Investigation of Certain Differential and Integro-differential Equations in Abstract Spaces

Summary

The governing partial differential equations of many physical phenomena can be modelled as evolution equations in Hilbert or more generally, in Banach spaces and may be viewed as ordinary differential equations in infinite dimensional spaces of functions. The work presented in this thesis demonstrates the applications of rich theory of functional analysis to study problems. By considering such problems in abstract spaces not only we establish the results for the problems under consideration, but also for the whole class of problems to which these prototype of problems belong. We concentrate on basic features those remain invariant when we switch from one problem to another belonging to the same class.

We first analyze the well-posedness of a solution of an evolution equation and then aim to find, the exact solution if possible, otherwise the approximate solution. We consider certain approximate equations in finite dimensional spaces associated with an evolution equation under consideration and establish the existence and uniqueness of solutions to these approximate equations. The convergence of the solutions of the approximate equations to the solution of the evolution equation is then established. Our tools for this are the theory of analytic semigroups and the contraction mapping theorem. We consider analytic semigroups generated by the operators appearing in evolution equations under consideration.

Although, our study concerns some special type of evolution equations, namely the Sobolev type equations and certain classes of second order semi-linear differential and integro-differential equations but the techniques are quite general which are extendable to other type of problems.

The thesis has seven chapters.

In **Chapter 1**, we give the introduction to the problems discussed in the subsequent chapters providing a motivation to the study carried out in this thesis. Then, we give a review of related works in the literature and the significance of the problems considered. Next, we mention some of the results required for the analysis in the ensuing chapters.

In **Chapter 2**, we consider the Sobolev type evolution equations in a separable Hilbert space and study the approximations of the solutions. The Sobolev type evolution equations are implicit evolution equations in which time derivatives of the unknown function appear

implicitly and these type of equations arise in many diffusion and fluid flow models. We consider an associated integral equation and a sequence of approximate integral equations using projection operators. Then we establish the existence of a unique solution to every approximate integral equation using the contraction mapping theorem. After proving some estimates for the solutions of approximate integral equations, we prove the convergence of the solutions of the approximate integral equations to the solution of the associated integral equation. Further, we show that the solution of associated integral equation can be extended to the maximal interval of existence and it is unique. Finally, we consider the Faedo-Galerkin approximations of solutions and prove some convergence results.

In **Chapter 3**, we consider a strongly damped semi-linear wave equation and reformulate it as a second order semi-linear evolution equation. We study the convergence of approximation of the solution to a second order semi-linear evolution equation in a separable Hilbert space by using similar techniques which are used in Chapter 2. First, with the help of a pair of associated integral equations and projection operators, we consider a pair of approximate integral equations and prove the existence and uniqueness of the solution to this pair. We then establish the convergence of the pair of approximate integral equations to the pair of associated integral equations and limit of the solution of approximate integral equations being the solution of associated integral equation.equations. Further, we show that the solution of the pair of associated integral equations can be extended to the maximal interval of existence and it is unique. Finally, we consider the Faedo-Galerkin approximations of the solutions and prove some convergence results.

In nuclear dynamics and thermo-elasticity, there is a need to reflect the effects of the "memory" of the system. This results in the inclusion of an integral term into the governing partial differential equation yielding a partial integro-differential equation (PIDE). This provides us a motivation to study the partial integro-differential equations. In the next two Chapters, we study a class of second order semi-linear integro-differential equations arising in the study of viscoelastic material with memory.

In **Chapter 4**, we consider an abstract second order semi-linear integro-differential equation in a Banach space and first prove the existence and uniqueness of a local classical solution with the help of the semigroup theory and the contraction mapping theorem. Further, under some additional growth conditions on the nonlinear maps we establish the continuation of this solution, the maximum interval of existence and the global existence.

In **Chapter 5**, we consider a strongly damped semi-linear integro-differential equation and reformulate it as a second order semi-linear integro-differential equation in a separable

Hilbert space and study the convergence of the approximation of the solution using similar techniques which are used in Chapters 2 and 3.

In **Chapter 6**, we consider a first order quasi-linear implicit integro-differential equation in a Banach space. First, we prove the existence, uniqueness and continuous dependence on the initial data of a strong solution by using the method of semi-discretization in time in which we discretize the time axis and replace the time derivatives by the correspond difference quotients. Then, we show that these discrete points lies in a ball whose radius is independent of the discretization parameters and prove some estimates. After defining the approximate solution in terms of these discrete points, we prove its convergence. Next, we establish the existence of a unique local mild solution and its regularity under some additional condition with the help of the semigroup theory and the contraction mapping theorem.

In **Chapter 7**, we conclude the thesis with some remarks and provide some insight into further research work in this and related areas.