;x)$ can be expressed in the matrix form as

$$\sigma_{\Phi^+}(\lambda;x) = \frac{1}{4}\begin{pmatrix} 1+x & 0 & 0 & 2x(1-\lambda)Z(\delta) \\ 0 & 1-x & 2x\lambda Z(\delta) & 0 \\ 0 & 2x\lambda Z(\delta) & 1-x & 0 \\ 2x(1-\lambda)Z(\delta) & 0 & 0 & 1+x \end{pmatrix}, \tag{200}$$

where $Z(\delta) = \cos^2\delta \sin^2\delta$. The eigenvalues are

$$E^1_{\Phi^+}(\lambda;x) = (1 - x(1-\lambda+\lambda\cos\delta))/4$$
$$E^2_{\Phi^+}(\lambda;x) = (1 + x(\lambda+\cos 4\delta - \lambda\cos\delta))/4,$$
$$E^3_{\Phi^+}(\lambda;x) = (1 - x(1+\lambda-\lambda\cos\delta))/4,$$
$$E^4_{\Phi^+}(\lambda;x) = (1 + x(2-\lambda-\cos 4\delta + \lambda\cos 4\delta))/4.$$

The eigenvalue $E^3_{\Phi^+}$ is negative when ever

$$\lambda > \frac{1-x}{x(1-\cos 4\delta)} = \frac{1-x}{2x}\frac{2}{1-\cos 4\delta}.$$

Using the fact that $0 < 1-\cos 4\delta < 2$, we find that the condition for λ_c takes the form

$$\lambda_c(x,\delta) = \frac{2}{1-\cos 4\delta}\cdot\lambda_c(x).$$

From above results, we can state that the maximally entangled state $|\Phi^+\rangle\langle\Phi^+|$ has been transformed to the non-maximally entangled state $\rho_{\Phi^+}(\delta)$. This implies that the proportion of the entanglement does not increase under local operations, while the quantitative dependence of λ_c on x remains unchanged as in Fig. 19. We now state that

$$1-\lambda_c(x,\delta) \leq 1-\lambda_c(x),$$

or

$$E_{PT}\left(\sum_{ij} A_i \otimes B_j \rho_{\Phi^+}(x) A_i^\dagger \otimes B_j^\dagger\right) \leq E_{PT}\left(\rho_{\Phi^+}(x)\right).$$

Let us consider the particular case of projection operators. If we set $\delta = 0$, the Kraus operators $\{M_1, M_2\}$ become

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\ ,\ M_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{201}$$

and Eqs. (193) and (199)turn into

$$\sigma_{\Phi^+}(x) = \frac{1}{4}\begin{pmatrix} 1+x & 0 & 0 & 0 \\ 0 & 1-x & 0 & 0 \\ 0 & 0 & 1-x & 0 \\ 0 & 0 & 0 & 1+x \end{pmatrix}, \tag{202}$$

$$\sigma^{PT}_{\Phi^+}(x) = \frac{1}{4}\begin{pmatrix} 1+x & 0 & 0 & 0 \\ 0 & 1-x & 0 & 0 \\ 0 & 0 & 1-x & 0 \\ 0 & 0 & 0 & 1+x \end{pmatrix}. \tag{203}$$

We see that all their eigenvalues are the positive and their convex combination becomes

$$\sigma_{\Phi^+}(\lambda; x) = \frac{1}{4}\begin{pmatrix} 1+x & 0 & 0 & 0 \\ 0 & 1-x & 0 & 0 \\ 0 & 0 & 1-x & 0 \\ 0 & 0 & 0 & 1+x \end{pmatrix}, \tag{204}$$

Obviously, the state Eq. (204) does not depend on the variable λ. This projective operations map the state $\rho_{\Phi^+}(x)$ to the separable area. That clearly means the amount of entanglement of the state $\rho_{\Phi^+}(x)$ does not increase under local projective operators.

B. Classical Communication: CC

Alice and Bob are allowed to talk each others via classical communication such as a telephone while performing their local quantum operations [3]. This means they can increase the classical correlations between the parts of the system. To combine the classical communication and the local operations, we arrive at LOCC operations which is important to distinguish between classical correlations and quantum one [3]. The details of LOCC operations have been introduced in the section 4.5.

C. Post-selection: PS

After LOCC operations, the state ρ_{AB} has been transformed into a new state ρ_m

$$\rho_{AB} \to \rho_m = \frac{A_m \otimes B_m \rho_{AB} A_m^\dagger \otimes B_m^\dagger}{\mathrm{Tr}\left(A_m \otimes B_m \rho_{AB} A_m^\dagger \otimes B_m^\dagger\right)} \tag{205}$$

where the denominator gives the normalization condition. We can collect all possible results to get [41]

$$\sigma_{AB} = \sum_m A_m \otimes B_m \rho_{AB} A_m^\dagger \otimes B_m^\dagger \tag{206}$$

Example 4.3.2: Basically, Alice and Bob can perform their local operations at different places. We now allow them to talk to each others via classical communications. In this section, we will show that after "LOCC+PS" operations, the amount of entanglement does not increase for a specific set of local operations.

Let us say that first, Alice talks to Bob and then secondly, Bob answers Alice. This process is refereed to *two-way communication.* Suppose Alice and Bob again use the Kraus operators which are based on $\{M_1, M_2\}$, see Example 4.3.1. We now define the operators

$$a_1 = a_{111} = a_{121} = a_{211} = a_{221} = \begin{pmatrix} \cos\delta & 0 \\ 0 & \sin\delta \end{pmatrix}, \tag{207}$$

$$a_2 = a_{112} = a_{122} = a_{212} = a_{222} = \begin{pmatrix} \sin\delta & 0 \\ 0 & \cos\delta \end{pmatrix}, \tag{208}$$

$$b_{11} = b_{21} = \begin{pmatrix} \cos\delta & 0 \\ 0 & \sin\delta \end{pmatrix}, \tag{209}$$

$$b_{12} = b_{22} = \begin{pmatrix} \sin\delta & 0 \\ 0 & \cos\delta \end{pmatrix}. \tag{210}$$

To help us to easily understand, we show a diagram of this process in Fig. 20. In this case, the LOCC+PS gives us eight possible outcomes. We now rewrite the operators for all cases as

$$a_{111}a_1 = \begin{pmatrix} \cos^2\delta & 0 \\ 0 & \sin^2\delta \end{pmatrix} \equiv A_1, \tag{211}$$

$$a_{112}a_1 = \begin{pmatrix} \cos\delta\sin\delta & 0 \\ 0 & \cos\delta\sin\delta \end{pmatrix} \equiv A_2, \tag{212}$$

$$a_{121}a_1 = \begin{pmatrix} \cos^2\delta & 0 \\ 0 & \sin^2\delta \end{pmatrix} \equiv A_3, \tag{213}$$

$$a_{122}a_1 = \begin{pmatrix} \cos\delta\sin\delta & 0 \\ 0 & \cos\delta\sin\delta \end{pmatrix} \equiv A_4, \tag{214}$$

$$a_{211}a_2 = \begin{pmatrix} \cos\delta\sin\delta & 0 \\ 0 & \cos\delta\sin\delta \end{pmatrix} \equiv A_5, \tag{215}$$

$$a_{212}a_2 = \begin{pmatrix} \sin^2\delta & 0 \\ 0 & \cos^2\delta \end{pmatrix} \equiv A_6, \tag{216}$$

$$a_{221}a_2 = \begin{pmatrix} \cos\delta\sin\delta & 0 \\ 0 & \cos\delta\sin\delta \end{pmatrix} \equiv A_7, \tag{217}$$

$$a_{222}a_2 = \begin{pmatrix} \sin^2\delta & 0 \\ 0 & \cos^2\delta \end{pmatrix} \equiv A_8, \tag{218}$$

and

$$\begin{aligned} B_1 = B_2 = B_5 = B_6 &= \begin{pmatrix} \cos\delta & 0 \\ 0 & \sin\delta \end{pmatrix} \\ B_3 = B_4 = B_7 = B_8 &= \begin{pmatrix} \sin\delta & 0 \\ 0 & \cos\delta \end{pmatrix} \end{aligned} \tag{219}$$

So, we obtain

$$\sigma_{\Phi^+} = \sum_{m=1}^{8} A_m \otimes B_m \rho_{\Phi^+} A_m^\dagger \otimes B_m^\dagger. \tag{220}$$

If we let $\rho_{\Phi^+} = I$, it is easily to show that

$$\sum_{m=1}^{8} (A_m \otimes B_m)\left(A_m^\dagger \otimes B_m^\dagger\right) = \sum_{m=1}^{8} A_m A_m^\dagger \otimes B_m B_m^\dagger = I. \tag{221}$$

Suppose the initial state, being shared between Alice and Bob, is given by

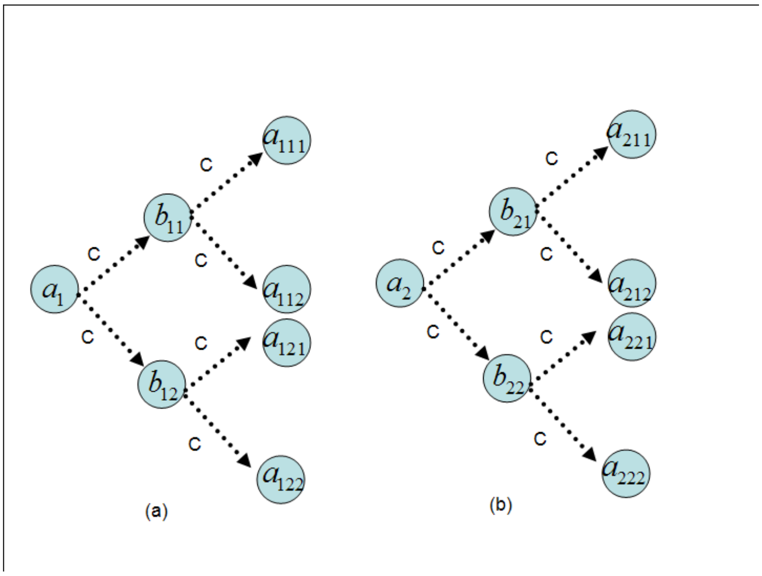

Figure 20: This two-way communication process. C represents the communication between Alice and Bob. First, Alice chooses the operators a_i and then tell Bob about her operators. After that Bob chooses the operators b_{ij} corresponding to the operators a_i and Bob also tells Alice about his operators. Finally, Alice chooses the operators a_{ijk} corresponding to both a_i and b_{ij}.

$$\rho_{\Phi^+}(x) = \frac{1}{4}\begin{pmatrix} 1+x & 0 & 0 & 2x \\ 0 & 1-x & 0 & 0 \\ 0 & 0 & 1-x & 0 \\ 2x & 0 & 0 & 1-x \end{pmatrix}. \tag{222}$$

The state σ_{Φ^+} becomes

$$\sigma_{\Phi^+} = \frac{1}{4}\begin{pmatrix} (1+x) & 0 & 0 & 16xc^3s^3 \\ 0 & (1-x) & 0 & 0 \\ 0 & 0 & (1-x) & 0 \\ 16xc^3s^3 & 0 & 0 & (1+x) \end{pmatrix}. \tag{223}$$

We can express the state σ_{Φ^+} in bra-ket form as

$$\sigma_{\Phi^+} = \frac{x}{2}\left(|00\rangle\langle 00| + |11\rangle\langle 11|\right) + 4x\cos^3\delta\sin^3\delta\left(|00\rangle\langle 11| + |00\rangle\langle 11|\right) + (1-x)\frac{I}{4}. \tag{224}$$

The partial transpose of σ_{AB} is

$$\sigma_{\Phi^+}^{PT} = \frac{1}{4}\begin{pmatrix} (1+x) & 0 & 0 & 0 \\ 0 & (1-x) & 16x\cos^3\delta\sin^3\delta & 0 \\ 0 & 16x\cos^3\delta\sin^3\delta & (1-x) & 0 \\ 0 & 0 & 0 & (1+x) \end{pmatrix}, \tag{225}$$

with eigenvalues

$$E^1_{\Phi^+} = E^2_{2\otimes 2} = (1+x)/4,$$
$$E^3_{\Phi^+} = \left(1 - x + 16x\cos^3\delta\sin^3\delta\right)/4,$$
$$E^4_{\Phi^+} = (2 - 2x - 3x\sin 2\delta + x\sin 6\delta)/8.$$

The third eigenvalue is negative as long as

$$x < \frac{1}{1 - 16\cos^3\delta\sin^3\delta},$$

where $\delta \in (\pi/2, \pi)$ and $(3\pi/2, 2\pi)$. We find that the range of a new critical value of x is

$$\frac{1}{3} < \frac{1}{1 - 16\cos^3\delta\sin^3\delta} < 1.$$

The fourth eigenvalue gives a negative sign as long as

$$x < \frac{2}{2 + 3\sin 2\delta - \sin 6\delta}$$

where $\delta \in (0, \pi/2)$ and $(\pi, 3\pi/2)$ and we can show that the range of a new critical value of x is

$$\frac{1}{3} < \frac{2}{2 + 3x\sin 2\delta - \sin 6\delta} < 1.$$

For $\delta = 0, \pi/2, \pi, 3\pi/2$, the state σ_{Φ^+} always is separabled. Next we will fine the condition for λ to distinguish the entanglement of the state σ_{Φ^+} after LOCC operations. The $\sigma_{\Phi^+}(\lambda; x)$ can be express in the matrix form as

$$\sigma_{\Phi^+}(\lambda; x) = \begin{pmatrix} 1+x & 0 & 0 & x(1-\lambda)B(\delta) \\ 0 & 1-x & x\lambda B(\delta) & 0 \\ 0 & x\lambda B(\delta) & 1-x & 0 \\ x(1-\lambda)B(\delta) & 0 & 0 & 1+x \end{pmatrix}, \tag{226}$$

where $B(\delta) = 16\cos^3\delta\sin^3\delta$. The eigenvalues are

$$E^1_{\Phi^+}(\lambda; x) = \frac{1}{4}\left(1 - x - 16x\lambda\cos^3\delta\sin^3\delta\right),$$

$$E^2_{\Phi^+}(\lambda; x) = \frac{1}{4}\left(1 - x + 16x\lambda\cos^3\delta\sin^3\delta\right),$$

$$E^3_{\Phi^+}(\lambda; x) = \frac{1}{4}\left(1 + x + 16x\cos^3\delta\sin^3\delta - 16x\lambda\cos^3\delta\sin^3\delta\right),$$

$$E^4_{\Phi^+}(\lambda; x) = \frac{1}{4}\left(1 + x - 16x\cos^3\delta\sin^3\delta + 16x\lambda\cos^3\delta\sin^3\delta\right).$$

The first eigenvalue gives a negative sign as long as

$$\lambda < \frac{1-x}{16x\cos^3\delta\sin^3\delta} = \frac{1-x}{2x}\frac{1}{8\cos^3\delta\sin^3\delta}$$

where $x \in (x_c, 1]$. x_c is the critical value which depends on the angle δ corresponding to $\delta \in (0, \pi/2) \cup (\pi, 3\pi/2)$. we obtain

$$\lambda_c(\delta) = \frac{\lambda_c}{8\cos^3\delta\sin^3\delta}$$

On the other hand, the second eigenvalue give a negative sign as long as the λ satisfies the inequality

$$\lambda < \frac{1-x}{-16x\cos^3\delta\sin^3\delta} = -\frac{1-x}{2x}\frac{1}{8\cos^3\delta\sin^3\delta}$$

where $x \in (x_c, 1]$. x_c is the critical value which depends on the angle δ corresponding to $\delta \in (\pi/2, \pi) \cup (3\pi/2, 2\pi)$. We obtain

$$\lambda_c(\delta) = -\frac{\lambda_c}{8\cos^3\delta\sin^3\delta}$$

Obviously, we still have the same structure of λ. Then we can conclude, for this particular example, that under the "LOCC+PS" operations, the proportion of entanglement in the Werner state does not increase. Finally, We can say that

$$1 - \lambda_c(\delta) \leq 1 - \lambda_c,$$

or

$$E_{PT}\left(\sum_{j=1}^{8} A_i \otimes B_j \rho_{\Phi^+}(x) A_i^\dagger \otimes B_j^\dagger\right) \leq E_{PT}\left(\rho_{\Phi^+}(x)\right).$$

7.7 The upper and lower limits of λ_c under LOCC operations

From a particular example of LOCC operations in the previous section, we find that the PT-entanglement measure does not increase after operating. In this section, we would like to show a possible way to prove the condition **E3.2** by the use of the variation theorem of eigenvalues [57]. To obtain the critical value of λ, we have to consider the eigenvalues of $\rho_{AB}(\lambda)$

$$E_k\left(\rho_{AB}\left(\lambda\right)\right) = E_k\left((1-\lambda)\rho_{AB} + \lambda\rho_{AB}^{PT}\right). \tag{227}$$

From Weyl's theorem [57]:

Theorem. Let A, B are n-by-n complex matrices and are Hermitian. The eigenvalues $E_k(A)$, $E_k(B)$, and $E_k(A+B)$ are arranged in increasing order. For each $k = 1, 2, ..., n$ we have

$$E_k\left(A\right) + E_1\left(B\right) \leq E_k\left(A+B\right) \leq E_k\left(A\right) + E_n\left(B\right), \tag{228}$$

or

$$E_k\left(B\right) + E_1\left(A\right) \leq E_k\left(A+B\right) \leq E_k\left(B\right) + E_n\left(A\right). \tag{229}$$

In our system, the maximum numbers n of eigenvalues are 4 or 6. Here, we restrict ourselves on $2 \otimes 2$ dimensions quantum system. Using Eq. (227), Eq. (228) becomes

$$E_k\left((1-\lambda)\rho_{AB}\right) + E_1\left(\lambda\rho_{AB}^{PT}\right) \leq E_k\left(\rho_{AB}\left(\lambda\right)\right) \leq E_k\left((1-\lambda)\rho_{AB}\right) + E_4\left(\lambda\rho_{AB}^{PT}\right), \tag{230}$$

or

$$(1-\lambda)E_k\left(\rho_{AB}\right) + \lambda E_1\left(\rho_{AB}^{PT}\right) \leq E_k\left(\rho_{AB}\left(\lambda\right)\right) \leq (1-\lambda)E_k\left(\rho_{AB}\right) + \lambda E_4\left(\rho_{AB}^{PT}\right). \tag{231}$$

From Eq. (152), if we set $\lambda = \lambda_c$, the lowest eigenvalue of $\rho_{AB}(\lambda_c)$ will be zero: $E_1\left(\rho_{AB}\left(\lambda_c\right)\right) = 0$. Then Eq. (231) becomes

$$(1-\lambda_c)E_1\left(\rho_{AB}\right) + \lambda_c E_1\left(\rho_{AB}^{PT}\right) \leq 0 \leq (1-\lambda_c)E_1\left(\rho_{AB}\right) + \lambda_c E_4\left(\rho_{AB}^{PT}\right), \tag{232}$$

and we use the fact that $E_1 \leq E_4$. Then Eq.(232) can be written

$$(1-\lambda_c)E_1\left(\rho_{AB}\right) + \lambda_c E_1\left(\rho_{AB}^{PT}\right) \leq 0 \leq (1-\lambda_c)E_4\left(\rho_{AB}\right) + \lambda_c E_4\left(\rho_{AB}^{PT}\right). \tag{233}$$

We obtain the lower and upper bounds of λ_c as follows:

$$\lambda_c^{(min)} = \frac{E_1\left(\rho_{AB}\right)}{E_1\left(\rho_{AB}\right) - E_1\left(\rho_{AB}^{PT}\right)} \leq \lambda_c \leq \frac{E_4\left(\rho_{AB}\right)}{E_4\left(\rho_{AB}\right) - E_4\left(\rho_{AB}^{PT}\right)} = \lambda_c^{(max)}. \quad (234)$$

Next we consider separable LOCC operations which have been described in the section 4.5.2. Suppose we have a set of operators $\{M_1, M_1, ..., M_N\}$ which $M_i = A_i \otimes B_i$ and

$$\sum_{i=1}^{N} M_i^\dagger M_i = I. \quad (235)$$

After operating, we find the partial transpose of new states as

$$M_i \rho_{AB} M_i^\dagger \rightarrow \left(M_i \rho_{AB} M_i^\dagger\right)^{PT} = \tilde{M}_i \rho_{AB}^{PT} \tilde{M}_i^\dagger, \quad (236)$$

where $\tilde{M}_i = (M_i)^{PT}$ are new operators which are taking into account the partial transpose operations. According to quantum operations, the state ρ_{AB} transforms to

$$\rho_{AB} \rightarrow \sigma_{AB} = \sum_{i=1}^{N} M_i \rho_{AB} M_i^\dagger. \quad (237)$$

We now form a convex combination of the state after performing LOCC operations

$$\sigma_{AB}\left(\lambda'\right) = (1-\lambda')\left(\sum_{i}^{N} M_i \rho_{AB} M_i^\dagger\right) + \lambda'\left(\sum_{i}^{N} \tilde{M}_i \rho_{AB}^{PT} \tilde{M}_i^\dagger\right). \quad (238)$$

In order to obtain the critical value of λ', we have to consider the eigenvalues of $\sigma_{AB}\left(\lambda'\right)$

$$E_k\left(\sigma_{AB}\left(\lambda'\right)\right) = E_k\left((1-\lambda')\left(\sum_{i}^{N} M_i \rho_{AB} M_i^\dagger\right) + \lambda'\left(\sum_{i}^{N} \tilde{M}_i \rho_{AB}^{PT} \tilde{M}_i^\dagger\right)\right). \quad (239)$$

Using again Weyl's theorem, we know that

$$(1-\lambda')E_k\left(\sum_{i}^{N} M_i \rho_{AB} M_i^\dagger\right) + \lambda' E_1\left(\sum_{i}^{N} \tilde{M}_i \rho_{AB}^{PT} \tilde{M}_i^\dagger\right) \leq E_k\left(\sigma_{AB}\left(\lambda'\right)\right) \leq ....$$

$$... \leq (1-\lambda')E_4\left(\sum_{i}^{N} M_i \rho_{AB} M_i^\dagger\right) + \lambda' E_k \sum_{i}^{N} \tilde{M}_i \rho_{AB}^{PT} \tilde{M}_i^\dagger. \quad (240)$$

If we set $\lambda' = \lambda'_c$, $E_k\left(\sigma_{AB}\left(\lambda'_c\right)\right) = 0$. Then Eq. (240) becomes

$$(1-\lambda'_c)E_1\left(\sum_{i}^{N} M_i \rho_{AB} M_i^\dagger\right) + \lambda'_c E_1\left(\sum_{i}^{N} \tilde{M}_i \rho_{AB}^{PT} \tilde{M}_i^\dagger\right) \leq 0 \leq ...$$

$$... \leq (1-\lambda'_c)E_4\left(\sum_{i}^{N} M_i \rho_{AB} M_i^\dagger\right) + \lambda'_c E_1\left(\sum_{i}^{N} \tilde{M}_i \rho_{AB}^{PT} \tilde{M}_i^\dagger\right). \quad (241)$$

Firstly, we consider the LHS of Eq. (241). We have

$$(1-\lambda_c')E_k\left(\sum_i^N M_i\rho_{AB}M_i^\dagger\right)+\lambda_c' E_1\left(\sum_i^N \tilde{M}_i\rho_{AB}^{PT}\tilde{M}_i^\dagger\right)\leq 0. \tag{242}$$

Consider $E_k\left(\sum_i^N M_i\rho_{AB}M_i^\dagger\right)$, we use Wely's theorem to obtain the inequality

$$E_k\left(M_1\rho_{AB}M_1^\dagger\right)+...+E_1\left(M_N\rho_{AB}M_N^\dagger\right)\leq E_k\left(\sum_{i=1}^N M_i\rho_{AB}M_i^\dagger\right). \tag{243}$$

and we also apply Wely's for $E_k\left(\sum_i^N \tilde{M}_i\rho_{AB}^{PT}\tilde{M}_i^\dagger\right)$

$$E_k\left(\tilde{M}_1\rho_{AB}^{PT}\tilde{M}_1^\dagger\right)+...+E_1\left(\tilde{M}_N\rho_{AB}^{PT}\tilde{M}_N^\dagger\right)\leq E_k\left(\sum_i^N \tilde{M}_i\rho_{AB}^{PT}\tilde{M}_i^\dagger\right). \tag{244}$$

Then we have

$$(1-\lambda_c')\left[\sum_{i=1}^N E_1\left(M_i\rho_{AB}M_i^\dagger\right)\right]+\lambda_c'\left[\sum_{i=1}^N E_1\left(\tilde{M}_i\rho_{AB}\tilde{M}_i^\dagger\right)\right]\leq 0. \tag{245}$$

From Ostrowski's theorem [57]:

Theorem. Let A and M_j are n-by$n$ complex matrices with A Hermitian and M_j nonsingular. Let the eigenvalues of A and $M_jM_j^\dagger$ be arranged in increasing order. For each $k=1,2,...$, there exists a positive real number Θ_k such that

$$E_k\left(M_i\rho_{AB}M_i^\dagger\right)=\Theta_k E_k\left(\rho_{AB}\right), \tag{246}$$

where

$$E_1\left(M_iM_i^\dagger\right)\leq\Theta_k\leq E_4\left(M_iM_i^\dagger\right). \tag{247}$$

Combining Eq. (246) and (247), we obtain

$$\begin{aligned}
&E_1\left(M_iM_i^\dagger\right)E_k\left(\rho_{AB}\right)\leq\Theta_k E_k\left(\rho_{AB}\right)\leq E_4\left(M_iM_i^\dagger\right)E_k\left(\rho_{AB}\right),\\
&E_1\left(M_iM_i^\dagger\right)E_k\left(\rho_{AB}\right)\leq E_k\left(M_i\rho_{AB}M_i^\dagger\right)\leq E_4\left(M_iM_i^\dagger\right)E_k\left(\rho_{AB}\right),
\end{aligned} \tag{248}$$

where $E_k\left(\rho_{AB}\right) \geq 0$. Similarly, we can show that, for partial transpose part, we have

$$E_k\left(\tilde{M}_i \rho_{AB}^{PT} \tilde{M}_i^{\dagger}\right) = \Theta_k' E_k\left(\rho_{AB}^{PT}\right), \tag{249}$$

where

$$E_1\left(\tilde{M}_i \tilde{M}_i^{\dagger}\right) \leq \Theta_k' \leq E_4\left(\tilde{M}_i \tilde{M}_i^{\dagger}\right). \tag{250}$$

If $E_1\left(\rho_{AB}^{PT}\right)$ is negative, we can show that

$$E_4\left(\tilde{M}_i \tilde{M}_i^{\dagger}\right) E_1\left(\rho_{AB}^{PT}\right) \leq E_1\left(\tilde{M}_i \rho_{AB}^{PT} \tilde{M}_i^{\dagger}\right) \leq E_1\left(\tilde{M}_i \tilde{M}_i^{\dagger}\right) E_1\left(\rho_{AB}^{PT}\right). \tag{251}$$

Using Eqs. (248) and (251), Eq. (245) becomes

$$(1-\lambda_c)\left[\sum_{i=1}^{N} E_1\left(M_i M_i^{\dagger}\right) E_1(\rho_{AB})\right] + \lambda_c\left[\sum_{i=1}^{N} E_4\left(\tilde{M}_i \tilde{M}_i^{\dagger}\right) E_1(\rho_{AB}^{PT})\right] \leq 0. \tag{252}$$

A new lower bound of the critical value of λ_c follows from this inequality:

$$\lambda_c' \geq \frac{E_1\left(\rho_{AB}\right)}{E_1\left(\rho_{AB}\right) - \kappa E_1\left(\rho_{AB}^{PT}\right)} = \lambda_c'^{(min)}, \tag{253}$$

where

$$\kappa = \frac{\sum_{i=1}^{N} E_4\left(\tilde{M}_i \tilde{M}_i^{\dagger}\right)}{\sum_{i=1}^{N} E_1\left(M_i M_i^{\dagger}\right)}. \tag{254}$$

In order to show that PT-entanglement measure does increase under LOCC operation on the average, we need to show that $\kappa \leq 1$. We know that the operators M_i require normalization condition Eq. (235). Then we find that

$$E_k\left(\sum_{i=1}^{N} M_i^{\dagger} M_i\right) = 1. \tag{255}$$

This condition holds for $\tilde{M}_i$ as well,

$$\sum_{i=1}^{N} (M_i^{\dagger} M_i)^{PT} = I \tag{256}$$

implies that

$$E_k\left(\sum_{i=1}^{N} \tilde{M}_i^{\dagger} \tilde{M}_i\right) = 1. \tag{257}$$

Again, using Weyl's theorem, we obtain

$$\begin{aligned} & E_k\left(M_1^\dagger M_1\right)+E_1\left(M_2^\dagger M_2\right)+...+E_1\left(M_N^\dagger M_N\right)\leq E_k\left(\sum_{i=1}^N M_i^\dagger M_i\right)\leq ... \\ & ...\leq E_k\left(M_1^\dagger M_1\right)+E_4\left(M_2^\dagger M_2\right)+...+E_4\left(M_N^\dagger M_N\right), \end{aligned} \tag{258}$$

or

$$\begin{aligned} & E_k\left(M_1^\dagger M_1\right)+E_1\left(M_2^\dagger M_2\right)+...+E_1\left(M_N^\dagger M_N\right)\leq 1\leq ... \\ & ...\leq E_k\left(M_1^\dagger M_1\right)+E_4\left(M_2^\dagger M_2\right)+...+E_4\left(M_N^\dagger M_N\right), \end{aligned} \tag{259}$$

and we also show that

$$\begin{aligned} & E_k\left(\tilde{M}_1^\dagger \tilde{M}_1\right)+E_1\left(\tilde{M}_2^\dagger \tilde{M}_2\right)+...+E_1\left(\tilde{M}_N^\dagger \tilde{M}_N\right)\leq 1\leq ... \\ & ...\leq E_k\left(\tilde{M}_1^\dagger \tilde{M}_1\right)+E_4\left(\tilde{M}_2^\dagger \tilde{M}_2\right)+...+E_4\left(\tilde{M}_N^\dagger \tilde{M}_N\right). \end{aligned} \tag{260}$$

Using the LHS of Eq. (259) for $k = 1$ and the RHS of Eq. (260) for $k = 4$, it follows that

$$E_1\left(\sum_{i=1}^N M_i^\dagger M_i\right)\leq 1\leq E_4\left(\sum_{i=1}^N \tilde{M}_i^\dagger \tilde{M}_i\right). \tag{261}$$

To obtain the value of κ, we need to show that $E_k\left(M_i^\dagger M_i\right)=E_k\left(M_i M_i^\dagger\right)$. We know that we can represent the matrices M_i in the polar form [57]

$$M_i = U_i H_i, \tag{262}$$

where U_i are the unitary matrices and H_i are the Hermitian matrices. Let us define $E_k\left(M_i M_i^\dagger\right)=\alpha_k^i\in\mathbb{R}$ and consider the equation

$$\left(M_i M_i^\dagger\right)\left|\Psi_k^i\right\rangle=\alpha_k^i\left|\Psi_k^i\right\rangle, \tag{263}$$

where $|\Psi_k^i\rangle$ are the eigenstates corresponding to α_k^i. Using Eq. (262), we have

$$\begin{aligned} \left(U_i H_i H_i U_i^\dagger\right)\left|\Psi_k^i\right\rangle &= \alpha_k^i\left|\Psi_k^i\right\rangle \\ H_i^2\left(U_i^\dagger\left|\Psi_k^i\right\rangle\right) &= \alpha_k^i\left(U_i^\dagger\left|\Psi_k^i\right\rangle\right). \end{aligned} \tag{264}$$

Next let us define $E_k\left(M_i^\dagger M_i\right)=\beta_k^i\in\mathbb{R}$ and consider the equation

$$\begin{aligned} \left(M_i^\dagger M_i\right)\left|\Phi_\alpha^i\right\rangle &= \beta_k^i\left|\Phi_\alpha^i\right\rangle \\ \left(H_i U_i^\dagger U_i H_i\right)\left|\Phi_\alpha^i\right\rangle &= \beta_k^i\left|\Phi_\alpha^i\right\rangle \\ H_i^2\left|\Phi_\alpha^i\right\rangle &= \beta_k^i\left|\Phi_\alpha^i\right\rangle. \end{aligned} \tag{265}$$

From Eqs. (264) and (265), we can state that $\beta_k^i = \alpha_k^i$ or $E_k\left(M_i^\dagger M_i\right) = E_k\left(M_i M_i^\dagger\right)$. This condition holds for $\tilde{M}_i$ as well. Then we can rewrite Eq. (261) as follows

$$E_1\left(\sum_{i=1}^{N} M_i M_i^\dagger\right) \le 1 \le E_4\left(\sum_{i=1}^{N} \tilde{M}_i \tilde{M}_i^\dagger\right). \tag{266}$$

Using Eq. (266), it is easy to show that

$$\kappa = \frac{E_4\left(\sum\limits_{i=1}^{N} \tilde{M}_i \tilde{M}_i^\dagger\right)}{E_1\left(\sum\limits_{i=1}^{N} M_i M_i^\dagger\right)} \ge 1. \tag{267}$$

Using the result of Eq. (267), we obtain

$$\frac{E_1\left(\rho_{AB}\right)}{E_1\left(\rho_{AB}\right) - E_1\left(\rho_{AB}^{PT}\right)} \ge \frac{E_1\left(\rho_{AB}\right)}{E_1\left(\rho_{AB}\right) - \kappa E_1\left(\rho_{AB}^{PT}\right)}. \tag{268}$$

Obviously, the lower bound of critical value of λ does not increase under LOCC operations. This means that the lower bound of λ_c' is below the lower bound of λ_c.

Next we would like to consider the upper bound of the critical value of λ_c'. The details of calculations can be found in Appendix B. The upper bound of λ_c' is

$$\lambda_c' \le \frac{E_4\left(\rho_{AB}\right)}{E_4\left(\rho_{AB}\right) - \kappa' E_4\left(\rho_{AB}^{PT}\right)} = \lambda_c'^{(max)}, \tag{269}$$

where

$$\kappa' = \frac{E_4\left(M_1^\dagger M_1\right) + \sum\limits_{i=2}^{N} E_1\left(\tilde{M}_i^\dagger \tilde{M}_i\right)}{\sum\limits_{i=2}^{N} E_4\left(M_i^\dagger M_i\right)}. \tag{270}$$

We can show that (see also Appendix B.)

$$\kappa' = \frac{E_4\left(M_1^\dagger M_1\right) + \sum\limits_{i=2}^{N} E_1\left(\tilde{M}_i^\dagger \tilde{M}_i\right)}{\sum\limits_{i=2}^{N} E_4\left(M_i^\dagger M_i\right)} \le 1. \tag{271}$$

Using the fact that of Eq. (307), we may find

$$\frac{E_4\left(\rho_{AB}\right)}{E_4\left(\rho_{AB}\right) - E_4\left(\rho_{AB}^{PT}\right)} \ge \frac{E_4\left(\rho_{AB}\right)}{E_4\left(\rho_{AB}\right) - \kappa' E_4\left(\rho_{AB}^{PT}\right)}. \tag{272}$$

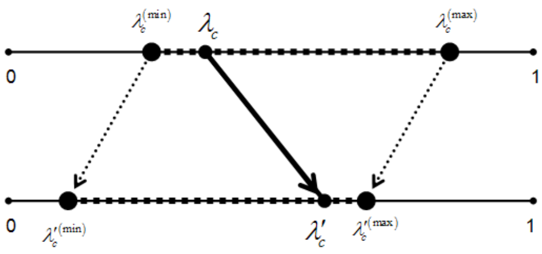

Figure 21: The upper and lower bounds, which are derived from Weyl's and Ostrowski's theorems, of λ'_c are decrease under LOCC operations (dash arrows). The arrow represents the result from the example in section 7.6.3.

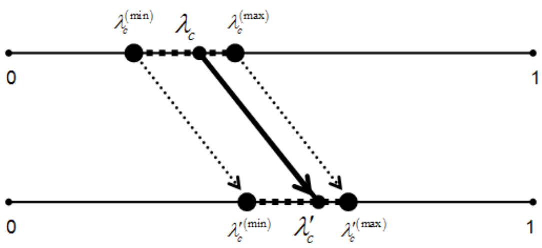

Figure 22: There should exist a better variation theorem of eigenvalues that the upper and lower bounds of λ_c do increase under LOCC operations (dash arrows). The arrow represents the result from the example in section 7.6.3.

We now conclude that the upper and lower limits of the critical value of λ do not increase under LOCC operations (see also Fig. 21.) Unfortunately, this result leads to the increasing of PT-entanglement measure at the lower and upper limits of the critical value

$$1-\lambda_c'^{(\min)} \geq 1-\lambda_c^{(\min)},$$
$$1-\lambda_c'^{(\max)} \geq 1-\lambda_c^{(\max)}.$$

However, the results of the examples in section 7 show that the PT-entanglement measure do not increase under LOCC operations. That means Weyl's variation theorem of eigenvalues is weak to show that the upper and lower limits of critical value of λ_c' do not increase under LOCC operations. We believe that there must exist the stronger variation theorem of eigenvalues in which leads to such conditions (see also Fig. 22)

$$1-\lambda_c'^{(\min)} \leq 1-\lambda_c^{(\min)},$$
$$1-\lambda_c'^{(\max)} \leq 1-\lambda_c^{(\max)}.$$

8 Conclusion

We have studied PPT-criterion to distinguish between entangled and separable states. We find that PPT is simple and works very well in low dimensional quantum systems. Then we use the PPT-concept to study how to differentiate the amount of entanglement for a given state. We form a convex combination between density matrix ρ_{AB} and ρ_{AB}^{PT};

$$\rho_{AB}(\lambda) = (1-\lambda)\rho_{AB} + \lambda\rho_{AB}^{PT}.$$

From this combination, we can define the entanglement measure called PT-entanglement measure

$$E_{PT}(\rho_{AB}) = 1-\lambda_c,$$

where λ_c has been defined in Eq. (152). To obtain the amount of entanglement, we have to search the value of λ, the largest value λ_c, in which the eigenvalues of the state $E_k(\rho_{AB}(\lambda))$ are positive. We find that for $2\otimes 2$ dimensional quantum system only one eigenvalue is negative. On the other hand, for $2\otimes 3$ there is that at least one eigenvalue is negative.

We have studied some characteristics of PT-entanglement measure for Werner and Bell diagonal states. We find that we always find the characteristic curves of the λ_c which is a function of the eigenvalues of the states ρ_{AB} and ρ_{AB}^{PT}. For the Werner states, it is the combination between entangled and separable states via a variable x, we find that the λ_c is a function of x (see Figs. 13 and 14)

$$\lambda_c = F(x).$$

If and only if the states ρ_{AB} and ρ_{AB}^{PT} commute, we obtain the explicit form of the function $F(x)$

$$F(x) = \frac{\nu_1}{\nu_1 - \mu_1},$$

where ν_1 is the lowest eigenvalue of ρ_{AB} and μ_1 is the lowest and negative eigenvalue of ρ_{AB}^{PT}.

For Bell diagonal states, it is the combination of the maximally entangled states (see Appendix A). Consider the convex combination

$$\sigma_{BD}(x) = (1-x)\rho_{BD}^{(1)} + x\rho_{BD}^{(2)},$$

where $\rho_{BD}^{(j)}$ are Bell diagonal states. Using PPT-criterion, we obtain two critical values of x. Then there are two regions of entangled states and there is one separable interval entangled regions. We also find that the critical value of λ is a function of the states $\sigma_{BD}(x)$ and $\sigma_{BD}^{PT}(x)$. In this case, we obtain two values of λ_c. The characteristic curves of the Bell diagonal states have been studied in section 7.3.3.

We have studied the basic properties of entanglement measures. In this paper, we mainly focus on three conditions:

E1. The entanglement measure $E(\rho)$ is zero if and only if the state ρ is separable.

E2. The entanglement measure $E(\rho)$ is invariant under local unitary transformations

$$E(U_{LO}\rho U_{LO}^{\dagger}) = E(\rho),$$

where $U_{LO} = U_A \otimes U_B$.

E3. The entanglement measure $E(\rho)$ does not increase under LOCC operations

$$E(\Phi_{LOCC}\rho) \leq E(\rho),$$

where Φ_{LOCC} is LOCC maps.

We have shown that for pure states, PT-entanglement measure can distinguish between pure separable and entangled states. The E_{PT} is zero if the pure state is separable. If the pure state is entangled, the E_{PT} is one. For pure entangled states, the PT-entanglement measure cannot differentiate the amount of entanglement among them. In the case of mixed states, we have shown that the E_{PT} is zero if and only if the mixed state is separable. The amount of entanglement can be differentiated by a function of a given state and its partial transpose,

$$E_{PT} = 1 - \lambda_c = 1 - F(x),$$

If the mixed state is entangled.

By using the Schmidt decomposition, we can show that the pure entangled states are invariant under local operations. For mixed states, we express the density in Fano expression which can be written in term of the Pauli's spin matrices. We know that under local operations, the eigenvalues of these matrices do not change. Then we show that the mixed states are also invariant under local transformations.

Under particular LOCC operations, we have shown that PT-entanglement measure does not increase in section 7.6.3

$$E_{PT}\left(\sum_{j=1}^{8} A_j \otimes B_j \rho_{AB} A_j^{\dagger} \otimes B_j^{\dagger}\right) \leq E_{PT}\left(\rho_{AB}\right).$$

It is still elusive in general whether PT-entanglement measure do increase or not. However, we use Wely's and Ostrowski's theorems of to derive the upper and lower bounds of the critical value of λ. Unfortunately, the PT-entanglement measure do increase at these bounds under LOCC operations.

We see that PT-entanglement measure can be shown to possess some fundamental properties attributed to a measure of entanglement. But some properties and problems remain open:

1. The λ_c is always zero if one of the eigenvalues for a considering state is zero either a pure or a mixed state. To solve this problem, we have to modify the convex combination between a given state and its partial transpose. It should be pointed out that we may form a convex combination between the identity matrix and the partial transpose of a considering state. Obviously, the problem of zero eigenvalues will be washed up. This work is in progress.

2. We have shown that in some specific types of LOCC operations, the PT-entanglement measure does not increase. By using Weyl's and Ostrowski's theorems of eigenvalues, we have shown that the critical value of λ does not increase *on the average* under LOCC operations (see Fig. 21). This result leads to increase of the PT-entanglement measure at these limits. However, this result do not agree with the examples for some certain types of LOCC operations. Then we expect that there must exist a stronger variation theorem of eigenvalues leading the PT-entanglement measure do decrease at the upper and lower limits (see Fig. 22).

3. In order to claim that PT-entanglement measure is monotone, we have to show that it also satisfies the convexity. This work is in progress.

4. PT-entanglement measure is restricted only on $2 \otimes 2$ and $2 \otimes 3$ because PPT-criterion is exact only in these dimensions. Then in higher dimensions, there are some states, which are called bound states, that PPT-criterion cannot detect and we believe that our measure will not appropriate in that dimensions as well. However, it is possible to modify our definition of entanglement measure by using reshuffling criterion [14] instance PPT-criterion. This part is also in progress.

9 Appendix A

In this section, we will present examples of PT-entanglement measure for Bell decomposable states. Before we start to solve the problem, we would like to give you briefly review on Bell-diagonal states (BD) and some their properties. A BD state is defined by

$$\rho_{BD} = p_1 \left|\Phi^+\right\rangle \left\langle\Phi^+\right| + p_2 \left|\Phi^-\right\rangle \left\langle\Phi^-\right| + p_1 \left|\Psi^+\right\rangle \left\langle\Psi^+\right| + p_1 \left|\Psi^-\right\rangle \left\langle\Psi^-\right|, \tag{273}$$

with $0 \leq p_i \leq 1$ and $\sum_{i=1}^{4} p_i = 1$. The Bell states are defined in Eq. (115) and (116).

According to the Fano form, we know that we can represent the state ρ_{BD} in term of Pauli's matrices. Then ρ_{BD} can be written as

$$\rho_{BD} = \frac{1}{4}\left(I_2 \otimes I_2 + \sum_{i=1}^{3} t_i \sigma_i \otimes \sigma_i\right), \tag{274}$$

where

$$\begin{aligned} t_1 &= p_1 - p_2 + p_3 + p_4, \\ t_2 &= -p_1 + p_2 + p_3 - p_4, \\ t_1 &= p_1 + p_2 - p_3 - p_4. \end{aligned} \tag{275}$$

For the state ρ_{BD} to be positive, $\rho_{BD} \geq 0$, the parameters t_i have to satisfy the following inequalities

$$\begin{aligned} 1 + t_1 - t_2 + t_3 &\geq 0, \\ 1 - t_1 + t_2 + t_3 &\geq 0, \\ 1 + t_1 + t_2 - t_3 &\geq 0, \\ 1 - t_1 - t_2 - t_3 &\geq 0. \end{aligned} \tag{276}$$

Above equations form a tetrahedron with its vertices located at $(1, -1, 1)$, $(-1, 1, 1)$, $(1, 1 - 1)$ and $(-1, -1, -1)$ [56]. We can see that its vertices represent the Bell states given in Eqs. (115) and (116), respectively (see Fig. 23)

According to the PPT-criterion for separability (see also section 5) [2, 25], a two-qubit state is separable if and only if its partial transpose is positive. That

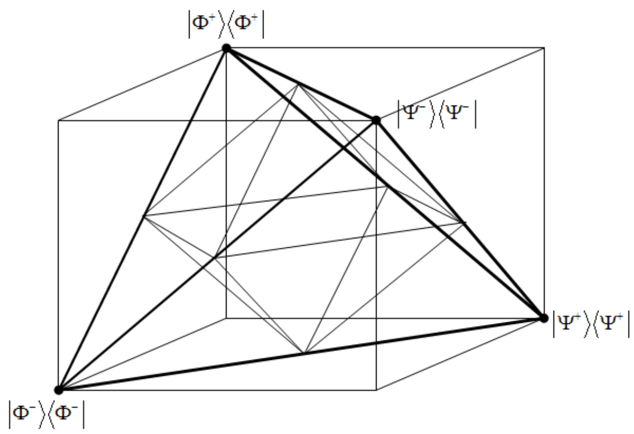

Figure 23: This figure shows the geometrical representation of Bell states.

means ρ_{BD} is separable if and only if t_i satify Eq. (276) and thus

$$\begin{aligned} 1+t_1+t_2+t_3 &\geq 0, \\ 1-t_1-t_2+t_3 &\geq 0, \\ 1+t_1-t_2-t_3 &\geq 0, \\ 1-t_1+t_2-t_3 &\geq 0. \end{aligned} \tag{277}$$

Inequalities (276) and (277) form an octahedron with its vertices located at $(\pm1,0,0)$, $(0,\pm1,0)$ and $(0,0,\pm1)$. We find that the tetrahedron of Eqs. (276) consists of five regions: a central region are separable region. There are other four smaller equivalent tetrahedral regions, at the corners, corresponding to entangled states (see Fig. 23).

In the following examples, we would like to show how the entanglement measure E_{PT} quantifies the entanglement for BD states.

Example A.1: Consider the convex combination of the states

$$\rho_1(p) = p\left|\Phi^+\right\rangle\left\langle\Phi^+\right| + (1-p)\left|\Phi^-\right\rangle\left\langle\Phi^-\right|, \tag{278}$$

$$\rho_2(q) = q\left|\Psi^+\right\rangle\left\langle\Psi^+\right| + (1-q)\left|\Psi^-\right\rangle\left\langle\Psi^-\right|, \tag{279}$$

where $p,q\in[0,1]$. We know that the mixing between two maximally entangled states will give a separable state. Then, obviously, the states ρ_1 and ρ_2 are separable if and only if p and q are $1/2$. We also observe that there exists a unitary transform $I_2\otimes Z$ such that

$$(I_2\otimes Z)\rho_1(p) = \rho_2(p), \tag{280}$$

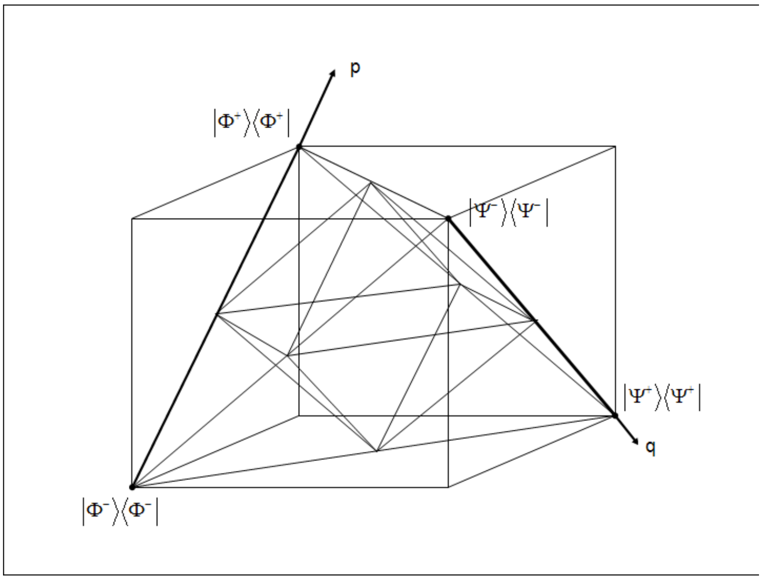

Figure 24: This figure shows geometrical representation of states $\rho_1(p)$ and $\rho_2(q)$.

where Z is a gate which is given by

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The geometric representation of above states can be found in Fig 24. Next we form a convex combination of ρ_1 and ρ_2 via the variable x as

$$\begin{aligned} \rho_{BD}(x,p,q) &= (1-x)\rho_1(p) + x\rho_2(q) \ , \quad x \in [0,1] \\ &= \frac{1}{2}\begin{pmatrix} 1-x & 0 & 0 & (1-x)(2p-1) \\ 0 & x & x(2q-1) & 0 \\ 0 & x(2q-1) & x & 0 \\ (1-x)(2p-1) & 0 & 0 & 1-x \end{pmatrix} \end{aligned} \tag{281}$$

and the eigenvalues of $\rho_{BD}(x,p,q)$ are

$$\begin{aligned} \nu^1_{BD}(x,p,q) &= qx, \\ \nu^2_{BD}(x,p,q) &= p(1-x), \\ \nu^3_{BD}(x,p,q) &= 1-p-x+px, \\ \nu^4_{BD}(x,p,q) &= x(1-q). \end{aligned} \tag{282}$$

Its partially transposed matrix is given by

$$\begin{aligned}
\rho_{BD}^{PT}(x,p,q) &= (1-x)\rho_1^{PT}(p) + x\rho_2^{PT}(q) \\
&= \frac{1}{2}\begin{pmatrix} 1-x & 0 & 0 & x(2q-1) \\ 0 & x & (1-x)(2p-1) & 0 \\ 0 & (1-x)(2p-1) & x & 0 \\ x(2q-1) & 0 & 0 & 1-x \end{pmatrix},
\end{aligned} \tag{283}$$

having eigenvalues

$$\begin{aligned}
\mu_{BD}^1(x,p,q) &= 1/2\,(2p-1+2x-2px)\,, \\
\mu_{BD}^2(x,p,q) &= 1/2\,(1-2p+2px)\,, \\
\mu_{BD}^3(x,p,q) &= 1/2\,(1-2qx)\,, \\
\mu_{BD}^4(x,p,q) &= 1/2\,(1-2x+2qx)\,.
\end{aligned} \tag{284}$$

Next, we will calculate the critical value of x of the state $\rho_{BD}(x,p,q)$. Later, the critical value of λ will be evaluated. To obtain the critical value of x, we now consider in three situations

1. If $0 \leq p < 1/2$ and $0 \leq q < 1/2$, we see that the first and fourth eigenvalues are negative when ever

$$\mu_{BD}^1 = 1/2\,(2p-1+2x+2px) \geq 0 \rightarrow x \geq \frac{2p-1}{2p-2},$$

$$\mu_{BD}^4 = 1/2\,(1-2x+2qx) \geq 0 \rightarrow x \leq \frac{1}{2-2q}.$$

Thus, the state $\rho_{BD}(x,p,q)$ is separable if

$$\frac{(2p-1)}{(2q-2)} \leq x \leq \frac{1}{(2-2q)},$$

otherwise it is entangled.

2. If $0 \leq p < 1/2$ and $1/2 < q \leq 1$, we see that the first and third eigenvalues are negative when ever

$$\mu_{BD}^1 = 1/2\,(2p-1+2x+2px) \geq 0 \rightarrow x \geq \frac{2p-1}{2p-2},$$

$$\mu_{BD}^3 = 1/2\,(1-2qx) \geq 0 \rightarrow x \leq \frac{1}{2q}$$

The state $\rho_{BD}(x)$ is separable if

$$\frac{(2p-1)}{(2q-2)} \leq x \leq \frac{1}{2q},$$

otherwise it is entangled.

3. If $1/2 < p \leq 1$ and $0 \leq q < 1/2$, we see that the first and third eigenvalues are negative when ever

$$\mu^1_{BD} = 1/2\,(2p - 1 + 2x + 2px) \geq 0 \rightarrow x \geq \frac{2p-1}{2p-2},$$

$$\mu^4_{BD} = 1/2\,(1 - 2x + 2qx) \geq 0 \rightarrow x \leq \frac{1}{2-2q}.$$

The state $\rho_{BD}(x)$ is separable if

$$\frac{(2p-1)}{(2q-2)} \leq x \leq \frac{1}{(2-2q)},$$

otherwise it is entangled.

In order to determine the PT-entanglement measure, we have to form the convex combination of the state $\rho_{BD}(x,p,q)$ with its partial transpose $\rho^{PT}_{BD}(x,p,q)$. We obtain; $\lambda \in [0,1]$

$$\begin{aligned}
\rho_{BD}(\lambda; x,p,q) &= (1-\lambda)\rho_{BD}(x,p,q) + \lambda\rho^{PT}_{BD}(x,p,q) \\
&= \begin{pmatrix} (1-x)/2 & 0 & 0 & B(p,q) \\ 0 & x/2 & A(p,q) & 0 \\ 0 & A(p,q) & x/2 & 0 \\ B(p,q) & 0 & 0 & (1-x)/2 \end{pmatrix}, \qquad (285)
\end{aligned}$$

where

$$A(p,q) = (p-1/2)\lambda + x(q - 1/2 + \lambda - p\lambda - q\lambda),$$
$$B(p,q) = (2p-1)(x-1)(\lambda-1)/2 + (q-1/2)x\lambda.$$

Its eigenvalues can be found as

$$\begin{aligned}
E^1_{BD}(\lambda; p,q,x) &= 1 - \lambda/2 + x(\lambda - 1 - q\lambda) + p(x - 1 - \lambda - x\lambda), \\
E^2_{BD}(\lambda; p,q,x) &= (p - 1/2 + x - px)\lambda + qx(1-\lambda), \\
E^3_{BD}(\lambda; p,q,x) &= p(x-1)(\lambda-1) + 1/2(1 + 2(q-1)x)\lambda, \\
E^4_{BD}(\lambda; p,q,x) &= x(1 + q(\lambda-1) + (p-1)\lambda) + \lambda/2(1-2p). \qquad (286)
\end{aligned}$$

According to the three conditions of the critical values of x, we can analyse the results as follows:

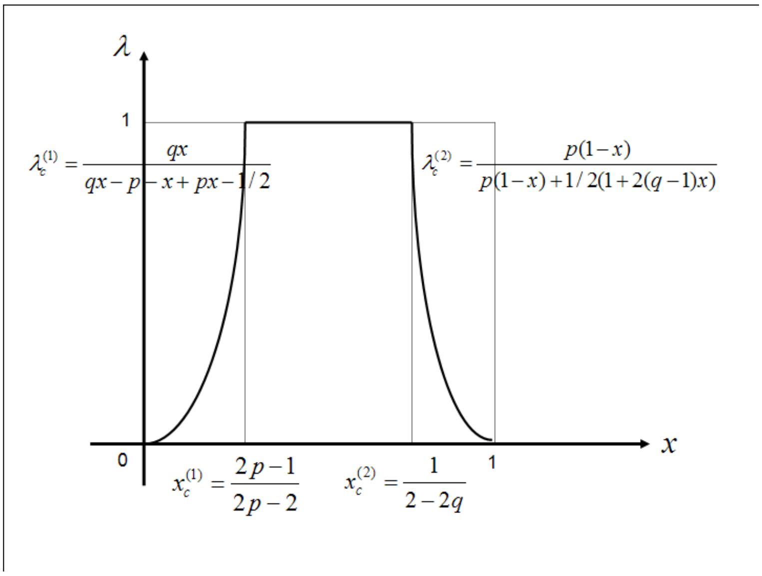

Figure 25: This figure shows the relation between x and λ for $0 \leq p < 1/2$ and $0 \leq q < 1/2$.

1. If $0 \leq p < 1/2$ and $0 \leq q < 1/2$, we see that the second eigenvalue is positive as long as

$$\lambda^{(1)} \leq \frac{qx}{qx - p - x + px - 1/2} \; ; \quad 0 \leq x \leq \frac{2p-1}{2p-2}$$

and

$$\lambda_c^{(1)} = \frac{qx}{qx - p + 1/2 - x + px}.$$

On the other hand, the third eigenvalue is positive as long as

$$\lambda^{(2)} \leq \frac{p(1-x)}{p(1-x) + (1 + 2(q-1)x)/2} \; ; \quad \frac{1}{2-2q} \leq x \leq 1$$

and

$$\lambda_c^{(2)} = \frac{p(1-x)}{p(1-x) + 1/2(1 + 2(q-1)x)}.$$

The geometric relation between x and λ can be found in Fig. 25.

2. If $0 \leq p < 1/2$ and $1/2 < q \leq 1$, we see that the second eigenvalue is positive as long as

$$\lambda^{(1)} \leq \frac{qx}{qx - p - x + px - 1/2} \; ; \quad 0 \leq x \leq \frac{2p-1}{2p-2}$$

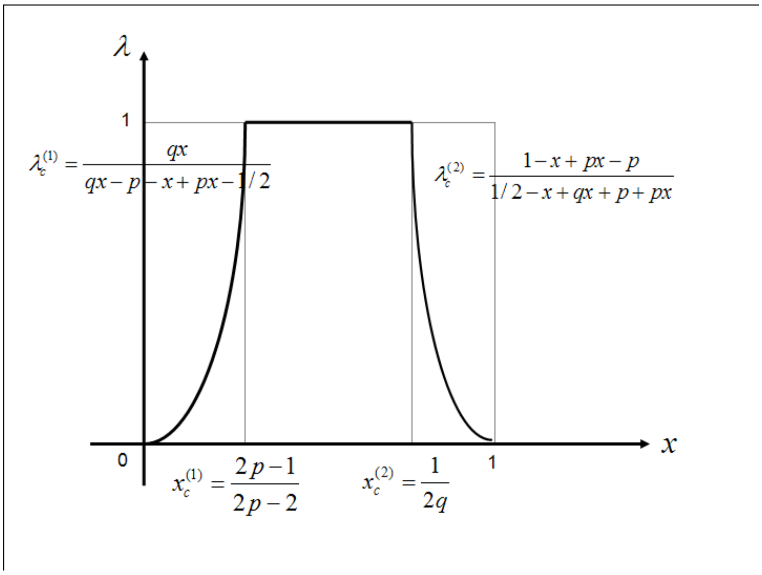

Figure 26: This figure shows the relation between x and λ_c for $0 \leq p < 1/2$ and $1/2 < q \leq 1$.

and

$$\lambda_c^{(1)} = \frac{qx}{qx - p + 1/2 - x + px}.$$

On the other hand, the first eigenvalue is positive as long as

$$\lambda^{(2)} \leq \frac{1 - x + px - p}{1/2 - x + qx + p + px} \quad ; \quad \frac{1}{2q} \leq x \leq 1$$

and

$$\lambda_c^{(2)} = \frac{1 - x + px - p}{1/2 - x + qx + p + px}.$$

The geometric relation between x and λ_c can be found in Fig. 26.

3. If $1/2 < p \leq 1$ and $0 \leq q < 1/2$, we see that the fourth eigenvalue is positive as long as

$$\lambda^{(1)} \leq \frac{(1-q)x}{p - 1/2 + x - xp - xq} \quad ; \quad 1 \leq x \leq \frac{1}{2 - 2q}$$

and

$$\lambda_c^{(1)} = \frac{(1-q)x}{p - 1/2 + x - xp - xq}.$$

On the other hand, the third eigenvalue is positive as long as

$$\lambda^{(2)} \leq \frac{1 - x + px - p}{1/2 - x + qx + p + px} \quad ; \quad \frac{1}{2q} \leq x \leq 1$$

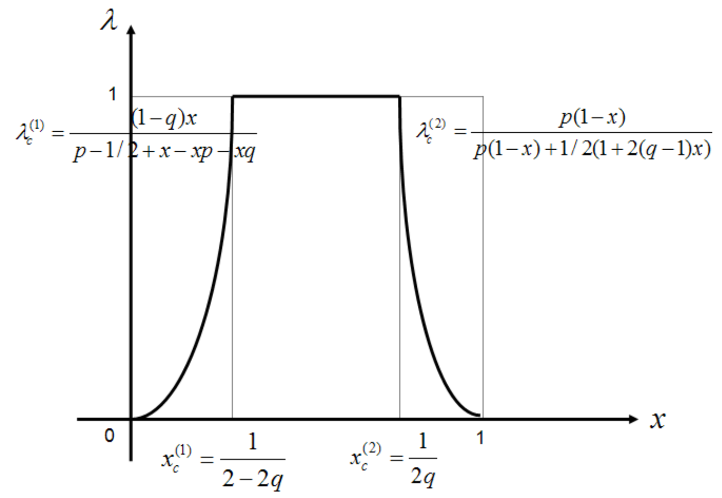

Figure 27: This figure shows the relation between x and λ for $1/2 < p \leq 1$ and $0 \leq q < 1/2$.

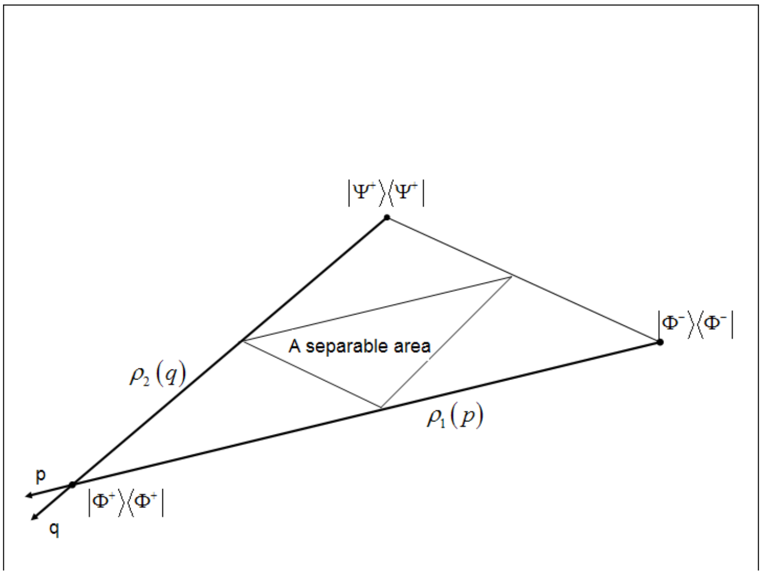

Figure 28: This figure shows geometrical representation of states $\rho_1(p)$ and $\rho_2(q)$.

and

$$\lambda_c^{(2)} = \frac{p(1-x)}{p(1-x)+1/2(1+2(q-1)x)}.$$

The geometric relation between x and λ_c can be found in Fig. 27.

This example shows that the symmetric distribution of Bell diagonal state. We find that there are two curves which are represented by $\lambda_c^{(1)}$ and $\lambda_c^{(2)}$, respectively. We will discuss in more details in the next section.

Example A.2: In this example, we will consider another convex combination of Bell diagonal states. Let us introduce

$$\sigma_1(p) = p\left|\Phi^+\right\rangle\left\langle\Phi^+\right| + (1-p)\left|\Phi^-\right\rangle\left\langle\Phi^-\right|, \tag{287}$$

$$\sigma_2(q) = q\left|\Phi^+\right\rangle\left\langle\Phi^+\right| + (1-q)\left|\Psi^+\right\rangle\left\langle\Psi^+\right|, \tag{288}$$

where $p, q \in [0,1]$. The states σ_1 and σ_2 are separable if p and q are 1/2, see Fig. 28. In this case, we cannot find a local unitary transforms between these states. We now form the convex combination

$$\begin{aligned} \sigma_{BD}(x,p,q) &= (1-x)\sigma_1(p) + x\sigma_2(q) \\ &= \frac{1}{2}\begin{pmatrix} 1+x(q-1) & 0 & 0 & a(p,q) \\ 0 & x(q-1) & x(q-1) & 0 \\ 0 & x(q-1) & x(q-1) & 0 \\ a(p,q) & 0 & 0 & 1+x(q-1) \end{pmatrix} \end{aligned} \tag{289}$$

where $a(p,q) = x(q+1) - 2p(x-1) - 1$ and $x \in [0,1]$. Its eigenvalues are

$$\begin{aligned} \nu_{BD}^1(x,p,q) &= 0, \\ \nu_{BD}^2(x,p,q) &= p - px, \\ \nu_{BD}^3(x,p,q) &= x - xq, \\ \nu_{BD}^4(x,p,q) &= 1 - p - x + px + qx. \end{aligned} \tag{290}$$

Its partial transpose can be found as

$$\begin{aligned} \sigma_{BD}^{PT}(x,p,q) &= (1-x)\sigma_1^{PT}(p) + x\sigma_2^{PT}(q) \\ &= \frac{1}{2}\begin{pmatrix} 1+x(q-1) & 0 & 0 & x(q-1) \\ 0 & x(q-1) & a(p,q) & 0 \\ 0 & a(p,q) & x(q-1) & 0 \\ x(q-1) & 0 & 0 & 1+x(q-1) \end{pmatrix} \end{aligned} \tag{291}$$

The eigenvalues of the partial transpose matrix are

$$\begin{aligned} \mu_{BD}^1(x,p,q) &= 1/2, \\ \mu_{BD}^2(x,p,q) &= (2p-1+2x(1-p))/2, \\ \mu_{BD}^3(x,p,q) &= (1-2p+2x(p-q))/2, \\ \mu_{BD}^4(x,p,q) &= (1-2x(1-q))/2. \end{aligned} \tag{292}$$

To obtain the condition for separability, we need to consider four cases separately:

1. If $0 \leq p < 1/2$ and $0 \leq q < 1/2$, the second and fourth eigenvalues are positive as long as

$$\mu^2_{BD}(x,p,q) = 1/2(2p - 1 + 2x(1-p)) \geq 0 \rightarrow x \geq \frac{1-p}{1-2p},$$

$$\mu^4_{BD}(x,p,q) = 1/2(1 - 2x(1-q)) \geq 0 \rightarrow x \leq \frac{1}{2(1-q)}.$$

The state σ_{BD} is separable if

$$\frac{(1-p)}{(1-2p)} \leq x \leq \frac{1}{(2(1-q))},$$

otherwise it is entangled.

2. If $1/2 < p \leq 1$ and $1/2 < q \leq 1$, the third eigenvalue always has a negative sign,

$$\mu^3_{BD}(x,p,q) = (1 - 2p + 2x(p-q))/2 \leq 0.$$

3. If $0 \leq p < 1/2$ and $1/2 < q \leq 1$, the second and third eigenvalues are positive as long as

$$\mu^2_{BD}(x,p,q) = 1/2(2p - 1 + 2x(1-p)) \geq 0 \rightarrow x \geq \frac{1-p}{1-2p},$$

$$\mu^3_{BD}(x,p,q) = 1/2(1 - 2p + 2x(p-q)) \geq 0 \rightarrow x \leq \frac{1-2p}{2(p-q)}.$$

The state σ_{BD} is separable if

$$\frac{(1-p)}{(1-2p)} \leq x \leq \frac{(1-2p)}{(2(p-q))},$$

otherwise it is entangled.

4. If $1/2 < p \leq 1$ and $1/2 < q \leq 1$, the third and fourth eigenvalues are positive as long as

$$\mu^3_{BD}(x,p,q) = 1/2(1 - 2p + 2x(p-q)) \geq 0 \rightarrow x \geq \frac{1-2p}{2(q-p)}.$$

$$\mu^4_{BD}(x,p,q) = 1/2(1 - 2x(1-q)) \geq 0 \rightarrow x \leq \frac{1}{2(1-q)}.$$

The state σ_{BD} is separable if

$$\frac{(1-2p)}{(q-p)} \leq x \leq \frac{1}{(2(1-q))},$$

otherwise it is entangled.

To quantify the amount of entanglement of σ_{BD}, we form the convex combination with its partial transpose,

$$\begin{aligned}\sigma_{BD}(\lambda; x, p, q) &= (1-\lambda)\sigma_{BD}(x,p,q) + \lambda\sigma_{BD}^{PT}(x,p,q) \\ &= \frac{1}{2}\begin{pmatrix} 1+x(q-1) & 0 & 0 & M(p,q) \\ 0 & x(1-q) & N(p,q) & 0 \\ 0 & N(p,q) & x(1-q) & 0 \\ M(p,q) & 0 & 0 & 1+x(q-1) \end{pmatrix},\end{aligned} \tag{293}$$

where $\lambda \in [0,1]$,and

$$M(p,q) = x(1-q) + 2p(x-1)(\lambda-1) + \lambda - 1,$$

$$N(p,q) = (2p-1)\lambda - x(2p\lambda + q - 1).$$

Its eigenvalues can be found as

$$\begin{aligned} E_{BD}^{1}(\lambda; x,p,q) &= \lambda(1-2p+2px-2qx)/2, \\ E_{BD}^{2}(\lambda; x,p,q) &= (2p-2x(p-q-p\lambda+q\lambda)+2p\lambda)/2, \\ E_{BD}^{3}(\lambda; x,p,q) &= (2-2p-2x(1-p+p\lambda-q\lambda)+2p\lambda)/2, \\ E_{BD}^{4}(\lambda; x,p,q) &= (2x(1-q-p\lambda+q\lambda)-\lambda+2p\lambda)/2. \end{aligned} \tag{294}$$

According to four parameter ranges introduced above, we analyze the results as follows:

1. If $0 \leq p < 1/2$ and $0 \leq q < 1/2$, the third eigenvalue is positive as long as

$$\lambda^{(1)} \leq \frac{1-p-x+xp}{xp-xq-p} \quad ; \quad 0 \leq x \leq \frac{1-p}{1-2q},$$

and thus

$$\lambda_c^{(1)} = \frac{1-p-x+xp}{xp-xq-p},$$

while the fourth eigenvalue is positive as long as

$$\lambda^{(2)} \leq \frac{2x(1-q)}{1-2p-2xq+2xp} \quad ; \quad \frac{1}{2(1-q)} \leq x \leq 1,$$

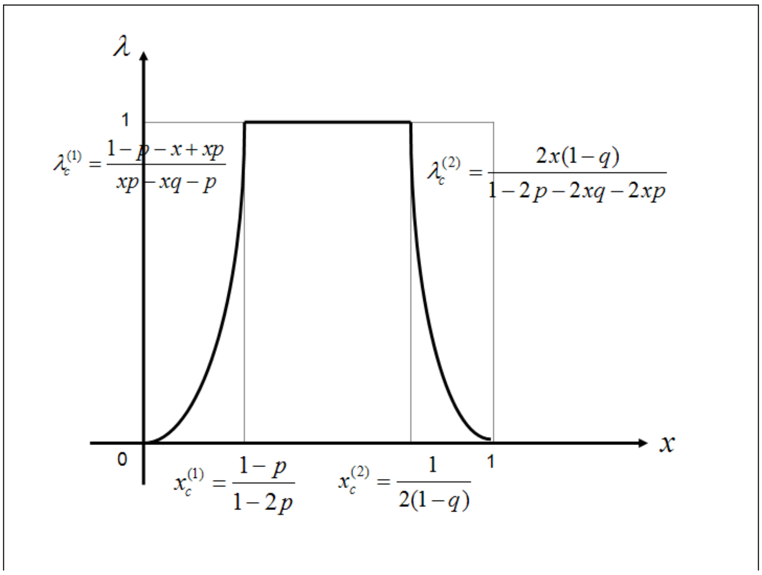

Figure 29: This figure shows the relation between x and λ for $0 \leq p < 1/2$ and $0 \leq q < 1/2$.

and

$$\lambda_c^{(2)} = \frac{2x(1-q)}{1-2p-2xq+2xp}.$$

The geometric relation between x and λ_c can be found in Fig. 29.

2. If $1/2 < p \leq 1$ and $1/2 < q \leq 1$, the first eigenvalue is always negative,

$$E^1_{BD}(\lambda; x, p, q) = \lambda(1 - 2p + 2px - 2qx)/2 \leq 0.$$

We observe that the first eigenvalue can be written in terms of $E^3_{BD}(x, p, q)$ as

$$E^1_{BD}(\lambda; x, p, q) = \lambda E^3_{BD}(x, p, q).$$

We know that $E^3_{BD}(x, p, q)$ takes negative values for all values of p and q in the second condition of separability. In this case, the state $\sigma_{BD}(\lambda; x)$ is a density matrix if and only if $\lambda = 0$. If we add a infinitesimal amount into λ as $\lambda = 0 + \varepsilon$, the first eigenvalue immediately produces a negative sign. That means that the critical value of λ is zero. We will discuss this result in the next section.

3. If $0 \leq p < 1/2$ and $1/2 < q \leq 1$, the first eigenvalue is always negative:

$$E^1_{BD}(\lambda; x, p, q) = \lambda(1 - 2p + 2px - 2qx)/2 \leq 0 \ ; \quad \frac{1-2p}{2(p-q)} \leq x \leq 1,$$

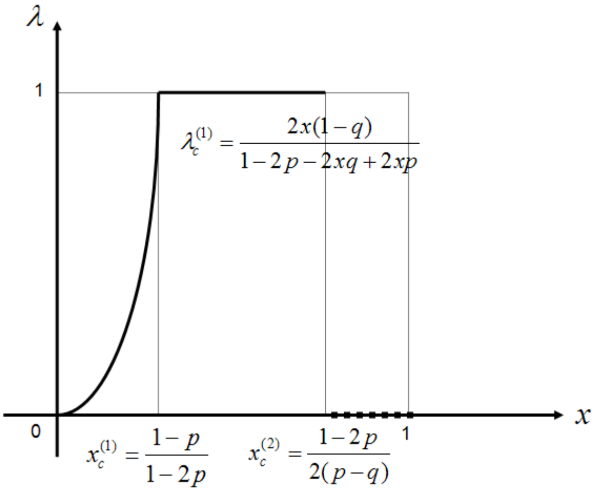

Figure 30: This figure shows the relation between x and λ for $0 \leq p < 1/2$ and $1/2 \leq q < 1$.

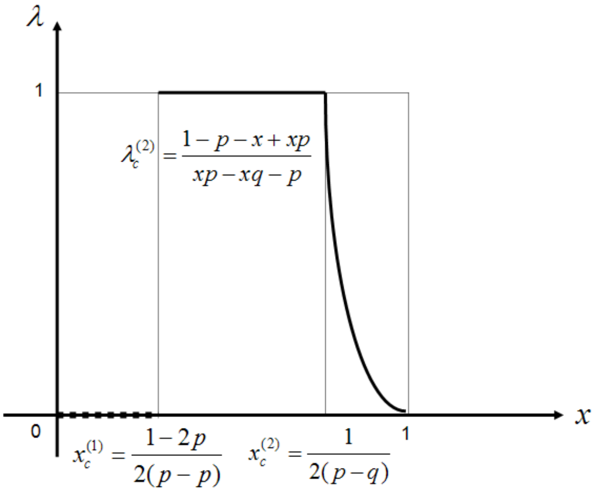

Figure 31: This figure shows the relarion between x and λ for $1/2 < p \leq 1$ and $0 \leq q < 1/2$.

and we also write

$$E^1_{BD}(\lambda;x,p,q)=\lambda E^3_{BD}(x,p,q),$$

while the fourth eigenvalue is positive as long as

$$\lambda^{(1)}\le\frac{2x(1-q)}{1-2p-2xq+2xp}\quad;\quad 0\le x\le\frac{1-p}{1-2p},$$

and

$$\lambda^{(1)}_c=\frac{2x(1-q)}{1-2p-2xq+2xp}.$$

The relation between x and λ_c is illustrated in Fig. 30.

4. If $1/2<p\le 1$ and $0\le q<1/2$, the first eigenvalue is always negative:

$$E^1_{BD}(\lambda;x,p,q)=\lambda(1-2p+2px-2qx)/2\le 0.\ ;\quad 1\le x<\frac{(1-2p)}{2(p-q)},$$

and we also write

$$E^1_{BD}(\lambda;x,p,q)=\lambda E^3_{BD}(x,p,q),$$

while the third eigenvalue is positive as long as

$$\lambda^{(1)}\le\frac{1-p-x+xp}{xp-xq-p}\quad;\quad \frac{1}{2(p-q)}\le x\le 1,$$

and

$$\lambda^{(1)}_c=\frac{1-p-x+xp}{xp-xq-p}.$$

The relation between x and λ_c has been shown in Fig. 31.

In this example, there are some cases that the critical value of λ is zero. This situation arises because the state $\sigma_{BD}(x)$ has zero eigenvalue. Then we will always find, for some appropriate values of p,q,x, that the eigenvalues of the state $\sigma_{BD}(\lambda;x)$ are negative values for any positive $\lambda>0$.

However, if we calculate the Negativity of the state $\sigma_{BD}(x)$, we find that

$$N(\sigma_{BD}(x))=\begin{cases}|E^3_{BD}(x,p,q)| & ;\ \frac{1}{2}<p\le 1\ ,\frac{1}{2}<q\le 1\\ 0 & ;\text{otherwise}\end{cases}.\qquad(295)$$

Then we can rewrite the first eigenvalue of the state $\sigma_{BD}(\lambda;x)$ as

$$E^1_{BD}(\lambda;x,p,q)=-\lambda N(\sigma_{BD}(x)).$$

Here we recover the negativity of entanglement measure.

10 Appendix B

In this section we would to calculate Eqs. (304) and (305). We now consider the RHS of Eq. (241)

$$0 \leq (1-\lambda_c')\, E_4\left(\sum_{i=1}^{N} M_i \rho_{AB} M_i^{\dagger}\right) + \lambda_c' E_1\left(\sum_{i=1}^{N} \tilde{M}_i \rho_{AB}^{PT} \tilde{M}_i^{\dagger}\right). \tag{296}$$

Using Weyl's theorm, we can show that

$$E_4\left(\sum_{i=1}^{N} M_i \rho M_i^{\dagger}\right) \leq \sum_{i=1}^{N} E_4\left(M_i \rho M_i^{\dagger}\right), \tag{297}$$

and

$$E_1\left(\sum_{i=1}^{N} \tilde{M}_i \rho^{PT} \tilde{M}_i^{\dagger}\right) \leq E_4\left(\tilde{M}_1 \rho^{PT} \tilde{M}_1^{\dagger}\right) + \sum_{i=2}^{N} E_1\left(\tilde{M}_i \rho^{PT} \tilde{M}_i^{\dagger}\right). \tag{298}$$

Inserting Eqs. (297) and (298) into Eq. (296), we have

$$0 \leq (1-\lambda_c')\left[\sum_{i=1}^{N} E_4\left(M_i \rho M_i^{\dagger}\right)\right] + \lambda_c'\left[E_4\left(\tilde{M}_1 \rho^{PT} \tilde{M}_1^{\dagger}\right) + \sum_{i=2}^{N} E_1\left(\tilde{M}_i \rho^{PT} \tilde{M}_i^{\dagger}\right)\right], \tag{299}$$

and we also use the relations in Eqs. (248) and (251). Then Eq. (299) becomes

$$\begin{aligned} 0 &\leq (1-\lambda_c')\left[\sum_{i=1}^{N} E_4\left(M_i M_i^{\dagger}\right) E_4\left(\rho_{AB}\right)\right] \\ &+ \lambda_c'\left[E_4\left(\tilde{M}_1 \tilde{M}_1^{\dagger}\right) E_4\left(\rho^{PT}\right) + \sum_{i=2}^{N} E_1\left(\tilde{M}_i \tilde{M}_i^{\dagger}\right) E_1\left(\rho^{PT}\right)\right]. \end{aligned} \tag{300}$$

We know that

$$E_1(\rho_{AB}^{PT}) \leq E_4(\rho_{AB}^{PT}). \tag{301}$$

Then we can write

$$\sum_{i=2}^{N} E_1\left(\tilde{M}_i \tilde{M}_i^{\dagger}\right) E_1\left(\rho^{PT}\right) \leq \sum_{i=2}^{N} E_1\left(\tilde{M}_i \tilde{M}_i^{\dagger}\right) E_4\left(\rho^{PT}\right). \tag{302}$$

Inserting Eq. (302) into Eq. (300), we obtain

$$\begin{aligned} 0 &\leq (1-\lambda_c')\left[\sum_{i=1}^{N} E_4\left(M_i M_i^{\dagger}\right) E_4\left(\rho_{AB}\right)\right] \\ &+ \lambda_c'\left[E_4\left(\tilde{M}_1 \tilde{M}_1^{\dagger}\right) E_4\left(\rho^{PT}\right) + \sum_{i=2}^{N} E_1\left(\tilde{M}_i \tilde{M}_i^{\dagger}\right) E_4\left(\rho^{PT}\right)\right], \end{aligned} \tag{303}$$

and the upper bound of λ'_c is

$$\lambda'_c \leq \frac{E_4\left(\rho_{AB}\right)}{E_4\left(\rho_{AB}\right) - \kappa' E_4\left(\rho_{AB}^{PT}\right)} = \lambda_c'^{(max)}, \tag{304}$$

where

$$\kappa' = \frac{E_4\left(M_1^\dagger M_1\right) + \sum_{i=2}^{N} E_1\left(\tilde{M}_i^\dagger \tilde{M}_i\right)}{\sum_{i=2}^{N} E_4\left(M_i^\dagger M_i\right)}. \tag{305}$$

Using the RHS of Eq. (260) for $k = 4$ and the LHS of Eq. (259) for $k = 1$, it follows that

$$\sum_{i=1}^{N} E_4\left(M_i M_i^\dagger\right) \geq E_4\left(\tilde{M}_1 \tilde{M}_1^\dagger\right) + \sum_{i=2}^{N} E_1\left(\tilde{M}_i \tilde{M}_i^\dagger\right), \tag{306}$$

and we also use the fact that $E_k(M_i M_i^\dagger) = E_k(M_i^\dagger M_i)$. Then we have

$$\kappa' = \frac{E_4\left(M_1^\dagger M_1\right) + \sum_{i=2}^{N} E_1\left(\tilde{M}_i^\dagger \tilde{M}_i\right)}{\sum_{i=2}^{N} E_4\left(M_i^\dagger M_i\right)} \leq 1. \tag{307}$$

Using the fact that of Eq. (307), we may find

$$\frac{E_4\left(\rho_{AB}\right)}{E_4\left(\rho_{AB}\right) - E_4\left(\rho_{AB}^{PT}\right)} \geq \frac{E_4\left(\rho_{AB}\right)}{E_4\left(\rho_{AB}\right) - \kappa' E_4\left(\rho_{AB}^{PT}\right)}. \tag{308}$$

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