Editorial Manager(tm) for International Journal of Modern Physics B
Manuscript Draft

Manuscript Number: JPB20070861R1

Title: The Path Integral Approach to N-particle in a PT-symmetric Harmonic Oscillator.

Article Type: Research paper

Section/Category: Condensed Matter Physics

Keywords: Path Integral, N-particles, PT-symmteric

Corresponding Author: Mr SIKARIN YOO-KONG, MSc

Corresponding Author's Institution:

First Author: SIKARIN YOO-KONG, MSc

Order of Authors: SIKARIN YOO-KONG, MSc

Manuscript Region of Origin:

Abstract: We study a path integral approach for a system of particles in a PT-symmetric harmonic oscillator: V(x)=mw^2(x^2+2iex)/2. The eigenvalues and eigenstates of the system have been calculated. We find that the total energy of the system is real. The connection between the non-Hermitian and Hermitian Hamiltonians has been discussed and we also establish this connection in the context of path integrals via a considering model.

Response to Reviewers: The title has been changed to "The path integral approach to an N-particle in a PT-symmetric Harmonic oscillator".

The vector notation has been added and also with the dimension of the system.

The references have been added.

The Path Integral Approach to an N-Particle in a PT-Symmetric Harmonic Oscillator

Sikarin Yoo-Kong

Department of Physics, King Mongkut's University of Technology Thonburi, Bangkok, 10140, Thailand

sikarin@maths.leeds.ac.uk

August 22, 2008

Abstract

We study a path integral approach to a system of particles in a PT-symmetric harmonic potential: $V(x) = m\omega^2(x^2 \pm 2i\varepsilon x)/2$. The eigenvalues and eigenstates of the system have been calculated. We find that the total energy of the system is real. The connection between the non-Hermitian and Hermitian Hamiltonians has been discussed and we also establish this connection in the context of path integrals via a considering model.

PACS: 71.38.+i, 63.20.K
Keywords: Path Integrals, N-particle, PT-symmetric

1 Introduction

Bender et al, [1, 2] has argued that hermiticity is not a natural physical requirement for Hamiltonians in quantum mechanics, but rather a mathematical criterion. This has become evident in his and collaborators' study of Hamiltonians that posses real eigenvalues. The matrix elements of the Hamiltonian can be complex in which the space-time reflection symmetry is preserved. They use the notation[1] $\hat{H} = \hat{H}^{\mathsf{PT}}$ instead of $\hat{H} = \hat{H}^{\dagger}$. A remarkable example of the quantum system is [2]

$$\hat{H} = \hat{p}^2 - (i\hat{x})^n, \tag{1.1}$$

[1] where P is linear and has the effect of changing the sign of the momentum operator $\hat{p}$ and the position operator $\hat{x}$: $\mathsf{P}\hat{p}\mathsf{P} = -\hat{p}$ and $\mathsf{P}\hat{x}\mathsf{P} = -\hat{x}$. The time reversal T is antilinear and satsifies $\mathsf{T}\hat{p}\mathsf{T} = -\hat{p}$, $\mathsf{T}\hat{x}\mathsf{T} = \hat{x}$ and $\mathsf{T}i\mathsf{T} = -i$.

where n is a continuous real parameter. They found that for $n \geq 2$ the eigenvalues of the Hamiltonian are real, positive and discrete. On the other hand, for $n < 2$, the spectrum is partly real and partly complex.

A system of a particle in PT-symmetric cubic oscillator has been studied by Bender et al., [3]. In this paper, the ground-state energy of the Hamiltonian $H = \frac{p^2}{2} + \frac{x^2}{4} + i\lambda x^3$ was calculated by using high-order Reyleigh-Schrodinger perturbation theory and they have found that the energy spectrum of this Hamiltonian is real. Later, the time evolution of this system has been studied by Figueira et al., [4]. They evaluated explicitly various transition amplitudes, for the situation when the systems are subjected to a monochromatic linearly polarized electric field. However, there are a number of examples of PT-symmetric quantum systems, i.e., [5, 6, 7, 8].

A question arises at this point, whether there is any connection between Hermitian Hamiltonian and a non-Hermitian PT-symmetric Hamiltonian. Mostafazadeh [9, 10, 11] has shown that there exists a Hermitian operator ρ, such that

$$h = \rho^{-1} H \rho, \tag{1.2}$$

where h is a Hermitian Hamiltonian and the operator ρ can be written in the form $\rho = e^{q/2}$. For a simple example, we consider

$$H = \frac{p^2}{2} + \frac{x^2}{2} + ix. \tag{1.3}$$

The q operator corresponds to the Hamiltonian Eq. (1.3) given by $q = -2p$, where p is the momentum and we can show that

$$h = e^p H e^{-p} = \frac{p^2}{2} + \frac{x^2}{2} + \frac{1}{2} \tag{1.4}$$

is a Hermitian Hamiltonian.

Another relation between a non-Hermitian and Hermitian Hamiltonian is that they have the same eigenvalues. Consider

$$H\Phi_n = E_n \Phi_n, \tag{1.5}$$

where Φ_n are the eigenstates corresponding to a non-Hermitian Hamiltonian. We multiply $\exp(-q/2)$ on the left of Eq. (1.5):

$$\begin{aligned} e^{-q/2} H e^{q/2} e^{-q/2} \Phi_n &= E_n e^{-q/2} \Phi_n \\ h\Psi_n &= E_n \Psi_n, \end{aligned} \tag{1.6}$$

where $\Psi_n = e^{-q/2}\Phi_n$ and $h = e^{-q/2} H e^{q/2}$.

More recently, Mostafazadeh [11] has studied the path integrals of non-Hermitian quantum mechanics. He considers the generating functions (partition function)

$$Z[\vec{J}] = \mathrm{Tr}\left(T \exp\left\{ \frac{i}{\hbar} \int_0^t \left(H - \vec{J} \cdot \vec{X} \right) dt \right\} \right), \tag{1.7}$$

where $\vec{J}\cdot\vec{X}$ represents the source term in quantum mechanics, $\vec{X} = (X_1, X_2, ..., X_m)$ are the pseudo-Hermitian dynamical variables and

$$X_m = e^{-q/2} x_m e^{q/2}, \tag{1.8}$$

where x_m are the dynamical configuration variable. By means of Eq. (1.8), he established the identity

$$T \exp\left\{ \frac{i}{\hbar} \int_0^t \left(h - \vec{J} \cdot \vec{x} \right) dt \right\} = e^{-q/2} T \exp\left\{ \frac{i}{\hbar} \int_0^t \left(H - \vec{J} \cdot \vec{X} \right) dt \right\} e^{q/2}. \tag{1.9}$$

Next, we would like to find an explicit relation of Eq. (1.9). In the discrete-time version, we can write the relation [12, 13]

$$\Phi(\vec{X}_{n+1}, t_{n+1}) = \int d\vec{X}_n K(\vec{X}_{n+1}, t_{n+1}; \vec{X}_n, t_n) \Phi(\vec{X}_n, t_n), \tag{1.10}$$

where Φ is the wave function corresponding to a non-Hermitian Hamiltonian and K is identical to the kernel of the quantum mechanical time-evolution operator (propagator)

$$K = \left\langle \vec{X}_{n+1} \left| \exp\left(-i \Delta t H / \hbar \right) \right| \vec{X}_n \right\rangle, \tag{1.11}$$

with $\Delta t = t_{n+1} - t_n$. Using Eq. (1.8), we obtain

$$\begin{aligned} e^{-q/2} \Phi(\vec{X}_{n+1}, t_{n+1}) &= \int e^{-q/2} d\vec{X}_n e^{q/2} e^{-q/2} K(\vec{X}_{n+1}, t_{n+1}; \vec{X}_n, t_n) e^{q/2} e^{-q/2} \Phi(\vec{X}_n, t_n) \\ \Psi(\vec{x}_{n+1}, t_{n+1}) &= \int d\vec{x}_n \tilde{K}(\vec{x}_{n+1}, t_{n+1}; \vec{x}_n, t_n) \Psi(\vec{x}_n, t_n), \end{aligned} \tag{1.12}$$

where

$$\Psi(\vec{x}_{n+1}, t_{n+1}) = e^{-q/2} \Phi(\vec{X}_{n+1}, t_{n+1}), \tag{1.13}$$

and

$$\tilde{K}(\vec{x}_{n+1}, t_{n+1}; \vec{x}_n, t_n) = e^{-q/2} K(\vec{X}_{n+1}, t_{n+1}; \vec{X}_n, t_n) e^{q/2} \tag{1.14}$$

is the propagator corresponding to a Hermitian Hamiltonian

$$\tilde{K} = \left\langle \vec{x}_{n+1} \left| \exp\left(-i \Delta t h / \hbar \right) \right| \vec{x}_n \right\rangle. \tag{1.15}$$

In this paper, the propagator of an N-particle in the complex potential will be calculated. The Hamiltonian under consideration is

$$H = \sum_{l=1}^{N} \left(\frac{\vec{p}_l^2}{2m} + \frac{m\omega^2}{2} \left(\vec{x}_l^2 \pm 2i\varepsilon \vec{x}_l \right) \right) + \frac{\kappa}{4} \sum_{lj}^{N} (\vec{x}_l - \vec{x}_j)^2 (1 - \delta_{lj}), \tag{1.16}$$

where $(\vec{p}_l, \vec{x}_l)$ are canonically conjugate variables of the ith particle in three dimensional spaces with unit vectors $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$, ε is a real constant and κ is a coupling constant. It is obvious to see that $H \neq H^\dagger$ but $H = H^{\mathsf{PT}}$. We choose a toy model Eq. (1.16) to study because the Lagrangian

$$L = \sum_{l=1}^{N} \left(\frac{m}{2}\dot{\vec{x}}_l^2 - \frac{m\omega^2}{2}(\vec{x}_l \pm 2i\varepsilon\vec{x}_l) \right) - \frac{\kappa}{4}\sum_{lj}^{N}(\vec{x}_l - \vec{x}_j)^2(1-\delta_{lj}) \tag{1.17}$$

is a function in the quadratic form of $(\dot{\vec{x}}_l, \vec{x}_l)$ and we know that the path integral Eq. (1.17) can be exactly computed.

In the next section, the path integral of the Hamiltonian Eq. (1.16) will be considered and the propagator will be exactly obtained. The eigenvalues and eigenstates of the system is also derived in section 3. The connection between the non-Hermitian and Hermitian Hamiltonians via the path integral will be discussed in section 4. We also provide a conclusion in the last section.

2 The propagator

The propagator of the system Eq. (1.16) can be expressed in the form

$$I = \prod_{l=1}^{N} K_l\left(\vec{x}_l(t), t; \vec{x}_l(0), 0\right), \tag{2.1}$$

where K_l is given by

$$K_l\left(\vec{x}_l(t), t; \vec{x}_l(0), 0\right) = \int_{\vec{x}_l(0)}^{\vec{x}_l(t)} D\left(\vec{x}_l(\tau)\right) \exp\left(\frac{i}{\hbar}\int_0^t d\tau L_l\right), \tag{2.2}$$

with

$$L_l = \frac{m\dot{\vec{x}}_l^2}{2} - \frac{m\omega^2}{2}\left(\vec{x}_l^2 \pm 2i\varepsilon\vec{x}_l\right) - \frac{\kappa}{4}\sum_{j}^{N}(\vec{x}_l - \vec{x}_j)^2(1-\delta_{lj}). \tag{2.3}$$

We now introduce the center of mass and relative coordinates

$$\vec{X} = \frac{1}{N}\sum_{l=1}^{N}\vec{x}_l \ , \quad \vec{X}_{lj} = \vec{x}_l - \vec{x}_j \tag{2.4}$$

Then the propagator Eq. (2.1) becomes

$$I = K_X\left(\vec{X}(t), t; \vec{X}(0), 0\right) \prod_{\gamma=1}^{3N-6} K^{\gamma}_{X_{lj}}\left(\vec{X}_{lj}(t), t; \vec{X}_{lj}(0), 0\right), \tag{2.5}$$

where $N > 2$ and

$$K_X\left(\vec{X}(t),t;\vec{X}(0),0\right)=\int\limits_{\vec{X}(0)}^{\vec{X}(t)} D\left(\vec{X}(\tau)\right)\exp\left(\frac{i}{\hbar}\int\limits_0^t d\tau L_X\right), \tag{2.6}$$

$$K^{\gamma}_{X_{lj}}\left(\vec{X}_{lj}(t),t;\vec{X}_{lj}(0),0\right)=\int\limits_{\vec{X}_{lj}(0)}^{\vec{X}_{lj}(t)} D\left(\vec{X}_{lj}(\tau)\right)\exp\left(\frac{i}{\hbar}\int\limits_0^t d\tau L_{X_{lj}}\right), \tag{2.7}$$

$$L_X=\frac{mN}{2}\dot{\vec{X}}^2-\frac{m\omega^2N}{2}\vec{X}^2\mp im\omega^2\varepsilon N\vec{X}, \tag{2.8}$$

$$L_{X_{lj}}=\frac{m}{2N}\dot{\vec{X}}^2_{lj}-\frac{m\omega^2}{2N}\vec{X}^2_{lj}+\frac{\kappa}{4}\vec{X}^2_{lj}. \tag{2.9}$$

2.1 The propagator for the center of mass

In this section, we would like to compute the propagator Eq. (2.6). We start to consider the equation of motion Eq. (2.8)

$$\ddot{\vec{X}}+\omega^2\vec{X}=\mp i\varepsilon\omega^2=\vec{F}, \tag{2.10}$$

which describes the harmonic motion of the center of mass with a constant (imaginary) driving force $\vec{F}=diag(\mp i\varepsilon\omega^2,\mp i\varepsilon\omega^2,\mp i\varepsilon\omega^2)$. The exact form of the propagator Eq. (2.6) can be found [12] as

$$K_X\left(\vec{X}(t),t;\vec{X}(0),0\right)=G_X\exp\left(\frac{i}{\hbar}S^{cl}_X\right), \tag{2.11}$$

where G_X is the prefactor and the classical action, S^{cl}_X, is given by

$$S^{cl}_X=\int\limits_0^t d\tau L_X=\frac{mN}{2}\left(\dot{\vec{X}}(t)\vec{X}(t)-\dot{\vec{X}}(0)\vec{X}(0)\right)\mp\frac{im\omega^2\varepsilon N}{2}\int\limits_0^t d\tau\vec{X}(\tau). \tag{2.12}$$

The classical path of the center of mass is easily obtained from Eq. (2.10)

$$\vec{X}(\tau)=\frac{\vec{X}(0)\sin\omega(t-\tau)+\vec{X}(t)\sin\omega\tau}{\sin\omega t}\mp i\varepsilon. \tag{2.13}$$

Substituting Eq. (2.13) into Eq. (2.12), we have

$$\begin{aligned} S_X^{cl} &= -\frac{m\omega^2\varepsilon^2 Nt}{2} + \frac{m\omega N}{2\sin\omega t}\left[\left(\vec{X}^2(t)+\vec{X}^2(0)\right)\cos\omega t - \vec{X}(t)\vec{X}(0)\right] \\ &\pm \frac{im\omega^2 N\varepsilon}{2\sin\omega t}\left(\cos\omega t - 1\right)\left(\vec{X}(t)-\vec{X}(0)\right). \end{aligned} \tag{2.14}$$

The prefactor can be obtained from

$$G_X = K_X\left(0,t;0,0\right) = \left(\frac{m\omega N}{i\pi\hbar\sin\omega t}\right)^{\frac{3}{2}} e^{\frac{-im\omega^2\varepsilon^2 Nt}{2\hbar}}. \tag{2.15}$$

2.2 The propagator for the relative coordinate

The equation of motion of the relative coordinate can be easily computed yielding

$$\ddot{\vec{X}}_{lj} + \Omega^2\vec{X}_{lj} = 0 \tag{2.16}$$

where $\Omega^2 = \omega^2 + N\kappa/2$. Eq. (2.16) describes a motion of a particle for a simple harmonic oscillator. The path integral of Eq. (2.7) can also be easily done

$$K^{\gamma}_{X_{lj}}\left(\vec{X}_{lj}(t),t;\vec{X}_{lj}(0),0\right) = G_{X_{lj}}\exp\left(\frac{i}{\hbar}S^{cl}_{X_{lj}}\right), \tag{2.17}$$

where $G_{X_{lj}}$ is the prefactor and $S^{cl}_{X_{lj}}$ is the classical action for the relative coordinate given by

$$S^{cl}_{X_{lj}} = \int_0^t d\tau L_{X_{lj}} = \frac{m}{2N}\left(\dot{\vec{X}}_{lj}(t)\vec{X}_{lj}(t) - \dot{\vec{X}}_{lj}(0)\vec{X}_{lj}(0)\right). \tag{2.18}$$

The classical path of the relative coordinate is given by

$$\vec{X}_{lj}(\tau) = \frac{\vec{X}_{lj}(0)\sin\Omega(t-\tau) + \vec{X}_{lj}(t)\sin\Omega\tau}{\sin\Omega t}. \tag{2.19}$$

Inserting Eq. (2.19) into Eq. (2.18), we have

$$S^{cl}_{X_{lj}} = \frac{m\Omega N}{2\sin\Omega t}\left[\left(\vec{X}^2_{lj}(t)+\vec{X}^2_{lj}(0)\right)\cos\Omega t - \vec{X}_{lj}(t)\vec{X}_{lj}(0)\right]. \tag{2.20}$$

The prefactor can be expressed in the form

$$G_{X_{lj}} = K^{\gamma}_{X_{lj}}\left(0,t;0,0\right) = \left(\frac{m\Omega}{i\pi\hbar N\sin\Omega t}\right)^{\frac{3}{2}}. \tag{2.21}$$

3 The eigenstates and eigenvalues

Next, we would like to calculate the energy spectrum of the Hamiltonian. We use the fact that

$$\mathrm{Tr}K(\vec{x},t;\vec{x},0)=\sum_{n=0}^{\infty}e^{-iE_nt/\hbar}, \tag{3.1}$$

where E_n is an energy eigenvalue.

From Eqs. (2.11) and (2.14), we find that

$$\mathrm{Tr}K_X(\vec{X},t;\vec{X},0)=\sum_{n=0}^{\infty}e^{-i(n+1/2)\omega t-imN\omega^2\varepsilon^2t/\hbar^2}. \tag{3.2}$$

Then the energy spectrum for the center of mass can be expressed

$$E_n=\left(n+\frac{1}{2}\right)\hbar\omega+mN\omega^2\varepsilon^2. \tag{3.3}$$

In the same fashion, we obtain the energy spectrum for the relative coordinate as

$$E_s=\left(s+\frac{1}{2}\right)\hbar\Omega, \tag{3.4}$$

and the total energy of the system is

$$E_{ns}=\left(n+\frac{1}{2}\right)\hbar\omega+\left(s+\frac{1}{2}\right)(3N-6)\hbar\Omega+mN\omega^2\varepsilon^2, \tag{3.5}$$

where $n,s=0,1,2,....$ We see that the energy eigenvalues of the system are real and have been shifted by a constant $+mN\omega^2\varepsilon^2$.

The eigenstates of the system can be evaluated by using the identity

$$K(\vec{x}(t),t;\vec{x}(0),0)=\sum_n\Psi_n(\vec{x}(t))\Psi_n(\vec{x}(0))e^{-iE_nt/\hbar}, \tag{3.6}$$

where $\Psi(\vec{x})$ is the wave function of the system and we also use the Mehler-formula [14]

$$e^{-\frac{(x^2+y^2)}{2}}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{z}{2}\right)^2H_n(x)H_n(y)=\frac{1}{\sqrt{1-z^2}}e^{\frac{4xyz-(x^2+y^2)(1+z^2)}{2(1-z^2)}}. \tag{3.7}$$

The wave function $\Psi_n(\vec{X})$ of the center of mass can be computed

$$\Psi_n(\vec{X})=\frac{1}{\sqrt{2^nn!}}\left(\frac{m\omega N}{2\pi\hbar}\right)^{\frac{1}{4}}\mathrm{H}_n\left(\sqrt{\frac{mN\omega}{\hbar}}(\vec{X}+i\varepsilon)\right)e^{-m\omega N(\vec{X}+i\varepsilon)^2/\hbar}, \tag{3.8}$$

where H_n are the Hermite polynomial.

The wave function $\Phi_s^{\gamma}(\vec{X}_{lj})$ of the relative coordinate is

$$\Phi_s^{\gamma}(\vec{X}_{lj}) = \frac{1}{\sqrt{2^s s!}} \left(\frac{m\Omega}{2N\pi\hbar}\right)^{\frac{1}{4}} \mathrm{H}_s \left(\sqrt{\frac{m\Omega}{2N\hbar}} \vec{X}_{lj}\right) e^{-m\Omega \vec{X}_{lj}^2/2N\hbar}, \tag{3.9}$$

and then we can write the wave function of the system in the form

$$\Upsilon_{ns}\left(\vec{X}, \vec{X}_{lj}, t\right) = \Psi_n(\vec{X}) \prod_{\gamma=1}^{3N-6} \Phi_s^{\gamma}(\vec{X}_{lj}) e^{-iE_{ns}t/\hbar}. \tag{3.10}$$

4 Transformations

Next, we will construct the transformation between a non-Hermitian and Hermitian Hamiltonian. Consider the non-Hermitian Hamiltonian of a particle in PT-symmetric harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2}\left(x^2 + 2i\varepsilon x\right). \tag{4.1}$$

The q operator can be found exactly in the form [1]: $q = -2\varepsilon p$ and we can show that

$$e^{\varepsilon p} H e^{-\varepsilon p} = \frac{p^2}{2m} + \frac{m\omega^2}{2} x^2 + \frac{m\omega^2 \varepsilon^2}{2} = h, \tag{4.2}$$

since $e^{\varepsilon p} x e^{-\varepsilon p} = x - i\varepsilon$. The Hamiltonian h satisfies the hermiticity property.

According to the change of coordinates Eq. (2.4), the non-Hermitian Hamiltonian can be written in the form

$$H = H_X + H_{X_{lj}}, \tag{4.3}$$

where

$$H_X = \frac{N\vec{P}^2}{2m} + \frac{m\omega^2 N}{2}\vec{X}^2 \pm m\omega^2 i\varepsilon N\vec{X}, \tag{4.4}$$

$$H_{X_{lj}} = \sum_{lj}^{N} \left(\frac{\vec{P}_{lj}^2}{2mN} + \frac{m\omega^2}{2N}\vec{X}_{lj}^2 - \frac{\kappa}{4}\vec{X}_{lj}^2\right)(1 - \delta_{lj}). \tag{4.5}$$

We observe that H_X is a non-Hermitian Hamiltonian. On the other hand, $H_{X_{lj}}$ is a Hermitian Hamiltonian. The q operator can be expressed in terms of the momentum of the center of mass: $q = -2\varepsilon P$, and we can show that

$$e^{\varepsilon P} H e^{-\varepsilon P} = e^{\varepsilon P} H_X e^{-\varepsilon P} + H_{X_{lj}} = h_X + H_{X_{lj}}, \tag{4.6}$$

where

$$h_X = \frac{N\vec{P}^2}{2m} + \frac{m\omega^2 N}{2}\vec{X}^2 \pm \frac{m\omega^2\varepsilon^2 N}{2}. \tag{4.7}$$

Using $q = -2\varepsilon P$, we can establish the relation for the propagator of the center of mass

$$e^{\varepsilon P} K_X(\vec{X}(t), t; \vec{X}(0), 0) e^{-\varepsilon P} = \tilde{K}_X(\vec{X}(t), t; \vec{X}(0), 0), \tag{4.8}$$

where $\tilde{K}_X$ is the propagator of the center of mass corresponding to the Hermitian Hamiltonian Eq. (4.7) given by

$$\tilde{K}_X(\vec{X}(t), t; \vec{X}(0), 0) = \left(\frac{m\omega N}{\pi i\hbar \sin\omega t}\right)^{\frac{3}{2}} e^{\frac{-im\omega^2\varepsilon^2 t}{\hbar}}$$
$$\times \exp\left\{\frac{i}{\hbar}\frac{m\omega N}{2\sin\omega t}\left(\left(\vec{X}^2(t) + \vec{X}^2(0)\right)\cos\omega t - 2\vec{X}(t)\vec{X}(0)\right)\right\} \tag{4.9}$$

We now write the propagator of the system after transforming

$$\tilde{I} = \tilde{K}_X\left(\vec{X}(t), t; \vec{X}(0), 0\right) \prod_{\gamma=1}^{3N-6} K^{\gamma}_{X_{lj}}\left(\vec{X}_{lj}(t), t; \vec{X}_{lj}(0), 0\right). \tag{4.10}$$

This propagator corresponds to the system Eq. (4.6). It is easy to observe that the energy spectrum of Eq. (4.10) is exactly the same as Eq. (2.5).

5 Conclusions

The explicit form of the propagator of the system Eq. (1.16) has been obtained by changing the coordinates via Eq. (2.4). The energy spectrum and the wave function are also calculated. The total energy of the system is still real even though the Hamiltonian does not satisfy the hermiticity. From Eq. (2.10), we can describe a motion of the center of mass of the system in a simple harmonic oscillator with an external constant field F in which the energy of the system will be shifted by the last term of Eq. (3.4). We observe that the Hermitian Hamiltonian and non-Hermitian Hamiltonians are related to each other through the *similarity transformation*, see Eq. (1.2). This relation also holds in the context of the path integral, see Eq. (4.8).

However, we would like to point out that the model considered in this paper is quite simple but it provides us a good background in mathematical details. We can study furthermore for complicated models but still in the quadratic form; for instance, N-particle in the electromagnetic field with PT-symmetric harmonic oscillators. The system of particles subjected into PT-symmetric potential type: $(ix)^n$ is also of interest,

$$H = \sum_{l=1}^{N} \left(\frac{\vec{p}_l^2}{2m} - \frac{m\omega^2}{2}(\vec{x}_l \pm 2\varepsilon(i\vec{x}_l)^n) \right) - \frac{\kappa}{4} \sum_{lj}^{N} (\vec{x}_l - \vec{x}_j)^2 (1 - \delta_{lj}). \tag{5.11}$$

Obviously, the path integral of this system cannot be solved exactly. We must introduce the method of the perturbation to deal with this problem. This work is in progress.

References

[1] C. M Bender. *Making sense of non-Hermitian Hamiltonian*. Rep. Prog. Phys. **70**, 947, 2007 and references therein.

[2] C. M Bender. *Introduction to* PT*-Symmetric Quantum Theory*. arXiv: quant-ph/0501052v1.

[3] C. M Bender and G. V. Dunne. *Large-order perturbation theory for a non-Hermitian* PT*-symmetric Hamiltonian*. J. Maths. Phys. **40**, 4616, 1999.

[4] C. Figueira de Morisson Faria and A. Frig. *Time evolution of non-Hermitian Hamiltonian systems*. J. Phys. A: Maths. Gen. **39**, 9269, 2006.

[5] G. A. Mezincescu. *Some properties of eigenvalues and eigenfunctions of the cubic oscillator with imaginary coupling constant*. arXiv: quant-ph/0002056V1.

[6] C. M Bender, Jun-Hua Chen and K. A. Milton. PT*-symmetric versus Hermitian formulations of quantum mechanics*. J. Phys. A: Maths. Gen. **39**, 1657, 2006.

[7] M. Znijil. PT*-symmetric harmonic oscillators*. arXiv: quant-ph/9905020v1.

[8] A. Jannussis, G. Brodimas, S. Baskoutas and A. Leodaris. *Non-Hermitian harmonic oscillator with discrete complex or real spectrum for non-unitary squeeze operators*. J. Phys. A: Maths. Gen. **36**, 2507, 2003.

[9] A. Mostafazadeh. PT*-Symmetric Quantum Mechanics: A Precise and Consistent Formulation*. arXiv: quant-ph/0407213v1.

[10] A. Mostafazadeh. *Time-Dependent Pseudo-Hermitian Hamiltonian Defining a Unitary Quantum System and Uniqueness of the Metric Operator*. arXiv: quant-ph/0706.1872v2.

[11] A. Mostafazadeh. *Path-Integral Formulation of Pseudo-Hermitian Quantum Mechanics and the Role of the Metric Operator*. arXiv: quant-ph/0708.397v1.

[12] L. S. Schulman. *Techniques and Application of Path Integration*. Dover Publication, Inc. Mineola, New York, 2005.

[13] C. Grosche and F. Steiner. *Handbook of Feynman Path Integrals*. Springer, 1998.

[14] A. Erdelyi, W. Magnus, F. Oberhettinger and F. G. Tricomi. *High Transcendental functions, Vol.I-III*. McGraw Hill, New York, 1955.