The impedance function of a confined polaron and bipolaron: The single path integral approach

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Abstract

We use the Single-Path-Integral to calculate the impedance function of the polaron and bipolaron in quantum confinement with the presence of the external field. The expectation values of the classical equation of motion is considered in order to obtain the impedance function. The mobility of the polaron and bipolaron in quantum confinement is also calculated in the direction parallel and perpendicular to the magnetic field. Without trapping, we also calculate the effective mass of the bipolaron in the magnetic field.

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1 Introduction

The polaron has long been studied by physicists. It is an electron moving in polar crystal together with the self-induced polarization of the lattice. The characteristics of polaron have been investigated by Feynman path integral [1].

In the past, most works on polarons were devoted to calculate the static properties such as the ground-state energy and the effective mass of the polaron at zero temperature [2, 3, 4, 5, 6, 7, 8]. The theory of Tyablikov [9] can prove results for both the weak and strong coupling limits, but this method cannot solve intermediate coupling behavior. Later, Feynman [1, 10] showed that path integral can recover the polaron problem for all coupling strength. The path integral method offers a unique advantage to solve the electron-phonon system [11, 12, 13, 14]. The phonon coordinates can be eliminated exactly, and the problem can be written in terms of the electron coordinates alone. However, written in terms of electron coordinates alone, the method cannot be completed without making some approximation. Feynman [1] attempted to simulate the exact influence function by an approximate harmonic one which imitated the exact functional as well as possible while still permitting the calculation to be completed. After deriving a variational principle which gave an upper bound to the ground-state energy of the polaron system, he used as an influence function the new well-know one-oscillator trial distribution whose strength and frequency are variational parameters used to minimize the ground-state energy at zero temperature. This method was soon generalized to finite temperature by Saitoh [15]. The difficulty arise again when we study the polaron in the magnetic field. The requirement that the actions of the polaron system are real after the transformation to imaginary time variables. This problem was already recognized by Feynman and Hibbs [10], who suspected that only a minor modification in the formulation of the Feynman inequality. The problem of the extension of the Feynman inequality to the case of a non-zero magnetic field has attracted particular attention because Feynman′s variational treatment of the polaron is superior for zero magnetic field. The approximation schemes [16, 17, 18, 19] have been developed for the free energy of a polaron in the magnetic field, based on the working hypothesis that the Feynman inequality remains valid for non-zero magnetic field. Although the argument of Larsen [19] is still a problem because the Feynman′ inequality indeed has no variational justification for a particle in the magnetic filed and does not provide an upper bound to the ground state for non-zero magnetic field [20]. Later, the problem of the ground-state energy of the polaron in the magnetic field has been solved by Devreese and Brosens [21]. They have extended the Feynman inequality to the case of a charged particle in a magnetic field.

The dynamical properties (optical absorption, mobility, response to a magnetic field) have also been calculated by using path integral formulation of the quadratic approximation [11, 12, 13]. The Double-Path-Integral [11, 12, 13, 14] has been used to analyze the response properties of a polaron to a constant electric field, the mobility, and to an electromagnetic field, the impedance function. Unfortunately, the Double-Path-Integral method is quite difficult because we have to perform double integrals (see appendix of the Ref. [11]). The alternative approaches have been investigated by many authors. Peeters and Devreese [47] have shown the alternative derivation of the Thornber-Feynman Theory [13]. They studied the polaron under the influence of the static external electric field. The Thornber-Feynman′s nonlinear conductivity was rederived by using Heisenberg equation of motion. At zero temperature and small coupling constant between electron and phonons, the average electron velocity is approximately two times the average velocity derived from a solution of the Boltzmann equation

[24]. Later, they have also redrived the Feynman-Hellwarth-Idding-Platzman theory [14] by using common operator techniques [23]. More recently, S. Yoo-kong [25] has applied the Single-Path-Integral method to solve the effective mass of the polaron in the presence of a magnetic and an electric fields. Obviously, the Single-Path-Integral is simpler than the Double-Path-Integral. Actually, Feynman's method of the polaron problem provides the method to calculate the effective mass of the polaron and we can use this to calculate such a property of a bipolaron.

Later, the bipolaron formulation was first studied by Pekar [4] and subsequently by Schultz [26]. The bipolaron is formed by two electrons or holes in polar crystal interacting with the lattice and themselves. Normally, the coupling must be strong enough in the crystal to overcome the Coulomb repulsion. Then the possibility to create the bipolaron depends on the competition between the Coulomb interactions of electrons or holes and attractive interactions via the distortion of the lattice induced by them. Consequently, the possibility problem of the bipolaron has been studied in the literature [27, 28, 29, 30, 31, 32, 33, 34, 35, 36]. The static properties of the bipolaron have been considered by various authors [37, 38, 39, 40, 41] and the problem of the bipolaron in the magnetic field has also been studied [42, 43, 44, 45].

Because of the development of microfrabrication technology, the dielectric slabs, heterojunction, quantum wires [49, 50] and quantum dots [51, 52] were created by using the synthesis polar semiconductor structure with low dimensionality. The electronics and optoelectronic device applications have been studied by many researchers. Recently, many authors have been directed toward exploring to the effect of the electron-phonon interaction on several electronic properties of quantum wire [53, 54, 55] and quantum dot [53, 56]. However, the properties of the polaron in bulk materials are drastically different from those in quantum wires and dots which confined the motion of the electrons in the plane transverse to the wire axis or in all the special direction of dot. More recently, the polarons and bipolarons in parabolic quantum low-dimensional systems have investigated by using Feynman path integral variational approach [57, 58, 59, 60, 61, 62]. It is all known that this method is very forceful and can be easily used to solve polaron problems.

In this contribution, we apply the Single-Path-Integral to analytically calculate the impedance function of the (bi)polaron in parabolic quantum dot and wire with the external fields, which could be used for further numerical calculation of other (bi)polaron properties. To obtain the impedance function, the equation of motion of the (bi)polaron in quantum confinement is calculated by using the Single-Path-Integral based on the minimization principle of the ground-state energy. Later, the mobility and the effective mass of the (bi)polaron are also studied. The Hamiltonian describing a bipolaron in quantum confinement with the external fields can be given by

$$
\begin{aligned}
H_{bi} &= \sum_{j=1,2}\left(\frac{\left(\vec{p}_j - q\vec{A}/c\right)^2}{2m} + q\vec{E}\left(\tau\right)\cdot\vec{r}\left(\tau\right) - V\left(\vec{\rho}_j, z_j\right)\right) + \frac{q^2}{\varepsilon_0\left|\vec{r}_1-\vec{r}_2\right|} \\
&+ \sum_{\vec{k}}\hbar\omega_{\vec{k}}a^{\dagger}_{\vec{k}}a_{\vec{k}} + \sum_{j=1,2}\sum_{\vec{k}}\left(C_{\vec{k}}a_{\vec{k}}e^{i\vec{k}\cdot\vec{r}_j(\tau)} + C^{*}_{\vec{k}}a^{\dagger}_{\vec{k}}e^{-i\vec{k}\cdot\vec{r}_j(\tau)}\right). \qquad (1)
\end{aligned}
$$

where $\vec{r}_j = [(\vec{\rho}_j = x_j, y_j), z_j]$ and $\vec{p}_j$ are the position and the momentum operators of the electrons. The gauge is $\vec{A} = (0, Bx, 0)$ with the z-axis in the direction of the magnetic field. $E(\tau)$ is the time dependent electric field which in this paper we take as $E(\tau) = E_0e^{-i\omega\tau}$, where E_0 is the amplitude and ω is the frequency. m is the mass of the electrons, $a_{\vec{k}}$ and $a^{\dagger}_{\vec{k}}$ are the creation and annihilation operators of a harmonic oscillator, and

$$
V\left(\vec{\rho}_j, z_j\right) = \frac{1}{2}\Omega_{\rho}^2\rho_j^2 + \frac{1}{2}\Omega_z^2 z_j^2
$$

is the confinement potential: $\Omega_z = 0$ for quantum wire and $\Omega_\rho = \Omega_z = \Omega$ for quantum dots, with Ω being in unit of $\omega_{\vec{k}}$, measuring the confining strength of the parabolic potential, and $C_{\vec{k}}$ is the Frohlich electron-phonon interaction

$$
C_{\vec{k}} = \frac{i\hbar\omega_{\vec{k}}}{\vec{k}}\sqrt{\frac{4\pi\alpha}{V}\sqrt{\frac{\hbar}{2m\omega_{\vec{k}}}}},
$$

with V being the crystal volume. The electron-phonon coupling constant α is given by

$$
\alpha = \frac{q^2}{\hbar\omega_{\vec{k}}}\sqrt{\frac{m\omega_{\vec{k}}}{2\hbar}}\left(\frac{1}{\varepsilon_\infty} - \frac{1}{\varepsilon_0}\right),
$$

where ε_0 is the static dielectric constant and ε_∞ is the high-frequency dielectric constant, and q is the electric charge of the electrons.

2 The impedance function

The propagator of the bipolaron can be expressed by

$$
K_{bi}\left\{\vec{r}_1(t), \vec{r}_2(t); \vec{r}_1(0), \vec{r}_2(0)\right\} = \int_0^t D\left(\vec{r}_1\right)\int_0^t D\left(\vec{r}_2\right)e^{\frac{i}{\hbar}S_{bi}\{\vec{r}_1(\tau),\vec{r}_2(\tau)\}}, \qquad (2)
$$

where the bipolaron action S_{bi}, which is obtained after the exact elimination of the phonon coordinates [1], is given by

$$\begin{aligned} S_{bi}\left\{\vec{r}_1(\tau),\vec{r}_2(\tau)\right\} &= \sum_{j=1,2}\int_0^t d\tau\left(\frac{m}{2}\dot{\vec{r}}_j^2(\tau)+\frac{m}{2}\dot{\vec{r}}_j(\tau)\cdot B\times\vec{r}_j(\tau)+q\vec{E}(\tau)\cdot\vec{r}_j(\tau)\right) \\ &- \int_0^t d\tau\left(\sum_{j=1,2}V\left(\vec{\rho}_j,z_j\right)+\frac{q^2}{\varepsilon_0\left|\vec{r}_1-\vec{r}_2\right|}\right) \\ &+ \frac{1}{2}\sum_{j,l=1,2}\sum_{\vec{k}}\left|C_{\vec{k}}\right|^2\int_0^t d\tau\int_0^t d\sigma G_{\omega_{\vec{k}}}\left(\tau-\sigma\right)e^{i\vec{k}\cdot\left(\vec{r}_j(\tau)-\vec{r}_l(\sigma)\right)}, \quad (3)\end{aligned}$$

where $G_{\omega_{\vec{k}}}$ is the memory function

$$G_{\omega_{\vec{k}}}\left(\vartheta\right)=\frac{\cos\omega_{\vec{k}}\left(\left|\vartheta\right|-t/2\right)}{\sin\omega_{\vec{k}}t/2}. \quad (4)$$

Let us introduce the center of mass $\vec{R}=[X,Y,Z]$ and the relative coordinates, $\vec{r}=[x,y,z]$,

$$\vec{R}=\frac{\vec{r}_1+\vec{r}_2}{2}\ ,\quad \vec{r}=\vec{r}_1-\vec{r}_2\ . \quad (5)$$

Using Eq. (5), we can represent the bipolaron action in term of $\vec{R}$ and $\vec{r}$ coordinates as

$$\begin{aligned} S_{bi} &= \int_0^t d\tau\left(m\dot{\vec{R}}^2(\tau)+m\dot{\vec{R}}(\tau)\cdot\vec{B}\times\vec{R}(\tau)+2q\vec{E}(\tau)\cdot\vec{R}(\tau)-V(X,Y,Z)\right) \\ &+ \int_0^t d\tau\left(\frac{m}{4}\dot{\vec{r}}^2(\tau)-\frac{q^2}{\varepsilon_0\left|\vec{r}(\tau)\right|}-V(x,y,z)\right) \\ &+ \sum_{\vec{k}}\left|C_{\vec{k}}\right|^2\int_0^t d\tau\int_0^t d\sigma G_{\omega_{\vec{k}}}\left(\tau-\sigma\right)e^{i\vec{k}\cdot\left(\vec{R}(\tau)-\vec{R}(\sigma)\right)}\left\{e^{i\vec{k}\cdot(\vec{r}(\tau)-\vec{r}(\sigma))}+e^{i\vec{k}\cdot(\vec{r}(\tau)+\vec{r}(\sigma))}\right. \\ &\left.+e^{-i\vec{k}\cdot(\vec{r}(\tau)-\vec{r}(\sigma))}+e^{-i\vec{k}\cdot(\vec{r}(\tau)+\vec{r}(\sigma))}\right\}, \quad (6)\end{aligned}$$

where

$$V\left(X,Y,Z\right)=\Omega_\rho^2\left(X^2+Y^2\right)+\Omega_z^2Z^2, \quad (7)$$

and

$$V\left(x,y,z\right)=\frac{1}{4}\Omega_\rho^2\left(x^2+y^2\right)+\frac{1}{4}\Omega_z^2z^2. \quad (8)$$

We now write the expectation values of the equation of motion of the center of mass coordinate and we set $q = c = 1$. We obtain

$$m\left\langle \ddot{\vec{R}}(t)\right\rangle - m\overleftrightarrow{\Omega}\cdot\left\langle \vec{R}(t)\right\rangle - \left\langle \dot{\vec{R}}(t)\right\rangle \times \vec{B} - \vec{E}(t) = \sum_{\vec{k}} \left|C_{\vec{k}}\right|^2 \vec{k} \left\langle \Xi_{\vec{k}}\right\rangle_t, \tag{9}$$

where $\overleftrightarrow{\Omega} = \mathrm{diag}(\Omega_\rho^2, \Omega_\rho^2, \Omega_z^2)$ and

$$\left\langle \Xi_{\vec{k}}\right\rangle_t = \frac{i}{2}\int_{-t}^{t} d\sigma G_{\omega_{\vec{k}}}\left(t-\sigma\right) D\left(t-\sigma\right), \tag{10}$$

and

$$D\left(t-\sigma\right) = \sum_{i=1}^{4} I_i, \tag{11}$$

with

$$I_1 = \left\langle e^{i\vec{k}\cdot\left(\vec{R}(\tau)-\vec{R}(\sigma)\right)} e^{i\vec{k}\cdot(\vec{r}(\tau)-\vec{r}(\sigma))}\right\rangle = \left\langle e^{i\vec{k}\cdot(\vec{r}_1(\tau)-\vec{r}_2(\sigma))}\right\rangle, \tag{12}$$

$$I_2 = \left\langle e^{i\vec{k}\cdot\left(\vec{R}(\tau)-\vec{R}(\sigma)\right)} e^{-i\vec{k}\cdot(\vec{r}(\tau)-\vec{r}(\sigma))}\right\rangle = \left\langle e^{i\vec{k}\cdot(\vec{r}_2(\tau)-\vec{r}_2(\sigma))}\right\rangle, \tag{13}$$

$$I_3 = \left\langle e^{i\vec{k}\cdot\left(\vec{R}(\tau)-\vec{R}(\sigma)\right)} e^{i\vec{k}\cdot(\vec{r}(\tau)+\vec{r}(\sigma))}\right\rangle = \left\langle e^{i\vec{k}\cdot(\vec{r}_1(\tau)-\vec{r}_2(\sigma))}\right\rangle, \tag{14}$$

$$I_4 = \left\langle e^{i\vec{k}\cdot\left(\vec{R}(\tau)-\vec{R}(\sigma)\right)} e^{-i\vec{k}\cdot(\vec{r}(\tau)+\vec{r}(\sigma))}\right\rangle = \left\langle e^{i\vec{k}\cdot(\vec{r}_2(\tau)-\vec{r}_1(\sigma))}\right\rangle. \tag{15}$$

To calculate $\langle ....\rangle$, we use the Feynman's method [1],

$$\langle ....\rangle = \int D\left(\vec{r}_1\right)\int D\left(\vec{r}_2\right)(....)e^{\frac{i}{\hbar}S_{bi}}. \tag{16}$$

Unfortunately, the path integral of Eq. (16) cannot be solved exactly because S_{bi} is not a quadratic function of $\vec{r}_j$ and $\dot{\vec{r}}_j$. To deal with this problem, we use the Feynman's approximation for the bipolaron,

$$\langle ....\rangle \simeq \int D\left(\vec{r}_1\right)\int D\left(\vec{r}_2\right)(....)e^{\frac{i}{\hbar}S_0}, \tag{17}$$

where S_0 is the trial action giving by

$$\begin{aligned} S_0 &= \sum_{j=1,2} \int_0^t d\tau \left(\frac{m}{2} \dot{\vec{r}}_j^2(\tau) + \frac{m}{2} \dot{\vec{r}}_j(\tau) \cdot \vec{B} \times \vec{r}_j(\tau) - V(\vec{\rho}_j, z_j) \right) \\ &- \int_0^t d\tau \frac{K}{2} \left(\vec{r}_1(\tau) - \vec{r}_2(\tau) \right)^2 \\ &- \frac{1}{2} \int_0^t d\tau \int_0^t d\sigma G_{\Omega_f}(\tau - \sigma) \left\{ C_1 \sum_{j=1,2} \left(\vec{r}_j(\tau) - \vec{r}_j(\sigma) \right)^2 + C_2 \left(\vec{r}_1(\tau) - \vec{r}_2(\sigma) \right)^2 \right\}. \end{aligned} \tag{18}$$

The Lagrangian which corresponds to the trial action S_0 is

$$\begin{aligned} L_0 &= \sum_{j=1,2} \left(\frac{m}{2} \dot{\vec{r}}_j^2(\tau) + \frac{m}{2} \dot{\vec{r}}_j(\tau) \cdot \vec{B} \times \vec{r}_j(\tau) - V(\vec{\rho}_j, z_j) + \frac{M}{2} \dot{\vec{Q}}_j^2(\tau) \right) \\ &- \frac{K}{2} \left(\vec{r}_1(\tau) - \vec{r}_2(\tau) \right)^2 \\ &- \frac{\kappa_1}{2} \sum_{j=1,2} \left(\vec{r}_j(\tau) - \vec{Q}_j(\tau) \right)^2 - \frac{\kappa_2}{2} \left(\left(\vec{r}_1(\tau) - \vec{Q}_2(\tau) \right)^2 + \left(\vec{r}_2(\tau) - \vec{Q}_1(\tau) \right)^2 \right), \end{aligned} \tag{19}$$

where $\vec{Q}_j$ is the position of the fictitious particle jth, and κ_1 and κ_2 and M are the spring constant and the mass of the fictitious particles, respectively:

$$\Omega_f = \sqrt{\frac{\kappa_1 + \kappa_2}{M}} \ , \quad C_1 = \frac{\kappa_1^2 + \kappa_2^2}{4M\Omega} \ , \quad C_2 = \frac{\kappa_1 \kappa_2}{M\Omega}. \tag{20}$$

We now start to calculate I_1. From Eq. (17), we have

$$I_1 = \left\langle e^{i\vec{k} \cdot (\vec{r}_1(\tau) - \vec{r}_1(\sigma))} \right\rangle = \int D(\vec{r}_1) \int D(\vec{r}_2) e^{\frac{i}{\hbar} S_0'}, \tag{21}$$

where

$$
\begin{aligned}
S_0' &= \sum_{j=1,2} \int_0^t d\tau \left(\frac{m}{2}\dot{\vec{r}}_j^2(\tau) + \frac{m}{2}\dot{\vec{r}}_j(\tau)\cdot\vec{B}\times\vec{r}_j(\tau) - V(\vec{\rho}_j, z_j) \right) \\
&+ \int_0^t d\tau \left(\sum_{j=1,2} \vec{f}_j'(\tau)\cdot\vec{r}_j(\tau) - \frac{K}{2}\left(\vec{r}_1(\tau)-\vec{r}_2(\tau)\right)^2 \right) \\
&- \frac{1}{2}\int_0^t d\tau \int_0^t d\sigma G_{\Omega_f}(\tau-\sigma) \left\{ C_1 \sum_{j=1,2} \left(\vec{r}_j(\tau)-\vec{r}_j(\sigma)\right)^2 + C_2\left(\vec{r}_1(\tau)-\vec{r}_2(\sigma)\right)^2 \right\},
\end{aligned}
\tag{22}
$$

with

$$G_{\Omega_f}(\vartheta) = \frac{\cos\Omega_f\left(|\vartheta| - t/2\right)}{\sin\Omega_f t/2}, \tag{23}$$

$$\vec{f}_1'(\tau) = i\vec{k}\left(\delta(\tau-t) - \delta(\tau-\sigma)\right) \ , \ \ \vec{f}_2'(\tau) = 0 \ . \tag{24}$$

The path integral of S_0' can be solved exactly. Brosens and Devreese [42] have solved the problem with $V(\vec{\rho}_j, z_j) = 0$ (see the appendix of this reference). Using the same manner, we obtain

$$I_1 = e^{-k_\perp^2 \Phi_{1,1}(t-\tau)} e^{-k_\parallel^2 \Upsilon_{1,1}(t-\tau)}, \tag{25}$$

where

$$\Phi_{1,1}(\tau-\sigma) = \frac{1}{4}\sum_{j=1}^{7} \lambda_j^2 \left(\frac{\cos s_j((\tau-\sigma)/2)\sin s_j\sigma/2\sin s_j((t-\tau)/2)}{\sin s_j t/2} \right), \tag{26}$$

$$\Upsilon_{1,1}(\tau-\sigma) = \frac{(\tau-\sigma)\Omega_f^2}{4v_1^2} + \frac{1}{4}\sum_{j=1}^{3} \Lambda_j^2 \left(\frac{\cos \upsilon_j((\tau-\sigma)/2)\sin \upsilon_j\sigma/2\sin \upsilon_j((t-\tau)/2)}{\sin \upsilon_j t/2} \right), \tag{27}$$

here λ_j^2 and Λ_j^2 are defined by Eq. (A31) and Eq. (A32) in the reference [42] and s_j are the solutions of the polynomials

$$\nu^3 + \nu^2 B - (\Omega_1^2 + \Omega_\rho^2)\nu - B\Omega_\rho^2 + \Omega_\rho^2\Omega_f^2 \ , \ \ j = 1,2,3 \ , \tag{28}$$

and

$$\nu^4 + \nu^3 B - (\Omega_2^2 + \Omega_3^2 + \Omega_\rho^2)\nu^2 - \nu B\Omega_f^2 + \Omega_2^2\Omega_3^2 + \Omega_\rho^2\Omega_f^2 \ , \ \ j = 4,5,6,7 \ , \tag{29}$$

and Ω_1^2, Ω_2^2, and Ω_3^2 are defined in Eqs. (A14), (A23) of reference [42], respectively. υ_j are the solution of the polynomials Eqs. (28) and (29) with $B = 0$.

The next term that will be calculate is

$$I_3 = \left\langle e^{i\vec{k}\cdot(\vec{r}_1(\tau)-\vec{r}_2(\sigma))}\right\rangle = \int D\left(\vec{r}_1\right)\int D\left(\vec{r}_2\right) e^{\frac{i}{\hbar}S_0''}, \tag{30}$$

where

$$\begin{aligned} S_0'' &= \sum_{j=1,2}\int_0^t d\tau\left(\frac{m}{2}\dot{\vec{r}}_j^2(\tau)+\frac{m}{2}\dot{\vec{r}}_j(\tau)\cdot\vec{B}\times\vec{r}_j(\tau)-V(\vec{\rho}_j,z_j)\right) \\ &+ \int_0^t d\tau\left(\sum_{j=1,2}\vec{f}_j''(\tau)\cdot\vec{r}_j(\tau)-\frac{K}{2}\left(\vec{r}_1(\tau)-\vec{r}_2(\tau)\right)^2\right) \\ &- \frac{1}{2}\int_0^t d\tau\int_0^t d\sigma G_{\Omega_f}\left(\tau-\sigma\right)\left\{C_1\sum_{j=1,2}\left(\vec{r}_j(\tau)-\vec{r}_j(\sigma)\right)^2+C_2\left(\vec{r}_1(\tau)-\vec{r}_2(\sigma)\right)^2\right\}, \end{aligned} \tag{31}$$

with

$$\vec{f}_1''(\tau) = i\vec{k}\left(\delta(\tau-\tau')\right) \ , \ \ \vec{f}_2''(\tau) = i\vec{k}\delta(\tau-\sigma) \ . \tag{32}$$

Similarly, we obtain

$$I_3 = e^{-k_\perp^2\Phi_{1,2}(t-\tau)}e^{-k_\parallel^2\Upsilon_{1,2}(t-\tau)}, \tag{33}$$

where

$$\begin{aligned} \Phi_{1,2}(\tau-\sigma) &= \frac{1}{4}\sum_{j=1}^{3}\lambda_j^2\left(\frac{\cos s_j((\tau-\sigma)/2)\sin s_j\sigma/2\sin s_j((t-\tau)/2)}{\sin s_j t/2}\right) \\ &+ \frac{1}{4}\sum_{j=4}^{7}\lambda_j^2\left(\frac{\sin s_j((\tau-\sigma)/2)\sin s_j\sigma/2\sin s_j((t-\tau)/2)}{\sin s_j t/2}\right), \end{aligned} \tag{34}$$

$$\begin{aligned} \Upsilon_{1,2}(\tau-\sigma) &= \frac{(\tau-\sigma)\Omega_f^2}{4\upsilon_1^2}+2\Lambda_1^2\left(\frac{\cos\upsilon_1((\tau-\sigma)/2)\sin\upsilon_1\sigma/2\sin\upsilon_1((t-\tau)/2)}{\sin\upsilon_1 t/2}\right) \\ &+ \frac{1}{4}\sum_{j=2}^{3}\Lambda_j^2\left(\frac{\sin\upsilon_j((\tau-\sigma)/2)\sin\upsilon_j\sigma/2\sin\upsilon_j((t-\tau)/2)}{\sin\upsilon_j t/2}\right). \end{aligned} \tag{35}$$

It is easy to show that

$$I_1 = e^{-k_\perp^2 \Phi_{1,1}(t-\tau)} e^{-k_\parallel^2 \Upsilon_{1,1}(t-\tau)} = I_2, \tag{36}$$

and

$$I_3 = e^{-k_\perp^2 \Phi_{1,2}(t-\tau)} e^{-k_\parallel^2 \Upsilon_{1,2}(t-\tau)} = I_4. \tag{37}$$

It can be illustrated that [46]

$$\left\langle \vec{R}(t) \right\rangle = -i \int_0^t d\tau \overleftrightarrow{Y}_0(t-\tau) \cdot \vec{E}(\tau). \tag{38}$$

Using $\vec{E}(\tau) = E_0 \exp(-i\omega\tau)$ and if a weak alternating electric field is applied to the system, we find that

$$\left\langle \vec{R}(t) \right\rangle = i \int_0^t d\xi \overleftrightarrow{Y}_0(\xi) e^{i\omega\xi} \cdot \vec{E}_0 e^{-i\omega t}. \tag{39}$$

Taking the Fourier transform of Eq. (39), we obtain

$$\vec{R}(\omega) = \overleftrightarrow{Y}_0(\omega) \cdot \vec{E}(\omega), \tag{40}$$

where $\overleftrightarrow{Y}_0(\omega)$ is the Fourier transform of the response function given by

$$\overleftrightarrow{Y}_0(\omega) = \int_0^\infty d\xi e^{i\omega\xi} i \int_0^t \frac{d\nu}{2\pi i} \left[\frac{1}{\overleftrightarrow{Z}_\nu} - \frac{1}{\overleftrightarrow{Z}_\nu^\dagger} \right] e^{i\nu\xi} \equiv \frac{1}{\overleftrightarrow{Z}_\omega}, \tag{41}$$

and $\overleftrightarrow{Z}_\omega$ is the impedance function [11, 12] of the center of mass of the bipolaron. We con now derive for $\overleftrightarrow{Z}_\omega$ from the equation of motion Eq. (9) without imposing the steady-state restriction and we make a change of variables $\vec{R} = \vec{J} + \bar{a}\exp(-i\omega\tau)$ and $t \to -i\beta$. Inserting Eqs. (25), (30), (36) and (37) in to Eq. (9) and we expand the result for $\bar{a}$ small , we obtain

$$\overleftrightarrow{Z}_\omega = -\omega^2 \overleftrightarrow{I} - \overleftrightarrow{\Omega} - i\omega \overleftrightarrow{\varepsilon} \cdot \vec{B} + \overleftrightarrow{\chi}(\omega), \tag{42}$$

where $\overleftrightarrow{I}$ is the identity matrix tensor and $\overleftrightarrow{\varepsilon}$ is the third reank totally antisymmetric tensor, and $\overleftrightarrow{\chi}(\omega)$ is a function that contains all of the correction due to the interaction with phonons,

$$\overleftrightarrow{\chi}(\omega) = \int_0^\infty d\xi \left(1 - e^{i\omega\xi}\right) \mathrm{Im} \overleftrightarrow{S}(\xi), \tag{43}$$

and

$$\overleftrightarrow{S}(\xi) = \int \frac{d^3k}{(2\pi)^3} \left|C_{\vec{k}}\right|^2 \vec{k}\cdot\vec{k} T_{\omega_{\vec{k}}}(\xi) \left\{ e^{-k_\perp^2 \Phi_{1,1}^\beta(\xi)} e^{-k_\parallel^2 \Upsilon_{1,1}^\beta(\xi)} + e^{-k_\perp^2 \Phi_{1,2}^\beta(\xi)} e^{-k_\parallel^2 \Upsilon_{1,2}^\beta(\xi)} \right\}, \tag{44}$$

where

$$T_{\omega_{\vec{k}}}(\xi) = \frac{e^{i\omega\xi}}{1-e^{-\beta\omega_{\vec{k}}}} + \frac{e^{-i\omega\xi}}{e^{\beta\omega_{\vec{k}}}-1}, \tag{45}$$

and

$$\Phi_{1,1}^\beta(\xi) = \frac{1}{4}\sum_{j=1}^{7} \lambda_j^2 \left(\frac{\cosh s_j(\beta/2) - \cos s_j(\xi - i\beta/2)}{\sinh s_j(\beta/2)} \right), \tag{46}$$

$$\Upsilon_{1,1}^\beta(\xi) = \frac{(\xi - i\beta/2)\Omega_f^2}{4\upsilon_1^2} + \frac{1}{2}\sum_{j=1}^{3} \Lambda_j^2 \left(\frac{\cosh \upsilon_j(\beta/2) - \cos \upsilon_j(\xi - i\beta/2)}{\sinh \upsilon_j(\beta/2)} \right), \tag{47}$$

$$\begin{aligned} \Phi_{1,2}^\beta(\xi) &= \frac{1}{4}\sum_{j=1}^{3} \lambda_j^2 \left(\frac{\cosh s_j(\beta/2) - \cos s_j(\xi - i\beta/2)}{\sinh s_j(\beta/2)} \right) \\ &+ \frac{1}{4}\sum_{j=4}^{7} \lambda_j^2 \left(\frac{\cosh s_j(\beta/2) + \cos s_j(\xi - i\beta/2)}{\sinh s_j(\beta/2)} \right), \end{aligned} \tag{48}$$

$$\begin{aligned} \Upsilon_{1,2}^\beta(\xi) &= \frac{(\xi - i\beta/2)\Omega_f^2}{4\upsilon_1^2} + \frac{1}{2}\Lambda_1^2 \left(\frac{\cosh \upsilon_1(\beta/2) - \cos \upsilon_1(\xi - i\beta/2)}{\sinh \upsilon_1(\beta/2)} \right) \\ &+ \frac{1}{2}\sum_{j=2}^{3} \Lambda_j^2 \left(\frac{\cosh \upsilon_j(\beta/2) + \cos \upsilon_j(\xi - i\beta/2)}{\sinh \upsilon_j(\beta/2)} \right). \end{aligned} \tag{49}$$

The analytic properties of $\overleftrightarrow{S}$ have been discussed by references [12, 13] allows one to write the expression for $\mathrm{Im}\overleftrightarrow{\chi}(\omega)$. We can now write a form more convenient for calculation as

$$\mathrm{Im}\overleftrightarrow{\chi}(\omega) = \mathrm{Im}\int_0^\infty d\mu \sin\omega\mu \overleftrightarrow{S}(\mu). \tag{50}$$

We may change the contour of integration in Eq. (50) as in reference [14]. The contribution from the remaining part of the contour gives

$$\mathrm{Im}\overleftrightarrow{\chi}(\omega) = \sinh(\beta\omega/2)\int_0^\infty du \cos(\omega u)\overleftrightarrow{\Sigma}(u), \tag{51}$$

where $\overleftrightarrow{\Sigma}(u) = \overleftrightarrow{S}(u - i\beta/2)$ and we set $\omega_{\vec{k}} = 1$. The mobility for the bipolaron in quantum confinement in the direction parallel and perpendicular with the magnetic field is given by

$$\begin{pmatrix} \frac{1}{\mu_\perp} \\ \frac{1}{\mu_\parallel} \end{pmatrix} = \lim_{\omega\to 0} \begin{pmatrix} \frac{\mathrm{Im}\chi_\perp(\omega)}{\omega} \\ \frac{\mathrm{Im}\chi_\parallel(\omega)}{\omega} \end{pmatrix} = \frac{\beta}{2}\int_0^\infty d\zeta \begin{pmatrix} \Sigma_\perp(\zeta) \\ \Sigma_\parallel(\zeta) \end{pmatrix}, \tag{52}$$

where

$$\mathrm{Im}\chi_\parallel(\omega) = \sinh(\beta\omega/2)\int_0^\infty du\cos(\omega u)\Sigma_\parallel(u), \tag{53}$$

and

$$\Sigma_\parallel(u) = \int \frac{d^3k}{(2\pi)^3}\left|C_{\vec{k}}\right|^2 k_\parallel^2 \frac{2\cos u}{\sinh\beta/2}\left\{e^{-k_\perp^2\Phi_{1,1}^\beta(u)}e^{-k_\parallel^2\Upsilon_{1,1}^\beta(u)} + e^{-k_\perp^2\Phi_{1,2}^\beta(u)}e^{-k_\parallel^2\Upsilon_{1,2}^\beta(u)}\right\}. \tag{54}$$

The integration in Eq. (54) can be done exactly. We obtain

$$\Sigma_\parallel(u) = \frac{2\alpha\cos u}{\sqrt{\pi}\sinh\beta/2}\left\{F\left(\Phi_{1,1}^\beta(u),\Upsilon_{1,1}^\beta(u)\right) + F\left(\Phi_{1,2}^\beta(u),\Upsilon_{1,2}^\beta(u)\right)\right\}, \tag{55}$$

where

$$F(x,y) = \frac{1}{(x-y)^{\frac{3}{2}}}\left(\ln\left(\frac{\sqrt{x}+\sqrt{x-y}}{\sqrt{x}-\sqrt{x-y}}\right) - \frac{2\sqrt{x-y}}{\sqrt{x}}\right). \tag{56}$$

We now introduce

$$\mathrm{Im}\chi(\omega) = \sinh(\beta\omega/2)\int_0^\infty du\cos(\omega u)\Sigma(u), \tag{57}$$

where

$$\Sigma(u) = \int \frac{d^3k}{(2\pi)^3}\left|C_{\vec{k}}\right|^2 k^2 \frac{2\cos u}{\sinh\beta/2}\left\{e^{-k_\perp^2\Phi_{1,1}^\beta(u)}e^{-k_\parallel^2\Upsilon_{1,1}^\beta(u)} + e^{-k_\perp^2\Phi_{1,2}^\beta(u)}e^{-k_\parallel^2\Upsilon_{1,2}^\beta(u)}\right\}. \tag{58}$$

Eq. (58) can also be solved exactly. We have

$$\Sigma(u) = \frac{\alpha\cos u}{\sqrt{\pi}\sinh\beta/2}\left\{\frac{1}{\Phi_{1,1}^\beta(u)\sqrt{\Upsilon_{1,1}^\beta(u)}} + \frac{1}{\Phi_{1,2}^\beta(u)\sqrt{\Upsilon_{1,2}^\beta(u)}}\right\}, \tag{59}$$

and we obtain

$$\Sigma_\perp(u) = \frac{\Sigma(u) - \Sigma_\parallel(u)}{2}. \tag{60}$$

Our expressions $\chi_\perp(\omega)$ and $\chi_\parallel(\omega)$ agree with reference [47] where they are called memory functions. There are three independent configuration were of interest.

i) Voigt configuration: The corresponding memory function is denoted by $\chi_\parallel(\omega)$ and gives the linear response to an electric field parallel to the magnetic field.

ii) Faraday configuration for the cyclotron resonance active mode ($+B$): The memory function in this case is $\chi_\perp(\omega)$ which describes the response to circular polarized light along the z-axis (which leads to an electric field perpendicular to the magnetic field).

iii) Faraday configuration for the cyclotron resonance inactive mode (-B): The memory function in this case is $-\chi_\perp(-\omega)$.

Next, we would like to consider the impedance function and the mobility for a polaron in quantum confinement. If we set $K = 0$ and C_2 is extremely small in Eq. (18), that means the electrons do not interact with each other directly but they do interact via fictitious particles which in this case we will ignore. The system L_0 describes the dynamic of an individual electron in the presence of the external fields and quantum confinement,

$$\begin{aligned} L_0 &= \sum_{j=1,2}\left(\frac{m}{2}\dot{\vec{r}}_j^{\,2}(\tau) + \frac{m}{2}\dot{\vec{r}}_j(\tau)\cdot\vec{B}\times\vec{r}_j(\tau) - V(\vec{\rho}_j, z_j) + \frac{M}{2}\dot{\vec{Q}}_j^{\,2}(\tau)\right) \\ &- \frac{\kappa_1}{2}\sum_{j=1,2}\left(\vec{r}_j(\tau)-\vec{Q}_j(\tau)\right)^2 - \frac{\kappa_2}{2}\left(\left(\vec{r}_1(\tau)-\vec{Q}_2(\tau)\right)^2 + \left(\vec{r}_2(\tau)-\vec{Q}_1(\tau)\right)^2\right). \end{aligned} \tag{61}$$

It is easy to show that

$$\Phi^\beta_{1,1}(u) \to \Phi^\beta_{1,2}(u) \to \Phi^\beta_\perp(u), \tag{62}$$

and

$$\Upsilon^\beta_{1,1}(u) \to \Upsilon^\beta_{1,2}(u) \to \Upsilon^\beta_\parallel(u). \tag{63}$$

where (see reference [48])

$$\Phi^\beta_\perp(u) = \sum_{j=1}^{3}\frac{(\tilde{s}_j^2-\Omega^2)}{m\prod_{j\neq i}(\tilde{s}_j-\tilde{s}_i)}\left(\frac{\cosh\tilde{s}_j(\beta/2)-\cos\tilde{s}_j(\xi-i\beta/2)}{\sinh\tilde{s}_j(\beta/2)}\right), \tag{64}$$

$$\Upsilon^\beta_\parallel(u) = \sum_{j=1}^{3}\frac{(\tilde{z}_j^2-\Omega^2)}{m\prod_{j\neq i}(\tilde{z}_j-\tilde{z}_i)}\left(\frac{\cosh\tilde{z}_j(\beta/2)-\cos\tilde{z}_j(\xi-i\beta/2)}{\sinh\tilde{z}_j(\beta/2)}\right), \tag{65}$$

here $\tilde{s}_j$ are the solution of the polynomial

$$m\nu^4 + \nu^3 B - m\nu^2(\upsilon_\perp^2 + \Omega_\rho^2) + \nu\Omega_f^2 B + m\Omega_\rho^2\Omega^2, \qquad (66)$$

and $\tilde{z}_j$ are the solution of

$$m\nu^4 - m\nu^2(\upsilon_\parallel^2 + \Omega_\rho^2) + m\Omega_z^2\Omega_f^2. \qquad (67)$$

Then the impedance function of the polaron in quantum confinement can be read

$$\overleftrightarrow{Z}_\omega^{(p)} = -\omega^2\overleftrightarrow{I} - \overleftrightarrow{\Omega} - i\omega\overleftrightarrow{\varepsilon}\cdot\vec{B} + \overleftrightarrow{\chi}^{(p)}(\omega), \qquad (68)$$

where

$$\mathrm{Im}\overleftrightarrow{\chi}^{(p)}(\omega) = \sinh(\beta\omega/2)\int_0^\infty du\cos\omega u\overleftrightarrow{\Sigma}^{(p)}(u), \qquad (69)$$

and

$$\overleftrightarrow{\Sigma}^{(p)}(u) = \int\frac{d^3k}{(2\pi)^3}\left|C_{\vec{k}}\right|^2\vec{k}\cdot\vec{k}\frac{2\cos u}{\sinh\beta/2}e^{-k_\perp^2\Phi_\perp^\beta(u)}e^{-k_\parallel^2\Upsilon_\parallel^\beta(u)}. \qquad (70)$$

We also find the mobility of the polaron in quantum confinement with the presence of the magnetic field

$$\begin{pmatrix}\frac{1}{\mu_\perp^{(p)}}\\ \frac{1}{\mu_\parallel^{(p)}}\end{pmatrix} = \frac{\beta}{2}\int_0^\infty\begin{pmatrix}\Sigma_\perp^{(p)}(u)\\ \Sigma_\parallel^{(p)}(u)\end{pmatrix}du, \qquad (71)$$

where

$$\begin{aligned}\Sigma_\parallel^{(p)}(u) &= \int\frac{d^3k}{(2\pi)^3}\left|C_{\vec{k}}\right|^2 k_\parallel^2\frac{2\cos u}{\sinh\beta/2}e^{-k_\perp^2\Phi_\perp^\beta(u)}e^{-k_\parallel^2\Upsilon_\parallel^\beta(u)}\\ &= \frac{2\alpha}{\sqrt{\pi}}\frac{\cos u}{\sinh\beta/2}F\left\{\Phi_\perp^\beta(u),\Upsilon_\parallel^\beta(u)\right\},\end{aligned} \qquad (72)$$

and we introduce

$$\begin{aligned}\Sigma^{(p)}(u) &= \int\frac{d^3k}{(2\pi)^3}\left|C_{\vec{k}}\right|^2 k^2\frac{2\cos u}{\sinh\beta/2}e^{-k_\perp^2\Phi_\perp^\beta(u)}e^{-k_\parallel^2\Upsilon_\parallel^\beta(u)}\\ &= \frac{2\alpha}{\sqrt{\pi}}\frac{\cos u}{\sinh\beta/2}\frac{1}{\Phi_\perp^\beta(u)\sqrt{\Upsilon_\parallel^\beta(u)}}.\end{aligned} \qquad (73)$$

We find that

$$\Sigma_{\perp}^{(p)}(u) = \frac{\Sigma^{(p)}(u) - \Sigma_{\parallel}^{(p)}(u)}{2}. \tag{74}$$

Recently, S. Yoo-kong [25] have calculated the effective of the polaron in the presence of the magnetic and electric fields by using the Single-Path-Integral. In this section, we would like to use the Single-Path-Integral to calculate the effective mass of the bipolaron and polaron in the presence of the external fields without trapping $\overleftrightarrow{\Omega} = 0$. We consider the case that $\omega < 1$ and $\beta \to \infty$. Then the impedance function Eq. (42) becomes

$$\overleftrightarrow{Z}_{\omega} = -\omega^2 I - i\omega \overleftrightarrow{\varepsilon} \cdot \vec{B} - \overleftrightarrow{\chi}'(\omega), \tag{75}$$

where

$$\begin{aligned}\overleftrightarrow{\chi}'(\omega) &= \int_0^{\infty} d\xi (1 - \cosh\omega\xi) e^{-\xi} \int \frac{d^3k}{(2\pi)^3} \left|C_{\vec{k}}\right|^2 \vec{k}\cdot\vec{k} \left\{ e^{-k_{\perp}^2 \tilde{\Phi}_{1,1}(\xi)} \, e^{-k_{\parallel}^2 \tilde{\Upsilon}_{1,1}(\xi)} \right. \\ &\quad \left. + e^{-k_{\perp}^2 \tilde{\Phi}_{1,2}(\xi)} e^{-k_{\parallel}^2 \tilde{\Upsilon}_{1,2}(\xi)} \right\}, \end{aligned} \tag{76}$$

with

$$\tilde{\Phi}_{1,1}(\xi) = \frac{1}{4} \sum_{j=1}^{7} \lambda_j^2 (1 - e^{-s'_j \xi}), \tag{77}$$

$$\tilde{\Phi}_{1,2}(\xi) = \frac{1}{4} \sum_{j=1}^{3} \lambda_j^2 (1 - e^{-s'_j \xi}) + \frac{1}{4} \sum_{j=4}^{7} \lambda_j^2 (1 + e^{-s'_j \xi}), \tag{78}$$

$$\tilde{\Upsilon}_{1,1}(\xi) = \frac{\xi \Omega_f^2}{4\upsilon_1^2} + \frac{1}{2} \sum_{j=1}^{3} \Lambda_j^2 (1 - e^{-\upsilon'_j \xi}), \tag{79}$$

$$\tilde{\Upsilon}_{1,2}(\xi) = \frac{\xi \Omega_f^2}{4\upsilon_1^2} + \frac{1}{2} \Lambda_1^2 (1 - e^{-\upsilon'_1 \xi}) + \frac{1}{2} \sum_{j=2}^{3} \Lambda_j^2 (1 + e^{-\upsilon'_j \xi}), \tag{80}$$

and s'_j are the solution of the polynomials

$$\nu^3 + \nu^2 B - \Omega_1^2 \nu^2 - B\Omega_f^2 \ , \quad j = 1, 2, 3 \ , \tag{81}$$

$$\nu^4 + \nu^3 B - (\Omega_2^2 + \Omega_3^2)\nu^2 - \nu B \Omega_f^2 + \Omega_2^2\Omega_3^2 \ , \quad j = 4,5,6,7 \ , \tag{82}$$

and υ_j' are the solution of the polynomials Eqs. (81) and (82) with $B = 0$.

If the frequency ω is extremely small, the $(1 - \cosh\omega\xi) \approx -\omega^2\xi^2/2$. We can obtain the effective mass of the bipolaron from the impedance function in the direction of parallel and perpendicular to the magnetic field. They are given by

$$m_\parallel^* = 1 + \frac{\alpha}{2\sqrt{\pi}} \int_0^\infty d\xi e^{-\xi}\xi^2 \left\{ F(\tilde{\Phi}_{1,1}(\xi), \tilde{\Upsilon}_{1,1}(\xi)) + F(\tilde{\Phi}_{1,2}(\xi), \tilde{\Upsilon}_{1,2}(\xi)) \right\}, \tag{83}$$

and

$$\begin{aligned} m_\perp^* &= 1 + \frac{\alpha}{4\sqrt{\pi}} \int_0^\infty d\xi e^{-\xi}\xi^2 \left\{ \frac{1}{\sqrt{\tilde{\Upsilon}_{1,1}(\xi)}\tilde{\Phi}_{1,1}(\xi)} - F(\tilde{\Phi}_{1,1}(\xi), \tilde{\Upsilon}_{1,1}(\xi)) \right. \\ &\left. + \frac{1}{\sqrt{\tilde{\Upsilon}_{1,2}(\xi)}\tilde{\Phi}_{1,2}(\xi)} - F(\tilde{\Phi}_{1,2}(\xi), \tilde{\Upsilon}_{1,2}(\xi)) \right\}. \end{aligned} \tag{84}$$

If we turn the magnetic off $B = 0$, we immediately get

$$\tilde{\Phi}_{1,1}(\xi) \to \tilde{\Upsilon}_{1,1}(\xi) \ , \quad \tilde{\Phi}_{1,2}(\xi) \to \tilde{\Upsilon}_{1,2}(\xi) \ , \tag{85}$$

and the effective mass of the bipolaron becomes

$$m^* = 1 + \frac{\alpha}{6\sqrt{\pi}} \int_0^\infty d\xi e^{-\xi}\xi^2 \left\{ \frac{1}{\left(\tilde{\Upsilon}_{1,1}(\xi)\right)^{3/2}} + \frac{1}{\left(\tilde{\Upsilon}_{1,2}(\xi)\right)^{3/2}} \right\}. \tag{86}$$

Next, we consider the effective mass of the polaron. If we set $K = 0$ and C_2 is extremely small in Eq. (18), the effective mass Eqs. (79) and (80) becomes

$$m_\parallel^* = 1 + \frac{\alpha}{2\sqrt{\pi}} \int_0^\infty d\xi e^{-\xi}\xi^2 \left\{ 2F(\tilde{\Phi}_\perp(\xi), \tilde{\Upsilon}_\parallel(\xi)) \right\}, \tag{87}$$

$$m_\perp^* = 1 + \frac{\alpha}{4\sqrt{\pi}} \int_0^\infty d\xi e^{-\xi}\xi^2 \left\{ \frac{2}{\tilde{\Phi}_\perp(\xi)\sqrt{\tilde{\Upsilon}_\parallel(\xi)}} + 2F(\tilde{\Phi}_\perp(\xi), \tilde{\Upsilon}_\parallel(\xi)) \right\}, \tag{88}$$

where

$$\tilde{\Phi}_{\perp}(\xi) = \sum_{j=1}^{7} \frac{(\tilde{s}_j^2 - \Omega^2)}{m \prod\limits_{j \neq l} (\tilde{s}_j - \tilde{s}_l)} (1 - e^{-\tilde{s}'_j \xi}), \tag{89}$$

$$\tilde{\Upsilon}_{\parallel}(\xi) = \sum_{j=1}^{4} \frac{(\tilde{z}_j^2 - \Omega^2)}{m \prod\limits_{j \neq l} (\tilde{z}_j - \tilde{z}_l)} (1 - e^{-\tilde{z}'_j \xi}), \tag{90}$$

where $\tilde{s}'_j$ and $\tilde{z}'_j$ are the solution of the polynomials Eqs. (66) and (67) for $\Omega_\rho = 0$ and $\Omega_z = 0$, respectively. Eqs. (89) and (90) are identical with Eqs. (3.9) and (3.10) for $N = 2$ in reference [25].

3 Conclusion

We use the Single-Path-Integral to obtain the impedance function of the bipolaron and polaron in quantum confinement with the presence of the magnetic field. The essential approximation to derive this lied in the procedure to calculate $\langle S_{bi} - S_0 \rangle$. In this case no approximation regarding the field strength, velocity, lattice coupling constant, or temperature was ever made. However, the part of the action Eqs. (22) and (31) describing the electrons-lattice interaction was approximated as close as possible to physical reality. Our impedance function for quantum wires can be directly obtained by setting $\Omega_z = 0$. A simple expression for quantum dots can be further obtained by setting $\Omega_\rho = \Omega_z = \Omega$.

However, the part of the action Eq. (18) describing the electrons-lattice interaction is approximated by using 2-variational parameters, κ_1 and κ_2. To improve the results in this paper, we may use the new Lagrangian in which the electrons-lattice interaction is approximated as closely as possible to physical reality with N-variational parameters,

$$\begin{aligned} L_0 &= \sum_{j=1,2} \left(\frac{m}{2} \dot{\vec{r}}_j^2 + \frac{m}{2} \dot{\vec{r}}_j \cdot B \times \vec{r}_j + V(\vec{\rho}_j, z_j) + \frac{M}{2} \dot{Q}_j^2 \right) - \frac{K}{2} (\vec{r}_1 - \vec{r}_2)^2 \\ &- \frac{\kappa_1}{2} \sum_{j=1,2} \left(\vec{r}_j - \vec{Q}_j \right)^2 - \sum_{j=2}^{N} \frac{\kappa_j}{2} \left(\vec{r}_1 - \vec{Q}_j \right)^2 - \sum_{j=2}^{N} \frac{\kappa_j}{2} \left(\vec{r}_2 - \vec{Q}_j \right)^2 . \end{aligned} \tag{91}$$

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