The Euler-Poinsot Problem

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Magic Course: Integrable systems
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Outlook

- Introduction
- Euler transformation
- Deprit's work and Kozlov's work

Andre Deprit: Free Rotation of a Rigid Studied in the Phase Plane.

V.V.Kozlov: Geometry of "Action-Angle" variables in Euler-Poinsot problem.

Introduction

One of the classical problems of mechanics is about a free motion of a rigid body called as the *Euler-Poinsot* problem. The formulation and solution of the problem can be performed in two steps.

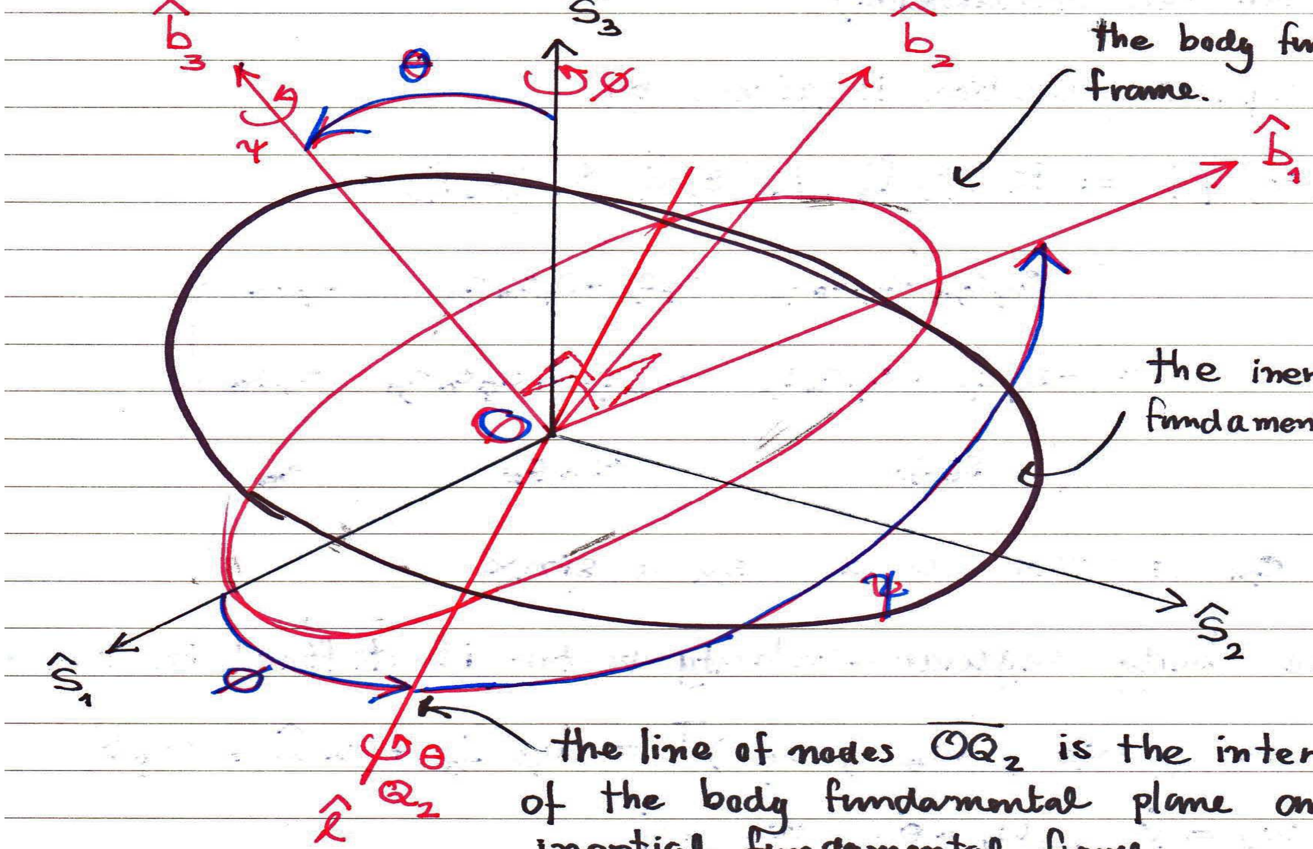
- The dynamics of the rotation are represented by differential equations for the components of the body angular velocity.
- The kinematic equations are created to transform the body angular velocity into a spatial inertial frame.

Eulerian Variables

In this section, we would like to briefly review Eulers transformation of the rigid body.

3-1-3 rotation

- $R(\phi, \widehat{s}_3)$, a rotation about $\widehat{s}_3$ by $0 \leq \phi \leq 2\pi$, mapping $\widehat{s}_1 \to \widehat{l}$
- $R(\theta, \widehat{l})$, a rotation about $\widehat{l}$ by $0 \leq \theta \leq \pi$, mapping $\widehat{s}_3 \to \widehat{b}_3$
- $R(\psi, \widehat{b}_3)$, a rotation about $\widehat{b}_3$ by $0 \leq \psi \leq 2\pi$, mapping $\widehat{l} \to \widehat{b}_1$



the body fundamental frame.

the inertial fundamental frame

the line of nodes $\overline{OQ_2}$ is the intersection of the body fundamental plane and the inertial fundamental frame.

We find that [Goldstein]

$$R(\phi,\hat{s}_3)=\begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}\ , \quad R(\theta,\hat{l})=\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}\ ,$$

$$R(\psi,\hat{b}_3)=\begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The composite rotation is given in the matrix form

$$R(\phi,\theta,\psi)=R(\psi,\hat{b}_3)R(\theta,\hat{l})R(\phi,\hat{s}_3)=\begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix},$$

where

$$v_1 = \begin{pmatrix} \cos\psi\cos\phi - \sin\psi\cos\theta\sin\phi \\ -\sin\psi\cos\phi - \cos\psi\cos\theta\sin\phi \\ \sin\theta\sin\phi \end{pmatrix},$$

$$v_2 = \begin{pmatrix} \cos\psi\sin\phi + \sin\psi\cos\theta\cos\phi \\ -\sin\psi\sin\phi + \cos\psi\cos\theta\cos\phi \\ -\sin\theta\sin\phi \end{pmatrix},$$

$$v_2 = \begin{pmatrix} \sin\psi\sin\theta \\ \cos\psi\sin\theta \\ \cos\theta \end{pmatrix}.$$

Angular-velocity vector

The body angular-velocity vector is defined in the form

$$\vec{\omega} = \left(\ \omega_\phi = \dot{\phi}, \ \ \omega_\theta = \dot{\theta}, \ \ \omega_\psi = \dot{\psi} \ \right)^T = \omega_\phi \hat{s}_3 + \omega_\theta \hat{l} + \omega_\psi \hat{b}_3.$$

Unfortunately the directions $\omega_\phi, \omega_\theta$ are not symmetrically placed. However, we can rotate them into the body frame

$$\vec{\omega} = \begin{pmatrix} \omega_{b_1} = \dot{\phi} \sin\theta \sin\psi + \dot{\theta} \sin\theta \sin\phi \\ \omega_{b_2} = \dot{\phi} \sin\theta \sin\psi + \dot{\theta} \sin\theta \sin\phi \\ \omega_{b_3} = \dot{\phi} \cos\theta + \dot{\psi} \end{pmatrix}$$

The Lagrangian and Hamiltonian

The Lagrangian of the system can be expressed in the form

$$\mathbb{L}(\phi,\theta,\psi,\dot{\phi},\dot{\theta},\dot{\psi}) = \frac{1}{2}\vec{\omega}\cdot\mathbb{I}\cdot\vec{\omega},$$

where $\mathbb{I}$ being the inertia tensor. We assume that the body axes concide with the principle axes of inertia, so

$$\mathbb{I} = diag(I_1, I_2, I_3),$$

then the Lagrangian can be written

$$\mathbb{L} = \frac{1}{2}\left(I_1\omega_{b_1}^2 + I_2\omega_{b_2}^2 + I_3\omega_{b_3}^2\right)$$

We now can find the generalized momenta

$$p_\phi = \frac{\partial \mathbb{L}}{\partial \dot{\phi}} = (I_1 \omega_{b_1} \sin\psi + I_2 \omega_{b_2} \cos\psi) \sin\theta + I_3 \omega_{b_3} \cos\theta,$$

$$p_\theta = \frac{\partial \mathbb{L}}{\partial \dot{\theta}} = I_1 \omega_{b_1} \cos\psi - I_2 \omega_{b_2} \sin\psi,$$

$$p_\psi = \frac{\partial \mathbb{L}}{\partial \dot{\psi}} = I_3 \omega_{b_3}.$$

The inverse relations will take the forms

$$I_1 \omega_{b_1} = \left(\frac{p_\phi - p_\psi \cos\theta}{\sin\theta} \right) \sin\psi + p_\theta \cos\psi,$$

$$I_2 \omega_{b_2} = \left(\frac{p_\phi - p_\psi \cos\theta}{\sin\theta} \right) \cos\psi - p_\theta \cos\psi,$$

$$I_3\omega_{b_3} = p_\psi.$$

In a free spin case, the Hamiltonian take a form

$$\mathbb{H} = p_\phi\dot{\phi} + p_\theta\dot{\theta} + p_\psi\dot{\psi} - \mathbb{L} = \frac{1}{2}\left(I_1\omega_{b_1}^2 + I_2\omega_{b_2}^2 + I_3\omega_{b_3}^2\right)$$

Using the inverse relations, the Hamiltonian can be rewritten

$$\begin{aligned}\mathbb{H}(-,\theta,\psi,p_\phi,p_\theta,p_\psi) &= \tfrac{1}{2}\left(\frac{\sin^2\psi}{I_1} + \frac{\cos^2\psi}{I_2}\right)\left(\frac{p_\phi - p_\psi\cos\theta}{\sin\theta}\right)^2 + \frac{p_\psi^2}{2I_3} \\ &+\tfrac{1}{2}\ \left(\frac{\cos^2\psi}{I_1} + \frac{\sin^2\psi}{I_2}\right)p_\theta^2 \\ &+\left(\frac{1}{I_1} - \frac{1}{I_2}\right)\ \left(\frac{p_\phi - p_\psi\cos\theta}{\sin\theta}\right)p_\theta\sin\psi\cos\psi\end{aligned} \quad . \tag{1}$$

The coordinate ϕ is cyclic, then p_ϕ is a constant of motion.

We now define $\vec{g} = \sum_i g_i \widehat{b}_i$ is the angular momentum in the body frame and we know that

$$\vec{g} = \mathbb{I}\vec{\omega} = \left(\begin{array}{ccc} I_1\omega_{b_1}, & I_2\omega_{b_2}, & I_3\omega_{b_3} \end{array} \right)^T .$$

We may show that from the conservation of angular momentum

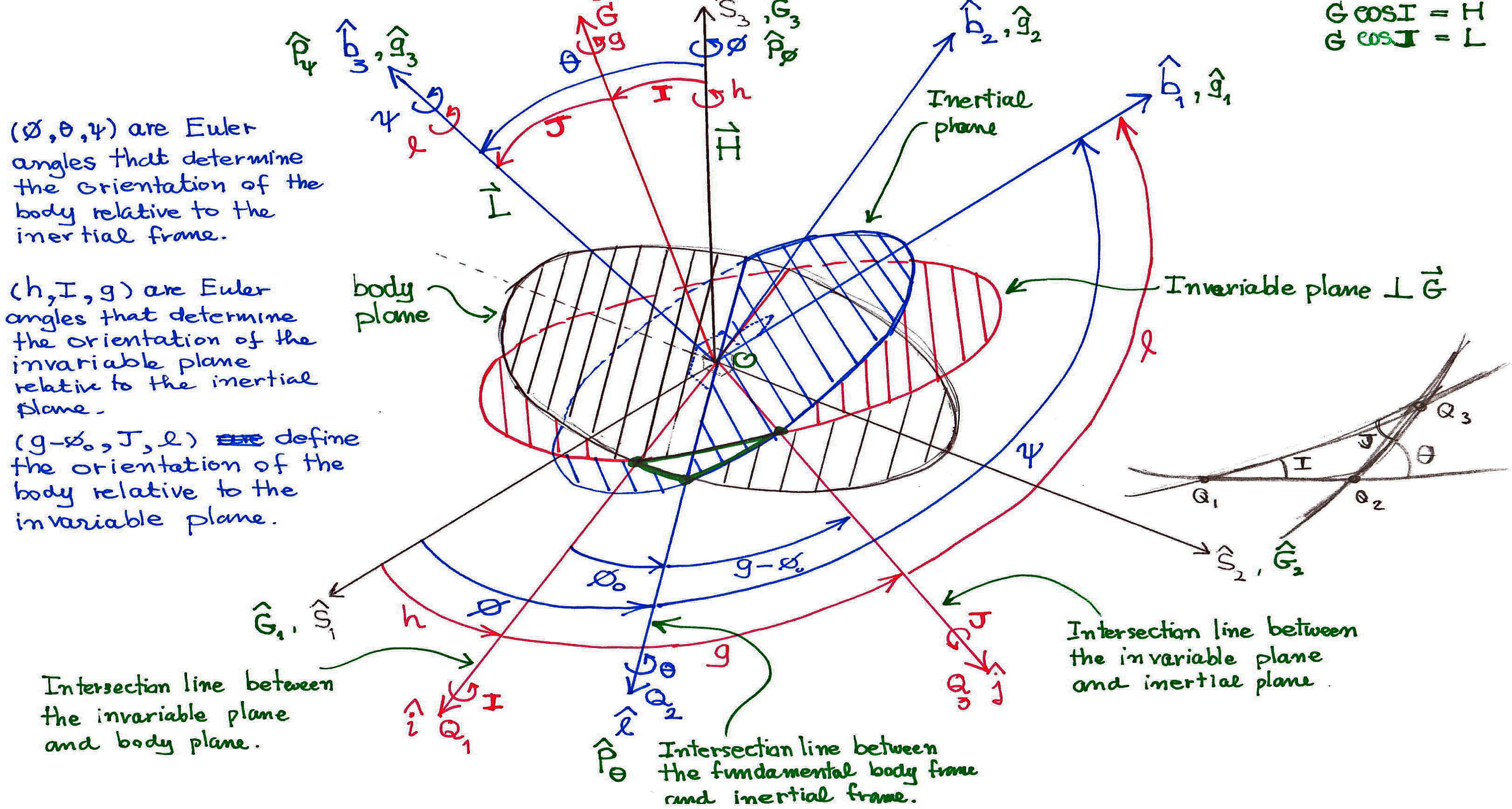
$$G^2 = g_1^2 + g_2^2 + g_3^2,$$

and the conservation of energy

$$\mathbb{H} = E = \frac{1}{2}\left(\frac{g_1^2}{I_1} + \frac{g_2^2}{I_2} + \frac{g_3^2}{I_3}\right).$$

These equations tell us that the trajectories will be confined on the level contour of the *energy ellipsoid*.

$G \cos I = H$
$G \cos J = L$



Invariable plane ⊥ $\vec{G}$

Intersection line between the invariable plane and inertial plane.

Intersection line between the invariable plane and body plane.

Intersection line between the fundamental body frame and inertial frame.

(ϕ, θ, ψ) are Euler angles that determine the orientation of the body relative to the inertial frame.

(h, I, g) are Euler angles that determine the orientation of the invariable plane relative to the inertial plane.

$(g-\phi_0, J, \ell)$ define the orientation of the body relative to the invariable plane.

The reduction of a number of variables

To reduce a number of variables, we need to make another rotation, Fig 2. We now define a 3-1-3-1-3 rotation sequence as follows:

- $R(h, \hat{s}_3)$, a rotation about $\hat{s}_3$ by $0 \leq h \leq 2\pi$, mapping $\hat{s}_1 \rightarrow \hat{\imath}$
- $R(I, \hat{\imath})$, a rotation about $\hat{\imath}$ by $0 \leq I \leq \pi$, mapping $\hat{s}_3 \rightarrow \vec{G}$
- $R(g, \vec{G}/G)$, a rotation about a unit vector pointing in the direction of the angular momentum by $0 \leq g \leq 2\pi$, mapping $\hat{\imath} \rightarrow \hat{\jmath}$

- $R(J,\hat{\jmath})$, a rotation about $\hat{\jmath}$ by $0 \le J \le \pi$, mapping $\hat{G} \to \vec{b}_3$

- $R(l,\hat{b}_3)$, a rotation about $\hat{b}_3$ by $0 \le J \le 2\pi$, mapping $\hat{\jmath} \to \vec{b}_1$

The composite rotation may be written as

$$R(h,I,g,J,l) = R(l,\hat{b}_3)R(J,\hat{\jmath})R(g,\vec{G}/G)R(I,\hat{\imath})R(h,\hat{s}_3).$$

From Fig 2, we define the transformation of momenta by the relations

$$\begin{aligned} p_\phi &= H = G\cos I \\ p_\theta &= G\sin J \sin(l-\psi) \\ p_\psi &= L = G\cos J \end{aligned} \qquad (2)$$

Inserting Eq. (2) into the inverse relations, we have

$$\begin{aligned} I_1\omega_{b_1} &= G\left[\frac{\cos I - \cos J\cos\theta}{\sin\theta}\sin\psi + \sin J\sin(l-\psi)\cos\psi\right] \\ I_2\omega_{b_2} &= G\left[\frac{\cos I - \cos J\cos\theta}{\sin\theta}\cos\psi - \sin J\sin(l-\psi)\sin\psi\right] \\ I_3\omega_{b_3} &= L. \end{aligned} \tag{3}$$

In the spherical triangle $Q_1Q_2Q_3$, we may use an identity of the type

$$\cos I = \cos J\cos\theta + \sin J\sin\theta\cos(\psi - l)$$

Then Eq. (3) becomes

$$\begin{aligned} I_1\omega_{b_1} &= G\sin J\sin l \\ I_2\omega_{b_2} &= G\sin J\cos l \\ I_3\omega_{b_3} &= L. \end{aligned} \tag{4}$$

And we use the fact that $H = G\cos I$ and $L = G\cos J$. Then

$$\begin{aligned} I_1\omega_{b_1} &= \sqrt{G^2 - L^2}\sin l \\ I_2\omega_{b_2} &= \sqrt{G^2 - L^2}\cos l \\ I_3\omega_{b_3} &= L. \end{aligned} \tag{5}$$

The Hamiltonian transforms

$$\mathbb{H}(l, -, -, L, G, -) = \frac{1}{2}\left(\frac{\sin^2 l}{I_1} + \frac{\cos^2 l}{I_2}\right)\left(G^2 - L^2\right) + \frac{L^2}{2I_3}.$$

Three of the phase variable are cyclic.

- H, so the longitude h of the node of the invariable plane in the fixed reference plane $O\widehat{s}_1\widehat{s}_2$ is a constant.

- g, so that the norm G of the angular momentum is a constant.

- h, so that the component H of G is a constant.

Now, the Euler-Poinsot problem is now reduced to conservative Hamiltonian system with only one degree of freedom l.

For $I_1 = I_2$, the Hamiltonian becomes

$$\mathbb{H} = \frac{1}{2}\left(\frac{1}{I_3} - \frac{1}{I_1}\right) L^2 + \frac{G^2}{2I_1}.$$

The variable l being ignorable, the momentum L is a constant,

$$\dot{l} = \frac{\partial \mathbb{H}}{\partial L} = \left(\frac{1}{I_3} - \frac{1}{I_1}\right) L$$

The body rotates around the axis of symmetry $O\widehat{b}_3$ at the uniform rate.

Whereas it precesses in the invariable plane at the uniform rate

$$\dot{g} = \frac{\partial \mathbb{H}}{\partial G} = \frac{G}{I_1}.$$

For $I_1 \neq I_2$, we observe that $-G \leq L \leq G$ and $0 \leq l \leq \pi$, and the equations of motion are

$$\begin{aligned} \dot{l} &= \frac{\partial \mathbb{H}}{\partial L} = \left(\frac{1}{I_3} - \frac{\sin^2 l}{I_1} - \frac{\cos^2 l}{I_2}\right) L && (6) \\ \dot{L} &= -\frac{\partial \mathbb{H}}{\partial l} = \frac{1}{2}\left(\frac{1}{I_2} - \frac{1}{I_1}\right)\left(G^2 - L^2\right) \sin 2l. && (7) \end{aligned}$$

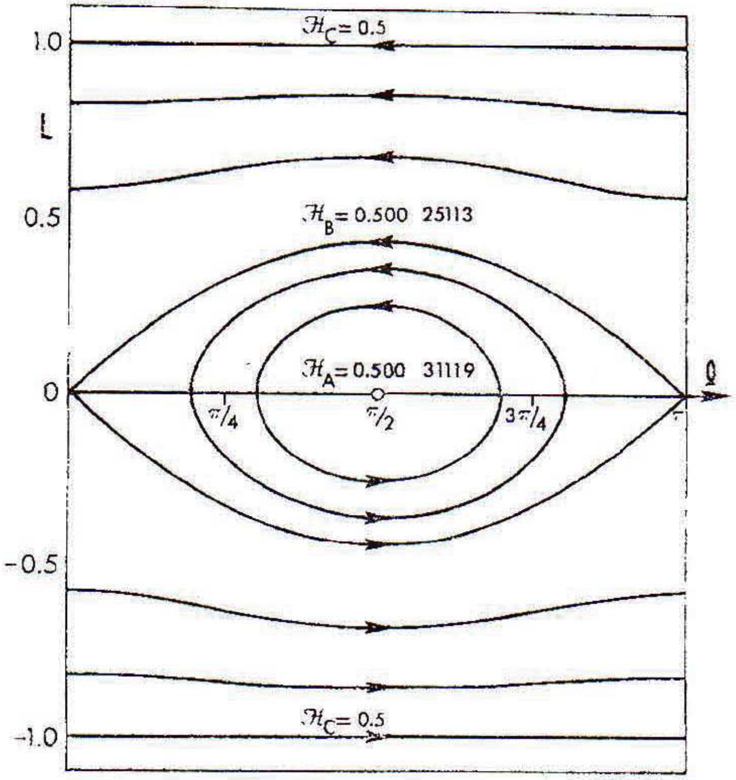


FIG. 2. Isoenergetic curves in the phase rectangle (l, L) of the Euler–Poinsot problem [C. F. Peters, Yale University Observatory].

Action-angle variables

$$\mathbb{H}\left(l, g, h, L, G, H\right) \rightarrow \mathbb{K}\left(\varphi_l, \varphi_g, \varphi_h, I_l, I_g, I_h\right).$$

Action-angle variables result from a type-2 canonical transformation in which the generating function $S = S(q_i, P_i = I_i, t)$. We now have

$$\mathbb{K} = \mathbb{H} + \frac{\partial S}{\partial t}$$

The Hamilton equations are

$$\frac{\partial \mathbb{K}}{\partial I_i} = \dot{\varphi}_i \ , \ \ \frac{\partial \mathbb{K}}{\partial \varphi_i} = -\dot{I}_i$$

The actions being

$$
\begin{aligned}
I_l &= \frac{1}{2\pi}\oint L dl && (8)\\
I_g &= \frac{1}{2\pi}\oint G dg = G && (9)\\
I_h &= \frac{1}{2\pi}\oint H dh = H && (10)
\end{aligned}
$$

The L can be directly obtained from the Hamiltonian

$$L = \sqrt{\frac{2\mathbb{H} - G^2\left(\frac{\sin^2 l}{I_1} + \frac{\cos^2 l}{I_2}\right)}{\frac{1}{I_3} - \left(\frac{\sin^2 l}{I_1} + \frac{\cos^2 l}{I_2}\right)}}$$

We can rewrite the function L in the form

$$L = \sqrt{I_3}\sqrt{\frac{a + b\,\mathsf{sin}^2 l}{c + d\,\mathsf{sin}^2 l}}$$

where

$$a = I_1\left(G^2 - 2EI_2\right)\ ,\ \ b = \ (I_2 - I_1)\,G^2 > 0$$
$$c = I_1\,(I_3 - I_2) > 0\ ,\ \ d = I_3\ \ (I_2 - I_1) > 0$$

Then the action I_l becomes

$$I_l = \frac{1}{2\pi}\oint\sqrt{I_3}\sqrt{\frac{a + b\,\mathsf{sin}^2 l}{c + d\,\mathsf{sin}^2 l}}dl.$$

The angles being

$$\dot{\varphi}_l = \frac{\partial \mathbb{K}}{\partial I_l} \ , \ \ \dot{\varphi}_g = \frac{\partial \mathbb{K}}{\partial I_g} \ \ , \ \ \dot{\varphi}_h = \frac{\partial \mathbb{K}}{\partial I_h} \ .$$

The generating function S of the transformation can be found through solving the corresponding Hamilton-Jacobi equation

$$\frac{1}{2}\left(\frac{\sin^2 l}{I_1} - \frac{\cos^2 l}{I_2}\right)\left[\left(\frac{\partial S}{\partial g}\right)^2 - \left(\frac{\partial S}{\partial l}\right)^2\right] + \frac{1}{2I_3}\left(\frac{\partial S}{\partial l}\right)^2 + \frac{\partial S}{\partial t} = 0.$$

Since the system is autonomous and variables g and h are cyclic, the generating function can be expressed

$$S = -\alpha t + I_g g + I_h h + W(l; I_l, I_g).$$

We find that

$$\begin{array}{l} L = \frac{\partial W}{\partial l} \ , \ \ \varphi_l = \frac{\partial W}{\partial I_l} \\ G = \frac{\partial S}{\partial g} = I_g \ , \ \ \varphi_g = \frac{\partial S}{\partial I_g} \ = g + \frac{\partial W}{\partial I_g} \ , \\ H = \frac{\partial S}{\partial h} = I_h \ , \ \ \varphi_h = \frac{\partial S}{\partial I_h} \ = h \end{array}$$

and

$$\mathbb{H} = -\frac{\partial S}{\partial t} = \alpha.$$

The function W can be directly obtained from

$$W(l; I_l, I_g) = \int\limits_{l_0}^{l} L(l; I_l, I_g) dl,$$

which can be expressed in term of the Elliptic functions.

Kozlov studied the geometric analysis of the actions

$$I_l = \frac{1}{2\pi}\oint \sqrt{\frac{2H - G^2\left(\frac{\sin^2 l}{I_1} + \frac{\cos^2 l}{I_2}\right)}{\frac{1}{I_3} - \left(\frac{\sin^2 l}{I_1} + \frac{\cos^2 l}{I_2}\right)}} dl = \frac{1}{2\pi}\oint \sqrt{I_3}\sqrt{\frac{a + b\sin^2 l}{c + d\sin^2 l}} dl,$$

$$I_g = \frac{1}{2\pi}\oint G dg = G$$