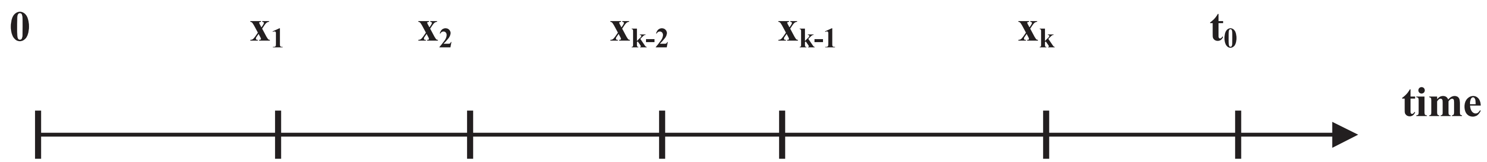
Stationary Palm's theorem

The result of this theorem is fundamental of supply and repair theory. Its demonstration herein is based on Barlow and Proschan, in their book of 1975 they did not call it Palm's theorem.



k events of Poisson process during $[0, t_0]$ with **constant** mean rate λ.

$x_1-0, x_2-x_1, \ldots, x_k-x_{k-1}, t_0-x_k$ successive intervals of occurrence with $0 \leq x_1 \leq x_2 \leq \ldots \leq x_{k-2} \leq x_{k-1} \leq x_k \leq t_0$.

Poisson process ⇒ by definition inter-event times are exponentially distributed with parameter λ.

The conditional joint density is:

$$\frac{p(\lambda, x_1-0)\,p(\lambda, x_2-x_1)\ldots.p(\lambda, x_k-x_{k-1})\,p(\lambda, t_0-x_k)}{p(\lambda, k, t_0)}=$$

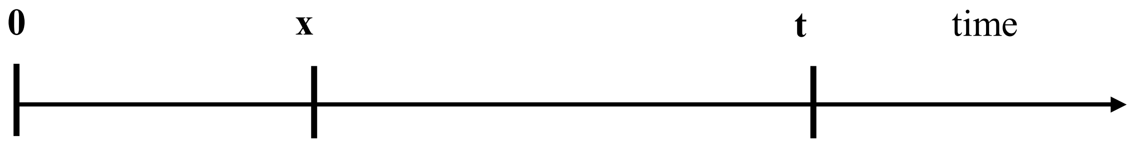
$$=\frac{\lambda * \exp[-\lambda(x_1-0)]\lambda * \exp[-\lambda(x_2-x_1)] * \ldots * \lambda * \exp[-\lambda(x_k-x_{k-1})]\lambda * \exp[-\lambda(t_0-x_k)]}{\frac{(\lambda t_0)^k}{k!}\exp(-\lambda t_0)}=$$

$$=\frac{\lambda^k * \exp(-\lambda t_0)}{\lambda^k {t_0}^k \exp(-\lambda t_0)/k!}=\frac{k!}{t_0^k}$$

and independent of $x_1, x_2-x_1, \ldots, x_k-x_{k-1}$. The occurrences are **iid** (independent and identically distributed among them) **rv** (random variable) and it is a nature of Poisson process.

time	t
arrivals	i + n

$$prob\ (\lambda t; n+i) = \frac{(\lambda t)^{n+i}}{(n+i)!} e^{-\lambda t}$$



time	t - x	t - x
arrivals for service	i = 0, 1, 2, 3, …	
Survivals		n
combination	$\binom{n+i}{i}$	$\binom{n+i}{n}$
probability of n survivals at time t		$(P(t))^n$
probability of i served at time t	$(1-P(t))^i$	
distribution (service / survival)	G(t-x)	$\bar{G}(t-x)$
time average value of $\bar{G}(t-x)$		$\int_0^t \frac{\bar{G}(t-x)}{t} dt$
P(t) under <u>steady state</u> (t → 0)		$\lim_{t\to\infty} \int_0^t \frac{\bar{G}(t-x)}{t} dt$

By n Bernoulli trials for iid rv:

$$\binom{n+i}{n} (P(t))^n (1 - P(t))^i$$

If failures occur according to a stationary Poisson process and repair times are independent, identically distributed random variables with a finite mean (MTTR), the steady-state probability distribution of the numbers of units in repair is Poisson and is independent of the particular service distribution.

$$p(\lambda t, P(t); n) = \sum_{i=0}^{\infty} \frac{(\lambda t)^{n+i}}{(n+i)!} e^{-\lambda t} \binom{n+i}{n} (P(t))^n (1-P(t))^i = \sum_{i=0}^{\infty} \frac{(\lambda t)^{n+i}}{(n+i)!} e^{-\lambda t} \frac{(n+i)!}{n!\,i!} (P(t))^n (1-P(t))^i =$$

$$= \frac{(\lambda t)^n}{n!} e^{-\lambda t} (P(t))^n \sum_{i=0}^{\infty} (\lambda t)^i \frac{(1-P(t))^i}{i!} = \frac{(\lambda t P(t))^n}{n!} e^{-\lambda t} \sum_{i=0}^{\infty} (\lambda t)^i \frac{(1-P(t))^i}{i!} =$$

$$= \frac{(\lambda t P(t))^n}{n!} e^{-\lambda t} \sum_{i=0}^{\infty} \frac{[\lambda t (1-P(t))]^i}{i!}$$

if $e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots$ then $\sum_{i=0}^{\infty} \frac{[\lambda t (1-P(t))]^i}{i!} = e^{\lambda t (1-P(t))}$

$$p(\lambda t, P(t); n) = \frac{(\lambda t P(t))^n}{n!} e^{-\lambda t} \sum_{i=0}^{\infty} \frac{[\lambda t (1-P(t))]^i}{i!} = \frac{(\lambda t P(t))^n}{n!} e^{-\lambda t} e^{\lambda t (1-P(t))} = \frac{(\lambda t P(t))^n}{n!} e^{-\lambda t P(t)}$$

From the table below

$$\lim_{t\to\infty} [p(\lambda t, P(t); n)] = \lim_{t\to\infty} \left[\frac{(\lambda t P(t))^n}{n!} e^{-\lambda t P(t)} \right] = \lim_{t\to\infty} \left[\frac{(\lambda \int_0^t \bar{G}(u) du)^n}{n!} e^{-\lambda \int_0^t \bar{G}(u) du} \right] = \frac{(\lambda \nu)^n}{n!} e^{-\lambda \nu} \Rightarrow p(\lambda, \nu; n) = \frac{(\lambda \nu)^n}{n!} e^{-\lambda \nu}$$

	General distribution	Exponential distribution
Life distribution	G(t) $t \geq 0$	$1-e^{-\frac{t}{\nu}}$ $t \geq 0$
Survival probability	$\bar{G}(t) = 1 - G(t)$ By definition: $\int_0^\infty \bar{G}(t)dt = \int_0^\infty (1-G(t))dt$ =E(t) = ν **ν = constant = mean repair time**	$e^{-\frac{t}{\nu}}$ $\lim_{T\to\infty} \int_0^T e^{-\frac{t}{\nu}} dt = \lim_{T\to\infty} \left[-\nu \left[e^{-\frac{t}{\nu}} \right]_0^T \right] = \lim_{T\to\infty} \left[(-\nu)\left(e^{-\frac{T}{\nu}} - 1 \right) \right] = \nu$
Density probability	$g(t) = \frac{d}{dt}[G(t)]$	$\left(\frac{1}{\nu}\right) e^{-\frac{t}{\nu}}$
p(t)	$\frac{\bar{G}(t-x)}{t}$	$\left(\frac{1}{t}\right) e^{-\frac{(t-x)}{\nu}}$
$P(t) = \int_0^t p(t)dt$ **Time average value of** $\bar{G}(t) = 1 - G(t)$	t-x = u ⇒ -dx = du x = 0 ⇒ u = t x = t ⇒ u = 0 $\int_0^t \frac{\bar{G}(t-x)}{t} dx = \int_t^o \frac{\bar{G}(u)}{t}(-du) =$ $= \int_0^t \frac{\bar{G}(u)}{t} du = \frac{1}{t}\int_0^t \bar{G}(u)du$	t-x = u ⇒ -dx = du x = 0 ⇒ u = t x = t ⇒ u = 0 $\int_0^t \frac{e^{-(t-x)/\nu}}{t} dx = \int_t^o \frac{e^{-u/\nu}}{t}(-du) = \frac{1}{t}\int_0^t e^{-u/\nu} du = -\frac{\nu}{t}\left[e^{-u/\nu}\right]_0^t =$ $= -\frac{\nu}{t}\left(e^{-t/\nu} - 1\right) = \frac{\nu}{t}\left(1 - e^{-t/\nu}\right)$
Under <u>steady state</u> $\lim_{t\to\infty}[tP(t)]$	$\lim_{t\to\infty} \int_0^t \bar{G}(u)du = \nu$	$\lim_{t\to\infty}\left[\nu\left(1 - e^{-t/\nu}\right)\right] = \nu$

<u>Note</u>: The Palm's theorem is proved for any distribution $\bar{G}(t)$ with mean ν (repair time, MTTR), the exponential time distribution is shown only for comparison purpose.