

How many claims to get ruined and recovered?

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Abstract

We consider in the classical surplus process the number of claims occurring up to ruin, by a different method presented by Stanford & Stroiński (1994). We consider the computation of Laplace transforms which can allow the computation of the probability function. Formulae presented are general.

The method uses the computation of the probability function of the number of claims during a negative excursion of the surplus process, in case it gets ruined. When initial surplus is zero this probability function allows us to completely define the recursion for the transform above. This uses the fact that in this particular case, conditional time to ruin has the same distribution as the time to recovery, given that ruin occurs.

We consider yet the computation of moments of the number of claims during recovery time, which with initial surplus zero allow us to compute the moments of the number of claims up to ruin.

Keywords: Probability of ruin; claim number up to ruin; claim number up to recovery; time to ruin; duration of negative surplus; severity of ruin; recursive methods.

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1 Introduction

In this work we consider the classical risk process, where claims occur as a Poisson process. Much has been studied and said in the actuarial literature over the classical model about ruin probabilities, either finite or infinite time. We know that if ruin is to occur it does at the instant of a claim. We can thus think on ruin not by directly addressing the *waiting time* of the event 'ruin' (finite time ruin probability), but think of the waiting time in terms of number of claims that occur until the process gets ruined, if it does. This isn't a new approach, for instance Stanford & Stroiński (1994) dealt with this problem in the classical model for phase-type distributed claim sizes. Recently Stanford, Stroiński & Lee (2000) extended the same approach to some non-Poisson claim processes. Both papers deal with the problem by studying the increment (positive or negative) on the risk reserve between two consecutive claims as the difference between the revenue earned and the claim amount. Their approach involves a recursive evaluation of Laplace-Stieltjes transforms allowing the calculation of the probability of ruin on the n -th claim occurrence ($n = 1, 2, \dots$), by evaluating the transform at the origin. The transform is based on the joint probability of non-ruin up to the n -th claim and the reserve remaining after the n -th claim occurrence less an appropriate level, say y . The authors considered phase-type distributed claim sizes, in particular exponential, mixtures of exponentials and Erlang.

We restrict here to the study of the same problem in the classical model, the evaluation of the probability of ruin occurring at the n -th claim ($n = 1, 2, \dots$) in the classical model on a more general way, using a completely different approach, more direct, say *classical*, by enhancing the relationship between time to ruin and duration of a negative surplus, once ruin has occurred, with an initial surplus zero. This has been explained by Egídio dos Reis (1993). We extend the study to the number of claims occurring during a first period of negative surplus.

Let $\{U(t), t \geq 0\}$ be the classical continuous time surplus process so that

$$U(t) = u + ct - S(t), \quad t \geq 0,$$

where $u (\geq 0)$ is the insurer's initial surplus, c is the constant insurer's rate of premium income per unit time, $S(t) = \sum_{j=1}^{N(t)} X_j$ the aggregate claim amount up to time t , $N(t)$ the number of claims in the same time interval having a Poisson (λt) distribution, $S(t) = 0$ if $N(t) = 0$, and $\{X_j\}_{j=1}^{\infty}$ a sequence of i.i.d. random variables representing the individual claim amounts. $\{X_j\}$ is independent of $\{N(t)\}$. We denote by $B(x)$ and $b(x)$ the common distribution and density function of X_j , respectively, with $B(x) = 0$, so that all claim amounts are positive. We also assume that the mean of X_j , which we denote by b_1 , is finite and that $c > \lambda b_1$. For simplicity we write $a = \lambda/c$. We will further assume in some parts of this paper the existence of the moment generating function of X_j , which we denote by $m(s) = E[e^{sX_j}]$, and we state that clearly where appropriate.

Define the time until ruin, denoted T , by

$$T = \begin{cases} \inf\{t : U(t) < 0\} \\ \infty & \text{if } U(t) \geq 0 \forall t \end{cases}$$

We denote by $T_c = T|T < \infty$ the conditional random variable time to ruin, given that ruin occurs. The probability of ultimate ruin from initial surplus u for this risk process is defined as

$$\psi(u) = \Pr\{U(t) < 0 \text{ for some positive } t | U(0) = u\} = \Pr\{T < \infty | U(0) = u\}$$

and let $\delta(u) = 1 - \psi(u)$ denote the survival probability. It is well known that $\psi(0) = ab_1$. If the moment generating function of X_j exists in an appropriate open interval, then the adjustment coefficient for this risk process is the unique positive number R such that

$$\lambda + cR = \lambda m(R) . \quad (1)$$

Let $G(u; x)$ and $g(u; x)$ be the (defective) distribution and density function of the probability and severity of ruin, respectively:

$$G(u; x) = \Pr \{T < \infty \text{ and } U(T) > -x | U(0) = u\} \text{ and } \frac{d}{dx} G(u; x) = g(u; x)$$

It is well known that $g(0; x) = a[1 - B(x)]$, see for instance Bowers, Gerber, Hickman, Jones & Nesbitt (1986). We denote by Y and Y_c the defective random variable of the severity of ruin and the conditional severity of ruin, given that ruin occurs, respectively.

Let $P(u; n)$ be the probability that ruin occurs before or at the n -th claim ($n = 1, 2, \dots$) from initial surplus $u \geq 0$ and $p(u; n)$ be the respective probability function. Denote the associated random variable by M . Obviously, we have that

$$\begin{aligned} p(u; 1) &= P(u; 1) \\ p(u; n+1) &= P(u; n+1) - P(u; n), \quad n \geq 1 \end{aligned}$$

and $\psi(u) = \lim_{n \rightarrow \infty} P(u; n)$.

Consider the surplus process ongoing even if ruin occurs at some instant. Once ruin has occurred, the process will be passing through negative values temporarily, as it will recover back to positive surplus values with probability one [please see Egídio dos Reis (1993)]. Let \tilde{T}_c be the duration of the surplus excursion through negative surplus values up to recovery or time to recovery, conditional on $T < \infty$. Denote by $q(u; n)$ as being the probability of having n claims before the surplus process recovers to non-negative values, given ruin has occurred and initial surplus u . The support of the r.v., say K , is the set $\{0, 1, 2, \dots\}$.

2 On the number of claims to get ruined

Considering the first claim occurrence we have that

$$\begin{aligned} P(u; 1) &= \int_0^\infty \lambda \exp\{-\lambda t\} \int_{u+ct}^\infty b(x) dx dt = \int_0^\infty \lambda \exp\{-\lambda t\} [1 - B(u+ct)] dt \\ P(u; n+1) &= P(u; 1) + \int_0^\infty \lambda \exp\{-\lambda t\} \int_0^{u+ct} b(x) P(u+ct-x; n) dx dt, \quad n \geq 1 \end{aligned}$$

and

$$\begin{aligned} p(u; n+1) &= \int_0^\infty \lambda \exp\{-\lambda t\} \int_0^{u+ct} b(x) p(u+ct-x; n) dx dt, \quad n \geq 1 \\ &= c^{-1} \int_{r=u}^\infty \lambda \exp\left\{-\left(\frac{r-u}{c}\right)\right\} \int_{x=0}^r b(x) p(r-x; n) dx dr \end{aligned} \quad (2)$$

putting $r = u + ct$. Similarly, we have that

$$p(u; 1) = ae^{au} \int_u^\infty e^{-ax} [1 - B(x)] dx . \quad (3)$$

Differentiating on u and setting $a = \lambda/c$, we have that for $n \geq 1$

$$\begin{aligned}\frac{d}{du}p(u; n+1) &= \int_{r=u}^{\infty} a^2 \exp\{-a(r-u)\} \int_{x=0}^r b(x)p(r-x; n)dxdt - a \int_0^u b(x)p(r-x; n)dx \\ &= ap(u; n+1) - a \int_0^u b(x)p(r-x; n)dx\end{aligned}\quad (4)$$

Let $\bar{p}(s; n+1) = \int_0^{\infty} e^{-su}p(u; n+1)du$ be the Laplace Transform (simply, LT) of $p(u; n+1)$. The LT of (4) comes

$$\begin{aligned}s\bar{p}(s; n+1) - p(0; n+1) &= a\bar{p}(s; n+1) - a\bar{b}(s)\bar{p}(s; n) \Leftrightarrow \\ \bar{p}(s; n+1) &= \frac{p(0; n+1) - a\bar{b}(s)\bar{p}(s; n)}{s-a}, \quad n \geq 1,\end{aligned}\quad (5)$$

where $\bar{b}(s)$ is the LT of $b(x)$. For $p(u; 1)$ we get its Laplace Transform using (3)

$$\begin{aligned}\bar{p}(s; 1) &= \int_0^{\infty} ae^{-(s-a)u} \int_u^{\infty} e^{-ax} [1 - B(x)] dxdu \\ &= \int_0^{\infty} ae^{-ax} [1 - B(x)] \int_0^x e^{-(s-a)u} du dx \\ &= (s-a)^{-1} \left(\int_0^{\infty} ae^{-ax} [1 - B(x)] dx - \int_0^{\infty} ae^{-sx} [1 - B(x)] dx \right), \quad s \neq a \\ &= (s-a)^{-1} (\bar{g}(0; a) - \bar{g}(0; s)), \quad s \neq a,\end{aligned}$$

where $\bar{g}(0; s)$ is the LT of the density $g(0, x)$. Note that

$$\lim_{s \rightarrow a} \bar{g}(0; s) = - \lim_{s \rightarrow a} \frac{d}{ds} \bar{g}(0; s) = \int_0^{\infty} xe^{-ax} g(0, x) dx,$$

using l'Hôpital's rule.

The transform $\bar{p}(s; n+1)$ ($n \geq 1$) is a recursive function and as a function of $p(0; n+1)$, to evaluate the recursion we need to compute $p(0; n+1)$. We can find an expression for $p(0; n+1)$, not practical though. From (2) setting $u = 0$ we get

$$\begin{aligned}p(0; n+1) &= \int_{s=0}^{\infty} ae^{-as} \int_{x=0}^s b(x)p(s-x; n)dxds \\ &= \int_{s=0}^{\infty} ae^{-as} b * p(s; n)ds \\ &= a\bar{p}(a; n)\bar{b}(a), \quad n = 1, 2, \dots\end{aligned}\quad (6)$$

That is, the above expression is the LT of the convolution $b * p(s; n)$ evaluated at $a = \lambda/c$. I recall that this is a positive constant. For $n = 1$ (and $u = 0$) we have immediately from (3)

$$p(0; 1) = \bar{g}(0; a) \quad (7)$$

If we put (6) into (5) and compute the limit as $s \rightarrow a$, we see that

$$\lim_{s \rightarrow a} \bar{p}(s; n+1) = - \lim_{s \rightarrow a} a \left[\bar{p}(s; n) \frac{d}{ds} \bar{b}(s) + \bar{b}(s) \frac{d}{ds} \bar{p}(s; n) \right], \quad n = 1, 2, \dots$$

using l'Hôpital's rule. We still need the starting value for the recursion in formula (6), $\bar{p}(a; 1)$. It can be got easily, as follows: From (3) we have that $p(u; 1) = ae^{au} \int_u^\infty e^{-ax} [1 - B(x)] dx$

$$\begin{aligned}\bar{p}(a; 1) &= \int_0^\infty e^{-au} ae^{au} \int_u^\infty e^{-ax} [1 - B(x)] dx du \\ &= \int_0^\infty \int_u^\infty ae^{-ax} [1 - B(x)] dx du \\ &= \int_0^\infty ae^{-ax} [1 - B(x)] \int_0^x du dx \\ &= \int_0^\infty xe^{-ax} g(0, x) dx = -\bar{g}'(0, a),\end{aligned}\tag{8}$$

where $\bar{g}'(0, a) = \frac{d}{ds} \bar{g}'(0, s) \Big|_{s=a}$. We'll see in subsequent sections how to better compute $p(0; n+1)$, $n = 1, 2, \dots$

3 On the number of claims to get recovered

Let's now consider the calculation of $q(u; n)$, the probability of having n claims before the surplus process recovers to non-negative values, given that ruin has occurred, or the (conditional) probability of having n claims during a negative surplus excursion. The support of the r.v. K is the set $\{0, 1, 2, \dots\}$, we note that we don't include the claim that has (just) caused ruin. This one is included on the number of "claims to get ruined". We can follow different approaches for the computation of $q(u; n)$.

First, we follow the paper by Gerber (1990) and consider first the particular case, when the surplus process starts with $u = 0$. Let T_x be the time of the first passage of the surplus process through a fixed positive level x starting from initial surplus zero. From Dickson & Gray (1984) we know that the process gets to positive level x without having occurred a single claim is equivalent to be for the first time at this level at time $T_x = x/c$, and the probability is

$$\Pr[T_x = x/c] = e^{-\lambda x/c} = e^{-ax}\tag{9}$$

Consider now back to the general surplus process with $u \geq 0$, and consider that ruin has occurred at some instant (T), with given deficit y . Given $Y_c = y$, the time that the process gets back to the zero level for the first time without any claim occurrence has probability $\exp\{-ay\}$. The proper distribution for the deficit at the time of ruin is $G(u, y)/\psi(u)$. Then to get the probability of having zero claims until the process recovers to level zero is got by *averaging* $\exp\{-ay\}$ over the distribution of Y_c . That is,

$$q(u; 0)\psi(u) = \int_0^\infty e^{-ax} g(u; x) dx = \bar{g}(u; a),\tag{10}$$

giving the LT of $g(u; x)$ evaluated at $s = a$. For a positive integer n we need to establish an equation for $q(u; n)$.

Consider first the calculation of $q(u; 1)$. Suppose that ruin occurs at time T with $-U(T) = y$ and restart the process from $-y$ and let the process upcross the level zero, or start from zero and cross the positive fixed level y . If no claim occurs the process will recover at time $T_y = y/c$. Consider that one claim occurs at instant t before recovery. The instant must lie in the interval

$(0, y/c)$ with density $\lambda \exp\{-\lambda t\}$. The amount of the claim is x with density $b(x)$. Just after this event the surplus will be $-y + ct - x$. No claim occurs until the process recovers from here. This has probability $\exp\{-a(x + y - ct)\}$ (see above). If we then average with the distribution of the conditional severity of ruin, we get that

$$\begin{aligned}\psi(u)q(u; 1) &= \int_{y=0}^{\infty} \int_{t=0}^{y/c} \lambda e^{-\lambda t} \int_{x=0}^{\infty} e^{-a(x+y-ct)} b(x) dx dt g(u; y) dy \\ &= \int_{y=0}^{\infty} e^{-ay} g(u; y) \int_{t=0}^{y/c} \lambda e^{-\lambda t} e^{\lambda t} \int_{x=0}^{\infty} e^{-ax} b(x) dx dt dy \\ &= \bar{b}(a) \int_{y=0}^{\infty} a y e^{-ay} g(u; y) dy = -a \bar{b}(a) \bar{g}'(u; a),\end{aligned}$$

where $\bar{g}'(u; a)$ is the derivative of the LT of $g(u; x)$ evaluated at $s = a$. For $n = 2, 3, \dots$ we can proceed recursively.

Let $q(n|y)$ be the conditional probability of having n claims before the process recovers to non-negative values, for a given severity $Y_c = y$, or the conditional probability of having exactly n claims before the process upcrosses the positive level y . Note that $q(n|y)$ is independent of u . Following the reasoning above we have that

$$\psi(u)q(u; n) = \int_{y=0}^{\infty} \int_{t=0}^{y/c} \lambda e^{-\lambda t} \int_{x=0}^{\infty} q(n-1|x+y-ct) b(x) dx dt g(u; y) dy, \quad n = 1, 2, \dots \quad (11)$$

Recall that one claim must occur at time, say t , before time y/c with amount x . At this instant the surplus will be at the (negative) value $-(x + y - ct)$, and then has to upcross level 0 after having occurred $(n-1)$ claims. Note that we can have the starting expression for $q(0|\cdot)$ from (9). To better evaluate (11) we can use direct results from Gerber (1990) just like what follows:

For a given positive value x and $S_k = X_1 + X_2 + \dots + X_k$ and $S_0 \equiv 0$, for $k = 0, 1, 2, \dots$, Gerber (1990) shows that

$$q(k|x, S_k) = x \frac{a^k (S_k + x)^{k-1}}{k!} e^{-a(S_k + x)},$$

i.e., the “conditional probability that the process $\{U(t)|u=0\}$ will pass *for the first time* through the level x between the k -th and the $(k+1)$ -th jump”, in his terminology. Hence,

$$q(n-1|x+y-ct) = \int_0^{\infty} (x+y-ct) \frac{a^{n-1} (z+x+y-ct)^{n-2}}{k!} e^{-a(z+x+y-ct)} b^{*(n-1)}(z) dz.$$

However, we don't need to evaluate the inner double integral in (11). We can compute directly

$$q(k|x) = \int_0^{\infty} x \frac{a^k (z+x)^{k-1}}{k!} e^{-a(z+x)} b^{*k}(z) dz, \quad k = 1, 2, \dots$$

We have that $(z+x)^{k-1} = \sum_{n=0}^{k-1} \binom{k-1}{n} z^n x^{k-1-n}$ ($k = 1, 2, \dots$). Then

$$\begin{aligned}q(k|x) &= \frac{x a^k}{k!} e^{-ax} \int_0^{\infty} \sum_{n=0}^{k-1} \binom{k-1}{n} z^n x^{k-1-n} e^{-az} b^{*k}(z) ds \\ &= \frac{x a^k}{k!} e^{-ax} \sum_{n=0}^{k-1} \binom{k-1}{n} x^{k-1-n} \int_0^{\infty} z^n e^{-az} b^{*k}(z) ds\end{aligned}$$

$$\begin{aligned}
&= \frac{xa^k}{k!} e^{-ax} \left(\sum_{n=1}^{k-1} \binom{k-1}{n} x^{k-1-n} (-1)^n \frac{d^n}{ds^n} \int_0^\infty e^{-sz} b^{*k}(z) dz \Big|_{s=a} + x^{k-1} \bar{b}(a)^k \right) \\
&= \frac{a^k}{k!} e^{-ax} \left(\sum_{n=1}^{k-1} \binom{k-1}{n} x^{k-n} (-1)^n \frac{d^n}{ds^n} \bar{b}(s)^k \Big|_{s=a} + x^k \bar{b}(a)^k \right)
\end{aligned}$$

Now,

$$\begin{aligned}
\psi(u)q(u; k) &= \int_0^\infty q(k|x)g(u, x)dx \\
&= \frac{a^k}{k!} \left(\sum_{n=1}^{k-1} \binom{k-1}{n} (-1)^n \left(\frac{d^n}{ds^n} \bar{b}(s)^k \Big|_{s=a} \right) \int_0^\infty x^{k-n} e^{-ax} g(u, x)dx \right. \\
&\quad \left. + \bar{b}(a)^k \int_0^\infty x^k e^{-ax} g(u, x)dx \right) \\
&= \frac{a^k}{k!} \left(\sum_{n=1}^{k-1} \binom{k-1}{n} (-1)^n \left(\frac{d^n}{ds^n} \bar{b}(s)^k \Big|_{s=a} \right) (-1)^{k-n} \left(\frac{d^{k-n}}{ds^{k-n}} \bar{g}(u; s) \Big|_{s=a} \right) \right. \\
&\quad \left. + \bar{b}(a)^k (-1)^k \left(\frac{d^k}{ds^k} \bar{g}(u; s) \Big|_{s=a} \right) \right) \\
&= \frac{(-a)^k}{k!} \left(\sum_{n=1}^{k-1} \binom{k-1}{n} \left(\frac{d^n}{ds^n} \bar{b}(s)^k \Big|_{s=a} \right) \left(\frac{d^{k-n}}{ds^{k-n}} \bar{g}(u; s) \Big|_{s=a} \right) \right. \\
&\quad \left. + \bar{b}(a)^k \left(\frac{d^k}{ds^k} \bar{g}(u; s) \Big|_{s=a} \right) \right) \\
&= \frac{(-a)^k}{k!} \left(\sum_{n=0}^{k-1} \binom{k-1}{n} \left(\frac{d^n}{ds^n} \bar{b}(s)^k \Big|_{s=a} \right) \left(\frac{d^{k-n}}{ds^{k-n}} \bar{g}(u; s) \Big|_{s=a} \right) \right),
\end{aligned}$$

where the symbol $\frac{d^0}{ds^0} \bar{b}(s)^k \Big|_{s=a} = \bar{b}(a)^k$.

If we apply Leibnitz's rule for derivatives of products we get

$$\psi(u)q(u; k) = \frac{(-a)^k}{k!} \left(\frac{d^{k-1}}{ds^{k-1}} \left(\bar{b}(s)^k \bar{g}'(u; s) \right) \Big|_{s=a} \right)$$

In summary we have

$$\begin{aligned}
q(u; 0) &= \bar{g}(u; a)/\psi(u) \\
q(u; 1) &= -a\bar{b}(a)\bar{g}'(u; a)/\psi(u) \\
q(u; n) &= \frac{(-a)^n}{\psi(u)n!} \left(\frac{d^{n-1}}{ds^{n-1}} \left(\bar{b}(s)^n \bar{g}'(u; s) \right) \Big|_{s=a} \right), \quad n = 2, 3, \dots
\end{aligned} \tag{12}$$

The calculation of the probabilities of the (conditional) number of claims during a negative excursion of the surplus process involves the computation of the Laplace transforms of the claim amount distribution and the distribution of the probability and severity of ruin, and, of course, the ultimate ruin probability. The LT's exist at least for $s \geq 0$, see Gerber (1979), and we should able

to compute them, at least numerically. In the subsequent section we will consider the particular case $u = 0$. Different authors have concerned about the computation of the distribution of the distribution $G(u, x)$. A reference paper is Gerber, Goovaerts & Kaas (1987). For other references please see Lin & Willmot (1999). See also Willmot (2000).

4 The case with zero initial surplus, $u = 0$

We consider here, in particular, the computation of $q(0; n)$, $n = 0, 1, 2, \dots$, which will allow us to compute $p(0; n)$, $n = 1, 2, \dots$ as we will establish a relationship between these two probability functions.

For the density of the (defective) severity of ruin with $u = 0$ we know that $g(0; x) = a[1 - B(x)]$ giving $\bar{g}(0; s) = a[1 - \bar{b}(a)]/s$ and $\bar{g}'(0; s) = -[\bar{g}(0; s) + a\bar{b}'(s)]/s$, so that $\bar{g}(0; a) = 1 - \bar{b}(a)$ and $\bar{g}'(0; a) = -\bar{g}(0; a)/a - \bar{b}'(a)$. These will be in $q(0; 0)$ and $q(0; 1)$. Knowing that $\psi(0) = ab_1$, we have that

$$\begin{aligned} q(0; 0) &= \bar{g}(0; a)/\psi(0) = [1 - \bar{b}(a)]/ab_1 \\ q(0; 1) &= -a\bar{b}(a)\bar{g}'(0; a)/\psi(0) = \bar{b}(a) [\bar{g}(0; a)/a + \bar{b}'(a)]/b_1 \\ q(0; n) &= -\frac{(-a)^{n-1}}{b_1 n!} \left(\frac{d^{n-1}}{ds^{n-1}} (\bar{b}(s)^n \bar{g}'(0; s)) \right) \Big|_{s=a}, \quad n = 2, 3, \dots \end{aligned} \quad (13)$$

We can establish easily a direct relation between the claims arriving during a negative surplus excursion and claim number until ruined when $u = 0$. Consider both the conditional random variables of the time to ruin and the time to recovery, given that $T < \infty$, T_c and \bar{T}_c with initial surplus $u = 0$. As explained by Egídio dos Reis (1993) This two have the same distribution in this particular case. Thus there is an obvious relation between the conditional r.v.'s $M|T < \infty$ and K in the same particularization. They don't have exactly the same distribution, the support set is different. Once ruin as occurred we may have zero claim occurrence until the process recovers. That is, the claim that causes ruin is not counted as a claim in the negative excursion until recovery. On the other hand, once the process starts with initial surplus zero it will never get ruined without any claim. So we need at least one claim, the claim that causes ruin, to have a negative deficit at T .

We recall that support of the r.v. K is the set $\{0, 1, 2, \dots\}$. Given the above reasons in the previous paragraph we can conclude that

$$\psi(0)q(0; n) = p(0; n + 1), \quad n = 0, 1, 2, \dots \quad (14)$$

We see from (7) and (13) that $p(0; 1) = \psi(0)q(0; 0) = \bar{g}(0; a)$. We could show an analytical proof by induction, using (2) and formulae (13). Thus, we can evaluate completely recursion (5) and then by inversion compute the probability function $\{p(0; n), n = 1, 2, \dots\}$.

5 On the moments of the number of claims to get recovered

In this subsection we assume that the moment generating function exists, in the classical model, so the adjustment coefficient R exists. Consider the surplus process ongoing even if ruin as occurred at some instant.

We first do a *re-cap* and take again the work by Gerber(1990) i.e., consider the particular case of the surplus process with $u = 0$. For this particular process, let \tilde{K} denote the number of claims occurring before the first upcrossing of the process at positive level x , having or not ruin occurred. Gerber (1990) showed that the moment generating function of \tilde{K} is given by

$$E[e^{s\tilde{K}}] = e^{g(s)x} , \quad (15)$$

where $g(s)$ is a function such that

$$s = \ln \frac{\lambda + cg(s)}{\lambda m(g(s))} , \quad (16)$$

and $\lambda/c < g(s) \leq 0$ and $s \leq 0$. This follows from the fact that $g(0) = 0$, the derivatives of both the numerator and the denominator in (16) are positive. The numerator is zero for $g(s) = -\lambda/c$, and the denominator is always positive. If there are no restrictions on the range of $g(s)$ in expression (16), we see that the numerator equals the denominator for $g(s) = 0$ or $g(s) = R$, and that for values of $g(s) < 0$ ($g(s) > -\lambda/c$) or $g(s) > R$, the fraction is between zero and one, and so the logarithm is negative. If we take the first two derivatives of the cumulant generating function, $\ln E[e^{s\tilde{K}}] = g(s)x$, and evaluate at them at $g(0) = 0$ we get easily

$$E[\tilde{K}] = \lambda x / c\delta(0) \text{ and } V[\tilde{K}] = \lambda x (c^2 + \lambda^2 \sigma_X^2) / [c\delta(0)]^3 ,$$

where $\sigma_X^2 = V[X_i] = b_2 - b_1^2$.

Consider now the general model with initial surplus $u \geq 0$. If we now follow what is developed by Egídio dos Reis (1993), Section 3, and take the expected value of (15) with respect to the conditional distribution of the severity of ruin, given $T < \infty$, we get the moment generating function of the number of claims occurring during a negative surplus K , given that ruin occurs. We denote this moment generating function as $M_K(u, s)$. Hence

$$M_K(u; s) = M_{Y_c}(u; g(s)) , \quad (17)$$

where $g(s)$ is defined as for (16). This mgf is simply got by setting $x = Y_c$ and taking expectations. If we take the first two derivatives of the cumulant generating function $\phi(s) = \ln M_{K_c}(u, s) = \ln M_{Y_c}(u, g(s))$, we get

$$\begin{aligned} \phi'(s) &= s'(g)^{-1} \frac{d}{dg} \ln M_{Y_c}(u, g) \\ \phi''(s) &= -\frac{s''(g)}{s'(g)^2} \phi'(s) + s'(g)^{-2} \frac{d^2}{dg^2} \ln M_{Y_c}(u, g) \end{aligned}$$

evaluate at $g(0) = 0$, knowing that

$$s'(0) = c\delta(0)/\lambda \text{ and } s''(0) = -(c^2 + \lambda^2 \sigma_X^2) / \lambda^2$$

we get

$$\begin{aligned} E[K_c] &= \lambda E[Y_c] / c\delta(0) = \lambda E[\tilde{T}_c | u] \\ V[K_c] &= \frac{\lambda}{[c\delta(0)]^2} \left(\frac{c^2 + \lambda^2 \sigma_X^2}{c\delta(0)} E[Y_c | u] + \lambda V[Y_c | u] \right) = \lambda \left(\frac{1 + \psi(0)}{1 - \psi(0)} E[\tilde{T}_c | u] + \lambda V[\tilde{T}_c | u] \right) , \end{aligned}$$

For the expressions for the moments $E[\tilde{T}_c|u]$ and $V[\tilde{T}_c|u]$ please see Egídio dos Reis (1993). If we look at the expected value formula for K we see that it equal mean claim occurrence for the process times the expected negative surplus duration per unit time. Expressions for the moments of the severity random variable can be got from Lin & Willmot (2000). Easy expressions for the same moments when $u = 0$ and limiting ones when $u \rightarrow \infty$ can be found in Egídio dos Reis (1993) and Egídio dos Reis (2000). This means that evaluation of the moments for K_c is available, provided they exist.

Furthermore, giving what is explained in the previous section concerning the distributions of K and M with $u = 0$, moments of the number of claims up to ruin are also available, for this particular case.

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