1

Let Z be a unit normal random variable, and for a fixed x, set

$$X = \begin{cases} Z & \text{if } Z > x \\ 0 & \text{otherwise} \end{cases}$$

Show that $E[X] = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$.

2

Urn 1 contains 5 white and 6 black balls, while urn 2 contains 8 white and 10 black balls. Two balls are randomly selected from urn 1 and are then put in urn 2. If 3 balls are then randomly selected from urn 2, compute the expected number of white balls in the trio.

HINT: Let $X_i = 1$ if the ith white ball initially in urn 1 is one of the three selected, and let $X_i = 0$ otherwise. Similarly, let $Y_i = 1$ if the ith white ball from urn 2 is one of the three selected, and let $Y_i = 0$ otherwise. The number of white balls in the trio can now be written as $\sum_1^5 X_i + \sum_1^8 Y_i$.

3

A fair die is successively rolled. Let X and Y denote, respectively, the number of rolls necessary to obtain a 6 and a 5. Find

(a) $E[X]$;

(b) $E[X|Y = 1]$;

(c) $E[X|Y = 5]$.

4

The joint density of X and Y is given by

$$f(x, y) = \frac{e^{-x/y}e^{-y}}{y} \qquad 0 < x < \infty, 0 < y < \infty$$

Compute $E[X^2|Y = y]$.

5

The joint density of X and Y is given by

$$f(x, y) = \frac{e^{-y}}{y} \qquad 0 < x < y, 0 < y < \infty$$

Compute $E[X^3|Y = y]$.

6

Consider the following dice game. A pair of dice are rolled. If the sum is 7, then the game ends and you win 0. If the sum is not 7, then you have the option of either stopping the game and receiving an amount equal to that sum or starting over again. For each value of i, $i = 2, \ldots, 12$, find your expected return if you employ the strategy of stopping the first time that a value at least as large as i appears. What value of i leads to the largest expected return?

HINT: Let X_i denote the return when you use the critical value i. To compute $E[X_i]$, condition on the initial sum.

7

A coin having probability p of coming up heads is continually flipped until both heads and tails have appeared. Find

(a) the expected number of flips;

(b) the probability that the last flip lands heads.

8

Let $U_1, U_2, \ldots$ be a sequence of independent uniform (0, 1) random variables. In Example 4h we showed that for $0 \le x \le 1$, $E[N(x)] = e^x$, where

$$N(x) = \min\left\{n: \sum_{i=1}^{n} U_i > x\right\}$$

This problem gives another approach to establishing this result.

(a) Show by induction on n that for $0 < x \le 1$ and all $n \ge 0$,

$$P\{N(x) \ge n + 1\} = \frac{x^n}{n!}$$

HINT: First condition on U_1 and then use the induction hypothesis.

(b) Use part (a) to conclude that

$$E[N(x)] = e^x$$

9

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed positive random variables. Find, for $k \le n$,

$$E\left[\frac{\sum_{i=1}^{k} X_i}{\sum_{i=1}^{n} X_i}\right]$$

10

Let $X_1, \ldots, X_n$ be independent and identically distributed random variables. Find

$$E[X_1 | X_1 + \cdots + X_n = x]$$

11

(a) Prove that

$$E[X] = E[X | X < a]P\{X < a\} + E[X | X \geq a]P\{X \geq a\}$$

HINT: Define an appropriate random variable and then compute $E[X]$ by conditioning on it.

(b) Use part (a) to prove Markov's inequality, which states that if $P\{X \geq 0\} = 1$, then for $a > 0$,

$$P\{X \geq a\} \leq \frac{E[X]}{a}$$

12

Let X be a normal random variable with parameters $\mu = 0$ and $\sigma^2 = 1$ and let I, independent of X, be such that $P\{I = 1\} = \frac{1}{2} = P\{I = 0\}$. Now define Y by

$$Y = \begin{cases} X & \text{if } I = 1 \\ -X & \text{if } I = 0 \end{cases}$$

In words, Y is equally likely to equal either X or $-X$.

(a) Are X and Y independent?
(b) Are I and Y independent?
(c) Show that Y is normal with mean 0 and variance 1.
(d) Show that $\text{Cov}(X, Y) = 0$.

13

If $Y = aX + b$, where a and b are constants, express the moment generating function of Y in terms of the moment generating function of X.