In any case, complete information description of complex systems is only a science fiction. Although we cannot say that all these systems have chaotic or fractal nature, a common feature of them is that *a part of their phase space is unaccessible so that complete and exhaustive exploit of the phase space is impossible.*

What can we do for probability and information calculations if we do not know how many states the system of interest has? Now I am discussing the method to characterize this incompleteness in statistics.

When we deal with a chaotic system (having or not fractal attractor), all considerations based on conventional normalization will necessarily lead to erroneous calculations because, simply, a considerable number of state points are not considered and that we can only write $\sum_{i=1}^{v} p_i = Q \neq 1$, where v, as mentioned previously, is only the number of accessible states. In this case, p_i is just the *incomplete distribution* and Q is *a constant characterizing the incompleteness* of the system and provides *a possible key to the introduction of incompleteness of information into statistical theory.* It should be supposed $Q = 1$ if information is complete.

The methodology of incomplete information theory we developed is to keep the frame of classical complete probability theory for incomplete information or probability distribution by introducing empirical parameters characterizing the incompleteness. This is just the same methodology as in the theory of chaos or fractals introducing fractal dimension to characterize the structures of space time.

In order to take advantage of the conventional probability theory, we need a "normalization" for incomplete distribution p_i. All we can write now is $\sum_i F_\omega(p_i) = 1$ which can be called generalized or *incomplete normalization.* The function F_ω should depend on the nature of the system and become identity function whenever information is supposed complete ($Q = 1$). The arithmetic average should now be given by $\bar{x} = \sum_i F_\omega(p_i) x_i$.

F_ω may be determined if the information measure and the distribution law are given. For example, with Hartley information measure and exponential distribution, F_ω can be showed to be identity function. In general, by entropy maximization through the functional

$$\delta[\sum_i F_\omega(p_i) I(p_i) + \beta \sum_i F_\omega(p_i) x_i] = 0 \qquad (1)$$

we get :

$$\frac{\partial \ln F_\omega(p_i)}{\partial p_i} = \frac{\partial I/\partial p_i}{I + \beta f_\omega^{-1}(p_i)} \tag{2}$$

or $F_\omega(p_i) = C\exp[\int \frac{\partial I/\partial p_i}{I+\beta f_\omega^{-1}(p_i)}dp_i]$ where β is the multiplier of Lagrange connected to expectation, $I(p_i)$ is the information measure, $p_i = f_\omega(x_i)$ the distribution function depending on the parameter ω, C the normalization constant of F_ω.

Incomplete normalization in fractal phase space

Now I will show that the "power normalization" is inevitable in fractal phase space.

For the sake of simplicity, let us consider a phase space in which the trajectory of a chaotic system forms a simple self-similar fractal structure, say, Sierpinski carpet. This means that the state point of the system can be found only on the black rectangular segments whose number is $W_k = 8^k$ at k^{th} iteration. Hence the total surface at this stage is given by $S_k = W_k s_k$ where $s_k = l_0/3^k$ is the surface of the segments at k^{th} iteration and l_0 the length of side of the square space at 0^{th} iteration. If the segments do not have same surface, we should write $S_k = \sum_{i=1}^{W_k} s_k(i)$. We suppose that the density of state is identical everywhere on the segments and that all points are equally probable, so that the probability for the system to be in the i^{th} segment may be defined as usual by $p_i = s_k(i)/S_k$. This probability is obviously normalized. The problem is that S_k is an indefinite quantity as $k \to \infty$ or as the time $t \to \infty$. So strictly speaking, it can not be used to define exact probability definition or all the probabilities would be null. In addition, S_k is not differentiable and contains inaccessible points. Thus the probability defined above makes no sense.

Alternatively, the probability may be reasonably defined on a integrable and differentiable support, say, the Euclidean space containing the fractal structure. To see how to do this, we write $S_k = l_0^2(\frac{1}{3^k})^{d-d_f}$ for identical segments or, for segments of variable size,

$$\sum_{i=1}^{W_k}[\frac{s_k(i)}{S_0}]^{d_f/d} = 1 \tag{3}$$

where $S_0 = l_0^d$ (here $d = 2$ for Sierpinski carpet) is a characteristic volume of the fractal structure embedded in a d-dimension Euclidean space, $d_f = \frac{\ln n}{\ln m}$ is the fractal dimension. $n = 8$ is the number of segments replacing a segment of the precedent iteration and $m = 3$ the scale factor of the iterations.

The microcanonical probability distribution at the k^{th} iteration can be defined as

$$p_i = \frac{s_k(i)}{S_0}$$

so that $\sum_{i=1}^{W_k} p_i^{d_f/d} = 1$. This is just the *incomplete normalization* with probabilities to the power $\omega = d_f/d$. The complete probability normalization $\sum_{i=1}^{W_k} p_i = 1$ can be recovered when $d_f = d$.

It should be noticed that, in Eq.(3), the sum over all the W_k segments at the k^{th} iteration does not mean the sum over all possible states of the system under consideration. This is because that the segment surface $s_k(i)$ does not represent the real number of state points on the segment which, as expected for any self-similar structure, evolves with k just as S_k. So at any given order k, the complete summation over all possible segments is not a complete summation over all possible states. But in any case, whatever is k, Eq.(3) and $\sum_{i=1}^{W_k} p_i^{\omega} = 1$ always holds for $\omega = d_f/d$.

In this simple case with self-similar fractal structure, the incompleteness of the *power normalization* is measured by the parameter $\omega = d_f/d$. $d_f > d$ means that the system has more states than W_k, the number of accessible states at given k. If $d_f < d$, the number of states is less than W_k. When $d_f = d$, the summation is complete at any order k, corresponding to complete information calculation.