ω and phase space expansion

As discussed in the case of chaotic phase space, $\omega = \ln n/d \ln m$ gives a measure of the incompleteness of the state counting in the d-dimension phase space. $\omega = 1$ means $d_f = d$ or $n = m^d$. In other word, at the k^{th} iteration, a segment of volume s_k is completely covered (replaced) by n segments of volume $s_{k+1} = s_k/m^d$. So the summation over all segments is equivalent to the sum over all possible states, making it possible to calculate complete information.

When $\omega > 1$ (or $\omega < 1$), $n > m^d$ (or $n < m^d$) and s_k is replaced by n segments whose total volume is more (or less) than s_k. So there is expansion (or negative expansion) of state volume when we refine the phase space scale. An estimation of this expansion at each scale refinement can be given by the ratio $r = \frac{ns_{k+1}-s_k}{s_k} = \frac{n}{m^d} - 1 = (\frac{1}{m^d})^{1-\omega} - 1 = (\omega - 1)\frac{(m^d)^{\omega-1}-1}{\omega-1}$. r describes *how much accessible states increase* at each step of the iteration or of the refinement of phase space. The physical content of ω is clear if we note that $\omega > 1$ and $\omega < 1$ correspond to an expansion ($r > 0$) and a negative expansion ($r < 0$), respectively, of the the accessible state volume at each step of the iteration.

When $\omega = 0$, we have $d_f = 0$ and $n = 1$, leading to $r = \frac{1}{m^d} - 1$. The iterate condition $n \geq 1$ means $\omega \geq 0$. $\omega < 0$ is impossible since it means $d_f < 0$ or $n < 1$ which obviously makes no sense. We can also write : $\omega - 1 = \ln(r+1)/\ln(m^d) = \ln(ns_{k+1}/s_k)/\ln(m^d)$, which implies that it is the difference $\omega - 1$ which directly measures the accessible state space expansion through the scale refinement.

ω and information growth

The expansion of the accessible state volume of a system in its phase space during the scale refinement should be interpreted as follows : the extra state points $\Delta = ns_{k+1} - s_k$ acquired at $(k+1)^{th}$ order iterate are just the number of unaccessible states at k^{th} order with respect to $(k+1)^{th}$ order. $\Delta > 0$ (or $\Delta < 0$) means that we have counted less (or more) states at k^{th} order than we should have done. Δ contains the *accessible information gain* (AIG) through the $(k+1)^{th}$ iterate.

To illustrate the relation between this "hidden information" and the parameter ω, let us suppose that *the distribution is scale-invariant*[1]. At the iterate of order k, the average information contained on s_k is given by $I_k = \int_{s_k} p^\omega I(1/p)ds$. At $k+1$ order, $I_{k+1} = \int_{ns_{k+1}} p^\omega I(1/p)ds$. Hence AIG is just $\Delta I = I_{k+1} - I_k = \int_{(ns_{k+1}-s_k)} p^\omega I(1/p)ds = \sigma_I \Delta$, where $\sigma_I = p^\omega I(1/p)$ is the information density or the average information carried by each state. The relative AIG is given by $\Delta I/I_k = r = (1-\omega)\frac{(1/m^d)^{1-\omega}-1}{1-\omega}$ which is independent of scale but dependent on scale changes. For given scaling factor m, the magnitude of ΔI or r increases with increasing difference $|1-\omega|$. The sign of r (or AIG) was discussed earlier. For given ω, $|\Delta I|$ increases with decreasing scaling. For $\omega = 1$ or $m = 1$, there is no information gain, corresponding to the case of complete information.

According to the relationship $\omega = d_f/d$ and the above discussions, it can be concluded that the incompleteness parameter ω may be considered as a measure of chaos. Certainly this is a conclusion on the basis of simple models and the relation between ω and the degree of chaos or fractal may be more complicated with more complex chaos and fractals, but it is consequent to say that more a system is chaotic, more its information is incomplete and more ω is different from unity.

References

[1] Scale invariant probability distribution means that the probability density in a fractal phase space is independent of the iteration order. In the case where any point on the phase space *segments* are equally visited, scale invariance implies that the probability for the system to be on a segment is proportional to the size of the segment, as I supposed in the section VIII when discussing probability normalization in fractal phase space. Scale invariance of probability density is a logical assumption since increasing or decreasing density may lead to infinite probability distributions in fractal phase spaces.