Stationary Palm´s theorem (ref Tijms, website Scribd)

A - Definition of Poisson process for constant λ:

$$\boldsymbol{p(\lambda,t,j)} = \frac{\boldsymbol{(\lambda t)^j}}{\boldsymbol{j}!}\boldsymbol{e^{-\lambda t}} = \frac{(\lambda t)^j}{j!}\left(1+\frac{-\lambda t}{1!}+\frac{(-\lambda t)^2}{2!}+\frac{(-\lambda t)^3}{3!}+\cdots\right) \quad j=0,1,2,3,\ldots$$

B - Properties of Poisson process (see numerical example on table below):

For Δt (or $\lambda\Delta t$) small, that is $\Delta t \to 0$ and the probability of j = 0 (zero) arrival in time Δt is

$$p(\lambda,\Delta t,0) = e^{-\lambda\Delta t} = 1-\lambda\Delta t+\frac{(-\lambda\Delta t)^2}{2!}+\frac{(-\lambda\Delta t)^3}{3!}+\cdots \Rightarrow \boldsymbol{p(\lambda,\Delta t,0) = 1-\lambda\Delta t+O(\lambda,\Delta t,0)}$$ where

$$O(\lambda,t,0) = \frac{(-\lambda\Delta t)^2}{2!}+\frac{(-\lambda\Delta t)^3}{3!}+\cdots \quad and \quad \lim_{\Delta t\to 0}\frac{O(\lambda,\Delta t,0)}{\Delta t}=0$$

For Δt (or $\lambda\Delta t$) small, that is $\Delta t \to 0$ and the probability of j = 1 (one) arrival in time Δt is

$$p(\lambda,\Delta t,1) = \lambda\Delta t e^{-\lambda\Delta t} = \lambda\Delta t(1-\lambda\Delta t+\cdots) \Rightarrow \boldsymbol{p(\lambda,\Delta t,1) = \lambda\Delta t+O(\lambda,\Delta t,1)}$$

For Δt (or $\lambda\Delta t$) small, that is $\Delta t \to 0$ and the probability of j = 2 (two) arrivals in time Δt is

$$p(\lambda,\Delta t,2) = \frac{(\lambda\Delta t)^2}{2!}e^{-\lambda\Delta t} = \frac{(\lambda\Delta t)^2}{2!}(1-\lambda\Delta t+\cdots) \Rightarrow \boldsymbol{p(\lambda,\Delta t,2) = O(\lambda,\Delta t,2)}$$

For Δt (or $\lambda\Delta t$) small, that is $\Delta t \to 0$ and the probabilities of j = 3, 4, … arrivals in time Δt as similar to $p_2(\Delta t)$ are

$$\boldsymbol{p(\lambda,\Delta t,j) = O(\lambda,\Delta t,j)} \; for\; j=3,4,5,\ldots..$$

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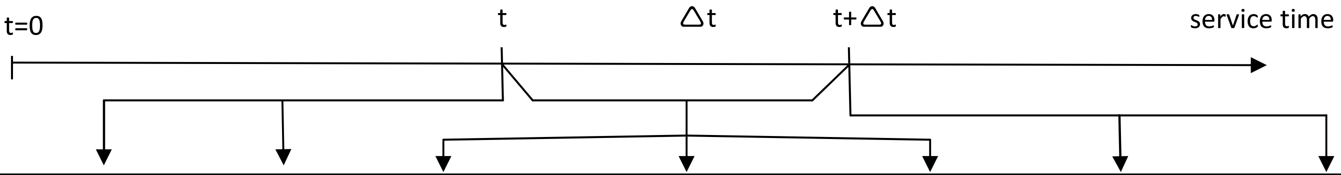
j	λ	Δt (small)	$\lambda\Delta t$	$poisson(j,\lambda\Delta t)$	Approximation	$\frac{poisson(j,\lambda\Delta t)}{\Delta t}$
0	2,0	0,001	0,002	9,98002E-01	$\mathbf{1-\lambda\Delta t = 1-0,002}$	9,98002E+02
1	2,0	0,001	0,002	1,99600E-03	$\mathbf{\lambda\Delta t = 0,002}$	1,99600E+00
2	2,0	0,001	0,002	1,99600E-06	**Very small**	1,99600E-03
3	2,0	0,001	0,002	1,33067E-09	**Very small**	1,33067E-06
4	2,0	0,001	0,002	6,65335E-13	**Very small**	6,65335E-10
5	2,0	0,001	0,002	2,66134E-16	**Very small**	2,66134E-13
6	2,0	0,001	0,002	8,87113E-20	**Very small**	8,87113E-17
7	2,0	0,001	0,002	2,53461E-23	**Very small**	2,53461E-20
8	2,0	0,001	0,002	6,33652E-27	**Very small**	6,33652E-24
9	2,0	0,001	0,002	1,40812E-30	**Very small**	1,40812E-27
10	2,0	0,001	0,002	2,81623E-34	**Very small**	2,81623E-31

C – Time dependent probability equation



time t	time t	interval △t	interval △t	interval △t	time $(t+\Delta t)$	time $(t+\Delta t)$
$p_j(t)$		0 arrival	$p(\lambda,\Delta t,0)$		$p_j(t+\Delta t)$	
$p_j(t)$	$G(t)$	1 arrival	$p(\lambda,\Delta t,1)$	1 service	$p_j(t+\Delta t)$	$G(t+\Delta t)$
$p_{j-1}(t)$	$G(t)$	1 arrival	$p(\lambda,\Delta t,1)$	0 service	$p_j(t+\Delta t)$	$G(t+\Delta t)$
		2 arrivals	$p(\lambda,\Delta t,2)=O(\lambda,\Delta t,2)$	……………		
		3 arrivals	$p(\lambda,\Delta t,3)=O(\lambda,\Delta t,3)$	……………		
		……….	……………	……………		

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$p_j(t)$, $p_{j-1}(t), p_j(t+\Delta t)$ are probabilities (**service time process**) of j, j-1, j servers busy respectively at times t, t, t+△t

$\boldsymbol{G}(t), \boldsymbol{G}(t+\Delta t)$ are ***<u>any</u>*** service time probability distribution respectively at times t, t+△t

$[1-\boldsymbol{G}(t)\ ], [1-\boldsymbol{G}(t+\Delta t)]$ are survival time probability distribution respectively at times t, t+△t

$p(\lambda, \Delta t, 0)$, $p(\lambda, \Delta t, 1), p(\lambda, \Delta t, 2)$, $p(\lambda, \Delta t, 3)$ are probabilities of 0, 1, 2, 3 arrivals (**Poisson process**) in interval △t

$O(\lambda, \Delta t, 2), O(\lambda, \Delta t, 3)$ are very small and negligible (**Poisson process**) values

Taking into account the above mutually exclusive cases and summarizing results from (B) above

<u>0 (zero) customer arrival in interval △t</u>

$$p(\lambda, \Delta t, 0) = 1 - \lambda\Delta t + O(\lambda, \Delta t, 0)$$

$$P\big(p(\lambda, \Delta t, 0), p_j(t)\ \big) = \ [1 - \lambda\Delta t]p_j(t) + O(\lambda, \Delta t, 0)p_j(t)$$

<u>1 customer arrival in interval △t and 1 service concluded in △t</u>

$$p(\lambda, \Delta t, 1) = \lambda\Delta t + O(\lambda, \Delta t, 1)$$

$$P\big(p(\lambda, \Delta t, 1), p_j(t), 1\ service\ concluded\ \big) = \ \lambda\Delta t p_j(t)G(t+\Delta t) + O(\lambda, \Delta t, 1)p_j(t)$$

<u>1 customer arrival in interval △t and 0(zero) service concluded in △t</u>

$$p(\lambda, \Delta t, 1) = \lambda\Delta t + O(\lambda, \Delta t, 1)$$

$$P\big(p(\lambda, \Delta t, 1), p_{j-1}(t), 0\ service\ concluded\ \big) = \ \lambda\Delta t p_{j-1}(t)[1 - G(t+\Delta t)] + O(\lambda, \Delta t, 1)p_{j-1}(t)$$

Summing the cases

$$p_j(t+\Delta t) = P\left(p(\lambda,\Delta t,0), p_j(t)\right) + P\big(p(\lambda,\Delta t,1), p_j(t), 1\ service\ concluded\ \big) +$$

$$+ P\big(p(\lambda,\Delta t,1), p_{j-1}(t), 0\ service\ concluded\ \big) + O(\lambda,\Delta t)$$

$$\boldsymbol{p_j(t+\Delta t) = \ (1-\lambda\Delta t)p_j(t) + \lambda\Delta t G(t+\Delta t)p_j(t) + \lambda\Delta t\big(1-G(t+\Delta t)\big)p_{j-1}(t) + O(\lambda,\Delta t)}$$

<u>D – Arrangement of differential equations system</u>

From (C) above

$$p_j(t+\Delta t) = \ (1-\lambda\Delta t)p_j(t) + \lambda\Delta t G(t+\Delta t)p_j(t) + \lambda\Delta t\big(1-G(t+\Delta t)\big)p_{j-1}(t) + O(\lambda,\Delta t)$$

Subtracting $p_j(t)$ from $p_j(t+\Delta t)$, dividing by Δt and doing $\Delta t \to 0$

$$p_j(t+\Delta t) - p_j(t) = \ -\lambda\Delta t p_j(t) + \lambda\Delta t G(t+\Delta t)p_j(t) + \lambda\Delta t\big(1-G(t+\Delta t)\big)p_{j-1}(t) + O(\lambda,\Delta t)$$

$$\frac{p_j(t+\Delta t) - p_j(t)}{\Delta t} = -\lambda p_j(t) + \lambda G(t+\Delta t)p_j(t) + \lambda\big(1-G(t+\Delta t)\big)p_{j-1}(t) + \frac{O(\lambda,\Delta t)}{\Delta t}$$

$$\lim_{\Delta t\to 0}\frac{p_j(t+\Delta t) - p_j(t)}{\Delta t} = \lim_{\Delta t\to 0}\left[-\lambda p_j(t) + \lambda G(t+\Delta t)p_j(t) + \lambda\big(1-G(t+\Delta t)\big)p_{j-1}(t) + \frac{O(\lambda,\Delta t)}{\Delta t}\right]$$

$$\lim_{\Delta t\to 0}\frac{p_j(t+\Delta t) - p_j(t)}{\Delta t} = p_j'(t) = -\big[\lambda\big(1-G(t)\big)\big]p_j(t) + \lambda[1-G(t)]p_{j-1}(t)$$

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Doing $\eta(t) = \lambda[1 - G(t)]$

$\boldsymbol{p_0'(t) = -\eta(t)p_0(t)}$

$\boldsymbol{p_1'(t) = -\eta(t)p_1(t) + \eta(t)p_0(t)}$

$\boldsymbol{p_2'(t) = -\eta(t)p_2(t) + \eta(t)p_1(t)}$

$\boldsymbol{p_3'(t) = -\eta(t)p_3(t) + \eta(t)p_2(t)}$

…………

$\boldsymbol{p_j'(t) = -\eta(t)p_j(t) + \eta(t)p_{j-1}(t)}$

<u>E – Solution of differential equations system</u>

Doing $\Lambda(t) = \int_0^t \eta(t)dt = \lambda \int_0^t [1 - G(t)]dt \Leftrightarrow \Lambda'(t) = \eta(t)$

$f(x) = e^{g(x)} \Leftrightarrow f'(x) = g'(x)e^{g(x)} = g'(x)f(x)$

<u>For j = 0</u>

$p_0'(t) = -\eta(t)p_0(t) = -\Lambda'(t)p_0(t) \Longrightarrow \boldsymbol{p_0(t) = e^{-\Lambda(t)}}$

For j = 1

$p_1'(t) = -\eta(t)p_1(t) + \eta(t)p_0(t) = \Lambda'(t)p_0(t) - \Lambda'(t)p_1(t) = \Lambda'(t)e^{-\Lambda(t)} - \Lambda'(t)p_1(t)$

$[\alpha\beta]' = \alpha'\beta + \alpha\beta'$ $\quad \alpha' = \Lambda'(t) \Rightarrow \alpha = \Lambda(t)\ and\ \beta = e^{-\Lambda(t)}$

$$[\alpha\beta]' = \left[\Lambda(t)e^{-\Lambda(t)}\right]' = \Lambda'(t)e^{-\Lambda(t)} + \Lambda(t)\big(-\Lambda'(t)\big)e^{-\Lambda(t)} \Rightarrow \boldsymbol{p_1(t) = \Lambda(t)e^{-\Lambda(t)}}$$

<u>For j = 2</u>

$$p_2'(t) = -\eta(t)p_2(t) + \eta(t)p_1(t) = \Lambda'(t)p_1(t) - \Lambda'(t)p_2(t) = \Lambda'(t)\Lambda(t)e^{-\Lambda(t)} - \Lambda'(t)p_2(t)$$

$$[\alpha\beta]' = \alpha'\beta + \alpha\beta' \qquad \alpha' = \Lambda'(t) \Rightarrow \alpha = \Lambda(t) \; and \; \beta = \Lambda(t)e^{-\Lambda(t)}$$

$$[\alpha\beta]' = \left[\Lambda^2(t)e^{-\Lambda(t)}\right]' = 2\Lambda(t)\Lambda'(t)e^{-\Lambda(t)} + \Lambda^2(t)\big(-\Lambda'(t)\big)e^{-\Lambda(t)}$$

$$\left[\frac{\alpha\beta}{2}\right]' = \Lambda(t)\Lambda'(t)e^{-\Lambda(t)} - \Lambda'(t)\left[\frac{\Lambda^2(t)e^{-\Lambda(t)}}{2}\right] \Rightarrow \boldsymbol{p_2(t) = \frac{\Lambda^2(t)}{2}e^{-\Lambda(t)}}$$

<u>For j = 3</u>

$$p_3'(t) = -\eta(t)p_3(t) + \eta(t)p_2(t) = \Lambda'(t)p_2(t) - \Lambda'(t)p_3(t) = \Lambda'(t)\frac{\Lambda^2(t)}{2}e^{-\Lambda(t)} - \Lambda'(t)p_2(t)$$

$$[\alpha\beta]' = \alpha'\beta + \alpha\beta' \qquad \alpha' = \Lambda'(t) \Rightarrow \alpha = \Lambda(t) \; and \; \beta = \frac{\Lambda^2(t)}{2}e^{-\Lambda(t)}$$

$$[\alpha\beta]' = \left[\frac{\Lambda^3(t)}{2}e^{-\Lambda(t)}\right]' = 3\frac{\Lambda^2(t)}{2}\Lambda'(t)e^{-\Lambda(t)} + \frac{\Lambda^3(t)}{2}\big(-\Lambda'(t)\big)e^{-\Lambda(t)}$$

$$\left[\frac{\alpha\beta}{3}\right]' = \frac{\Lambda^2(t)}{2}\Lambda'(t)e^{-\Lambda(t)} - \Lambda'(t)\left[\frac{\Lambda^3(t)e^{-\Lambda(t)}}{2\cdot 3}\right] \Rightarrow \boldsymbol{p_3(t) = \frac{\Lambda^3(t)}{2\cdot 3}e^{-\Lambda(t)}}$$

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For j, by induction

$$p_j(t) = \frac{\Lambda^j(t)}{j!} e^{-\Lambda(t)}$$

F – Conclusion

Under condition of stationary and steady states $(t \to \infty)$ the service probabilities are

$$\Lambda(t \to \infty) = \lambda \left[\lim_{t \to \infty} \int_0^t [1 - G(t)]dt \right] = \lambda \int_0^\infty [1 - G(t)]dt$$

By definition $\int_0^\infty [1 - G(t)]dt = E[G(t)] = \mu = expeced\ or\ mean\ service\ time$

$$p(\lambda, \mu, j) = \frac{(\lambda\mu)^j}{j!} e^{-\lambda\mu}$$ **for j = 0, 1, 2, 3, …….**

G – Completion

The dynamic and not stationary conditions of theorem (without demonstration) are

$$p(j, t) = \frac{[\Lambda(t)]^j}{j!} e^{-\Lambda(t)}$$

$$\lambda = \lambda(\tau)$$

$$\eta(\tau, t) = \lambda(\tau)[1 - G(t - \tau)]$$

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$$\Lambda(t) = \int_0^t \eta(\tau, t)d\tau = \int_0^t \lambda(\tau)[1 - G(t-\tau)]d\tau$$

$$\boldsymbol{p}(\mathbf{j}, \mathbf{t}) = \frac{\left[\int_0^t \lambda(\tau)[1 - G(t-\tau)]d\tau\right]^{\mathbf{j}}}{\mathbf{j}!} e^{-\int_0^t \lambda(\tau)[1-G(t-\tau)]d\tau}$$

Doing $\lambda(t) = \lambda = constant$

$\theta = t - \tau \Rightarrow d\theta = -d\tau, \quad \tau = 0 \Rightarrow \theta = t, \; \tau = t \Rightarrow \theta = 0 \;\; and \; t \to \infty \; (steady\ state)$

$$\lim_{t\to\infty}\left[\int_0^t \lambda(\tau)[1 - G(t-\tau)]d\tau\right] = \lim_{t\to\infty}\left[\lambda\int_t^0 [1 - G(\theta)](-d\theta)\right] = \lambda \lim_{t\to\infty}\left[\int_0^t [1 - G(\theta)]d\theta\right] = \lambda\int_0^\infty [1 - G(\theta)]d\theta$$

And the result is the same as got in (F) above.

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