

# ON DISJOINT SETS OF NATURAL CLASSES

Heakyung Lee

Department of Mathematics

Winthrop University

Rock Hill, SC 29733

leeh@winthrop.edu

Peter L. Vachuska

Department of Mathematical Sciences

University of Wisconsin–Washington County

West Bend, WI 53095

pvachuska@uwc.edu

## Abstract

An  $M$ -natural class is any subclass of  $\sigma[M]$  which is closed under (1) submodules, (2) isomorphic copies, (3) direct sums and (4)  $M$ -injective envelopes. Let  $\mathcal{C}$  be any set of pairwise disjoint  $M$ -natural classes. We define the  $\mathcal{C}$ -dimension of an  $R$ -module and examine how finite  $\mathcal{C}$ -dimension is related to certain injectivity conditions in  $\sigma[M]$ . We also define a  $\mathcal{C}$ -chain and relate ACC on  $\mathcal{C}$ -chains again to certain injectivity conditions.

## 1 Introduction

Let  $M$  be any  $R$ -module and  $\sigma[M]$  be the class of  $R$ -modules subgenerated by  $M$  (See [10, page 118] for example). Recall that an  $M$ -natural class is any subclass of  $\sigma[M]$  which is closed under (1) submodules, (2) isomorphic copies, (3) direct sums and (4)  $M$ -injective envelopes. If  $M = R$ , we will just use the term natural class. These classes were first used by S. S. Page and Y. Zhou [7, 8, 9, 11] in classifying when direct sums of  $M$ -injective modules contained in a certain class are again  $M$ -injective along with many other results. J. Dauns earlier in [3] had considered the case when  $M = R$  and called these classes saturated classes. These classes arise naturally in many situations (see for examples section 4 of [3]). In [11], Zhou defined the type dimension of a module, and has showed that  $R$  has finite type dimension if and only if for every family  $\{M_i : i \in I\}$  of pairwise orthogonal nonsingular modules, the direct sum of their injective envelopes  $\bigoplus_{i \in I} E(M_i)$  is injective. In this paper we use  $M$ -natural classes to generalize the definition of type dimension, to  $\mathcal{C}$ -dimension, where  $\mathcal{C}$  is a set of disjoint  $M$ -natural classes.

In section 2, we provide basic properties of  $\mathcal{C}$ -dimension, and investigate the connections between finite  $\mathcal{C}$ -dimension and certain direct sums preserving certain injectivity conditions. In section 3, we define a  $\mathcal{C}$ -chain and examine finiteness conditions on  $\mathcal{C}$ -chains.

All rings in this paper are associative with identity and all modules are unitary right  $R$ -modules. For a background of properties of  $M$ -natural classes see [1] and [3]. For an  $M$ -natural class  $\alpha$  we will call any element of  $\alpha$  an  $\alpha$ -module, and for a module  $N$ , we let  $N_{(\alpha)}$  be any maximal  $\alpha$ -module in  $N$ . Unless otherwise specified  $\mathcal{C}$  will always be a pairwise ‘disjoint’ set of  $M$ -natural classes.

## 2 $\mathcal{C}$ -Dimension

Zhou introduced type dimension and studied the basic properties of modules with finite type dimension in [11], [12] and [13]. The *type dimension* of a module is the maximum number of orthogonal atomic submodules. Two modules  $H$  and  $K$  are said to be *orthogonal* (written  $H \perp K$ ) if they have no nonzero isomorphic submodules and to be *parallel* if  $H$  is not orthogonal to any nonzero submodule of  $K$  and  $K$  is not orthogonal to any nonzero submodule of  $H$ . Note that if two modules are parallel and one is in a natural class, then so is the other. An *atomic module* is a nonzero module with the property that any two nonzero submodules are parallel.

In this section we consider any set of disjoint  $M$ -natural classes  $\mathcal{C}$  (we consider classes to be ‘disjoint’ if their intersection contains only the zero module) and look at the maximum number of submodules having the properties that every nonzero submodule is in some natural class of  $\mathcal{C}$ , but no two are from the same natural class of  $\mathcal{C}$ .

If we let  $\mathcal{A}$  be the set of natural classes generated by atomic modules (i.e., they consist of all modules which are parallel to an atomic module), then the notion of  $\mathcal{A}$ -dimension and type dimension coincide. This set of disjoint natural classes is maximal in the sense that no member contains a nontrivial union of disjoint natural classes. We then have that the type dimension of any module in  $\sigma[M]$  is at least as large as the  $\mathcal{C}$ -dimension for any set of disjoint natural classes  $\mathcal{C}$ . This provides the starting point of the results in this section.

**Definition 1** Let  $\mathcal{C}$  be a pairwise disjoint set of  $M$ -natural classes and for any module  $N \in \sigma[M]$ , define the  $\mathcal{C}$ -support of  $N$  to be  $\mathcal{C}\text{-Supp}(N) = \{\alpha \in \mathcal{C} | N_{(\alpha)} \neq 0\}$  and the  $\mathcal{C}$ -dimension of  $N$  to be  $\mathcal{C}\text{-dim}(N) = |\mathcal{C}\text{-supp}(N)|$ .

**Lemma 2.1** Let  $L$  be a submodule of  $N \in \sigma[M]$ . Then

- (a).  $\mathcal{C}\text{-dim}(L) \leq \mathcal{C}\text{-dim}(N)$ .
- (b).  $\mathcal{C}\text{-supp}(\oplus X_i) = \cup \mathcal{C}\text{-supp}(X_i)$ , which implies  $\mathcal{C}\text{-dim}(\oplus X_i) \leq \sum \mathcal{C}\text{-dim}(X_i)$ .
- (c). If  $L \leq_e N$ , then  $\mathcal{C}\text{-dim}(L) = \mathcal{C}\text{-dim}(N)$ .
- (d). If  $L$  and  $N/L$  have both finite  $\mathcal{C}$ -dimension, then  $\mathcal{C}\text{-dim}(N) \leq \mathcal{C}\text{-dim}(L) + \mathcal{C}\text{-dim}(N/L)$ .

**Example 2.2** While it is true that  $\mathcal{C}\text{-supp}(\sum X_i) \geq \cup \mathcal{C}\text{-supp}(X_i)$ , we do not in general have equality when the sum is not direct. Suppose  $R = \mathbb{Z}$  is the ring of integers. Let  $M = \mathbb{Z}$ . Let  $\alpha_1$  be the set of  $R$ -modules which are isomorphic to a submodule of a direct sum of copies of  $E(\mathbb{Z})$ . For each prime integer  $p$ , let  $\alpha_p$  be the set of  $R$ -modules which are isomorphic to a submodule of a direct sum of copies of  $E(\mathbb{Z}/p\mathbb{Z})$ . Then  $\mathcal{C} = \{\alpha_n \mid n = 1 \text{ or } n \text{ is prime}\}$  is a set of disjoint natural classes. Call the submodules of  $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$  generated by  $(1, 0)$  and  $(1, 1 + 2\mathbb{Z})$   $C$  and  $D$  respectively. Then  $\mathcal{C}\text{-supp}(C) = \mathcal{C}\text{-supp}(D) = \{\alpha_1\}$ , but  $\mathcal{C}\text{-supp}(C + D) = \{\alpha_1, \alpha_2\}$ .

**Definition 2** Let  $N \in \sigma[M]$  and  $L$  be a submodule of  $N$ .  $L$  is called a  $\mathcal{C}$ -submodule of  $N$  if  $L \leq P \leq N$  with  $\mathcal{C}\text{-supp}(L) = \mathcal{C}\text{-supp}(P)$  implies  $L = P$ .

We say that  $\mathcal{C}$  spans  $\sigma[M]$  provided that for every nonzero module  $N \in \sigma[M]$ ,  $N_{(\alpha)} \neq 0$  for some  $\alpha \in \mathcal{C}$ .

**Lemma 2.3** For any  $N \in \sigma[M]$ :

- (a). If  $L$  is a  $\mathcal{C}$ -submodule of  $N$ , then  $L$  is closed in  $N$ .
- (b). If  $L$  is a  $\mathcal{C}$ -submodule of  $N$ , and  $P$  is a complement of  $L$ , then  $\mathcal{C}\text{-supp}(L) \cap \mathcal{C}\text{-supp}(P) = \emptyset$ .
- (c). For any  $\alpha \in \mathcal{C}$ ,  $N_{(\alpha)} \neq 0$  is a  $\mathcal{C}$ -submodule of  $N$ .

**Proof.** (a) This follows from Lemma 2.1(c).

(b) Suppose  $L$  is a  $\mathcal{C}$ -submodule of  $N$ ,  $P$  is a complement of  $L$  in  $N$  and  $\alpha$  is any element of  $\mathcal{C}$ . If  $\alpha \in \mathcal{C}\text{-supp}(L)$ , then  $\mathcal{C}\text{-supp}(L) = \mathcal{C}\text{-supp}(L \oplus P_{(\alpha)})$ . It follows that  $L = L \oplus P_{(\alpha)}$  and thus  $\alpha \notin \mathcal{C}\text{-supp}(P)$ .

(c) Let  $L$  be a submodule of  $N$  such that  $0 \neq N_{(\alpha)} \subseteq L \subseteq N$  with  $\mathcal{C}\text{-supp}(N_{(\alpha)}) = \mathcal{C}\text{-supp}(L)$ . Since  $\mathcal{C}\text{-supp}(N_{(\alpha)}) = \alpha$ ,  $\mathcal{C}\text{-supp}(L) = \alpha$ . Hence  $0 \neq L_{(\alpha)} \leq_e L$ . Therefore  $L \in \alpha$ . Since  $N_{(\alpha)}$  is a maximal  $\alpha$ -submodule of  $N$ ,  $N_{(\alpha)} = L$ .

**Lemma 2.4** If  $N \in \sigma[M]$  has finite  $\mathcal{C}$ -dimension and  $L$  is a  $\mathcal{C}$ -submodule of  $N$ , then both  $L$  and  $N/L$  have finite  $\mathcal{C}$ -dimension and

$$\mathcal{C}\text{-dim}(N) = \mathcal{C}\text{-dim}(L) + \mathcal{C}\text{-dim}(N/L).$$

**Proof.** Let  $P$  be a complement of  $L$  in  $N$ . Then  $L \oplus P \leq_e N$  and hence  $P$  embeds as an essential submodule of  $N/L$ . Since  $L$  is a  $\mathcal{C}$ -submodule,  $\mathcal{C}\text{-supp}(L) \cap \mathcal{C}\text{-supp}(P) = \emptyset$  by Lemma 2.3(b). Therefore  $\mathcal{C}\text{-dim}(N) = \mathcal{C}\text{-dim}(L \oplus P) = \mathcal{C}\text{-dim}(L) + \mathcal{C}\text{-dim}(P) = \mathcal{C}\text{-dim}(L) + \mathcal{C}\text{-dim}(N/L)$ .

A module  $N \in \sigma[M]$  is said to be a  $\mathcal{C}$ -atom if  $N_{(\alpha)} = N$  for some  $\alpha \in \mathcal{C}$  and is said to be  $\mathcal{C}$ -indecomposable if it cannot be written as the direct sum of two nonzero modules with disjoint  $\mathcal{C}$ -supports.

**Lemma 2.5** If  $\mathcal{C}$  spans  $\sigma[M]$ , then for any  $N \in \sigma[M]$ ,  $N$  is a  $\mathcal{C}$ -atom if and only if  $E_M(N)$  is  $\mathcal{C}$ -indecomposable.

**Proof.** Suppose  $N$  is a  $\mathcal{C}$ -atom such that  $E_M(N)$  is not  $\mathcal{C}$ -indecomposable. Then  $E_M(N) = N_1 \oplus N_2$  where  $\mathcal{C}\text{-supp}(N_1) \cap \mathcal{C}\text{-supp}(N_2) = \emptyset$ . Since  $N \in \alpha$  for some  $\alpha \in \mathcal{C}$ ,  $E_M(N) \in \alpha$  and thus both  $N_1$  and  $N_2$  must be in  $\alpha$ , which is a contradiction.

Conversely, suppose  $N$  is not a  $\mathcal{C}$ -atom. Since  $\mathcal{C}$  spans  $\sigma[M]$ , for some  $\alpha \in \mathcal{C}$ ,  $N_{(\alpha)}$  is a nonzero (non-essential) submodule of  $N$ . Let  $N'$  be a (nonzero) complement of  $N_{(\alpha)}$  in  $N$ . Then  $\alpha \notin \mathcal{C}\text{-supp}(N')$  and hence  $E_M(N) = E_M(N_{(\alpha)}) \oplus E_M(N')$  with  $\mathcal{C}\text{-supp}(E_M(N_{(\alpha)})) \cap \mathcal{C}\text{-supp}(E_M(N')) = \emptyset$ .

**Lemma 2.6** *Let  $\mathcal{C}$  span  $\sigma[M]$ . For a module  $N \in \sigma[M]$ , the following are equivalent.*

- (1).  $N$  has finite  $\mathcal{C}$ -dimension.
- (2).  $N$  has ACC on  $\mathcal{C}$ -submodules.
- (3).  $N$  has DCC on  $\mathcal{C}$ -submodules.

**Proof.**

(1)  $\Rightarrow$  (2) Let  $N_1 \subset N_2 \subset \dots$  be a strictly ascending chain of  $\mathcal{C}$ -submodules of  $N$ . For  $i \geq 1$ , let  $L_i$  be a complement of  $N_i$  in  $N_{i+1}$ . Since  $N_i$  is closed in  $N$  by Lemma 2.3(a),  $L_i \neq 0$  for any  $i$ . Claim:  $\mathcal{C}\text{-supp}(L_i) \cap \mathcal{C}\text{-supp}(L_j) = \emptyset$  if  $i \neq j$ . If  $\mathcal{C}\text{-supp}(L_i) \cap \mathcal{C}\text{-supp}(L_j) \neq \emptyset$  for some  $i \neq j$ , assume that  $i < j$ . Then there exists  $\alpha \in \mathcal{C}$  such that  $(L_i)_{(\alpha)} \neq 0$  and  $(L_j)_{(\alpha)} \neq 0$ , and  $(L_i)_{(\alpha)} \subset (N_{i+1})_{(\alpha)} \subset (N_j)_{(\alpha)}$ . Hence  $(N_j)_{(\alpha)} \neq 0$ , and this implies that  $\alpha \in \mathcal{C}\text{-supp}(N_j) \cap \mathcal{C}\text{-supp}(L_j)$ . This is a contradiction which proves the claim. Since  $\mathcal{C}$  spans  $\sigma[M]$ ,  $L_i \cap L_j = 0$  for all  $i \neq j$  and therefore  $N$  contains  $\bigoplus_{j=1}^{\infty} L_j$ , which implies  $N$  has infinite  $\mathcal{C}$ -dimension.

(2)  $\Rightarrow$  (3). Assume (2) and let  $N_1 \supseteq N_2 \supseteq \dots$  be a descending chain of  $\mathcal{C}$ -submodules of  $N$ . Let  $P_0 = 0$  and let  $P_i$  be a complement of  $N_i$  containing  $P_{i-1}$  in  $N$  for  $i > 0$ . Suppose  $L_i$  is a submodule of  $N$  such that  $P_i \subset L_i \leq N$  with  $\mathcal{C}\text{-supp}(P_i) = \mathcal{C}\text{-supp}(L_i)$ . Then  $L_i \cap N_i \neq 0$  by the maximality of  $P_i$ . Since by Lemma 2.3(b),  $\mathcal{C}\text{-supp}(P_i) \cap \mathcal{C}\text{-supp}(N_i) = \emptyset$ , it follows that  $\mathcal{C}\text{-supp}(P_i) \cap \mathcal{C}\text{-supp}(L_i \cap N_i) = \emptyset$ , and hence  $\mathcal{C}\text{-supp}(L_i) \cap \mathcal{C}\text{-supp}(L_i \cap N_i) = \emptyset$ , which cannot happen since  $\mathcal{C}$  spans  $\sigma[M]$ . Hence  $P_i$  is a  $\mathcal{C}$ -submodule of  $N$ , and  $P_1 \subseteq P_2 \subseteq \dots$ . By assumption, there exists an  $k \in \mathbb{Z}^+$  such that  $P_k = P_{k+1} = P_{k+2} = \dots$ . Let  $n \geq k$ . Then  $N_{n+1} \oplus P_n = N_{n+1} \oplus P_{n+1} \leq_e N$ . This implies that  $N_{n+1} \oplus P_n \leq_e N_n \oplus P_n$  and  $(N_{n+1} \oplus P_n) \cap N_n \leq_e (N_n \oplus P_n) \cap N_n$ . By the modular law,  $N_{n+1} \leq_e N_n$ , which implies  $\mathcal{C}\text{-supp}(N_{n+1}) = \mathcal{C}\text{-supp}(N_n)$ . Therefore  $N_{n+1} = N_n$  since  $N_i$  is a  $\mathcal{C}$ -submodule of  $N$ .

(3)  $\Rightarrow$  (1). Suppose that  $N$  has infinite  $\mathcal{C}$ -dimension. Then  $N$  contains  $\bigoplus_{i=1}^{\infty} N_{(\alpha_i)}$  with  $N_{(\alpha_i)} \neq 0$  for some  $\alpha_i \in \mathcal{C}$ . Let  $N_i = N_{(\alpha_i)}$ ,  $X_1 = N$ , and for each  $i \geq 2$ , let  $X_i$  be a complement of  $N_{i-1}$  in  $X_{i-1}$  such that  $\bigoplus_{j=1}^{\infty} N_j \subseteq X_i$ . Then

$$N = X_1 \supset X_2 \supset X_3 \supset \dots$$

is a strictly descending chain, since  $N_i \subseteq X_i$ , while  $N_i \not\subseteq X_{i+1}$  for all  $i \geq 1$ .

We claim that  $X_i$  is a  $\mathcal{C}$ -submodule of  $N$  for each  $i \geq 1$ . Suppose, for any  $i \geq 1$ ,  $L_i$  is a submodule of  $N$  such that  $X_i \subseteq L_i \subseteq N$  with  $\mathcal{C}\text{-supp}(X_i) =$

$\mathcal{C}\text{-supp}(L_i)$ . If  $L_i$  strictly contains  $X_i$ , then  $L_i \cap N_{i-1} \neq 0$  because  $X_i$  is a complement of  $N_{i-1}$  in  $X_{i-1}$ . It follows that  $\{\alpha_{i-1}\} = \mathcal{C}\text{-supp}(L_i \cap N_{i-1}) \subseteq \mathcal{C}\text{-supp}(L_i) = \mathcal{C}\text{-supp}(X_i)$ . But  $X_i$  is a complement of  $N_{i-1}$ , so  $\mathcal{C}\text{-supp}(X_i)$  does not contain  $\alpha_{i-1}$ . Hence  $X_i = L_i$ . This implies that  $X_i$  is a  $\mathcal{C}$ -submodule of  $N$ .

A module  $N \in \sigma[M]$  is said to be *weakly injective in  $\sigma[M]$*  (respectively, *weakly  $R$ -injective in  $\sigma[M]$* ) if for any finitely generated (respectively, cyclic) submodule  $F$  in  $E_M(N)$  there exists a submodule  $L$  of  $E_M(N)$  such that  $F \leq L \cong N$ .

The following Lemma is used in several places and relates how preservation of various types of injectivity is related.

**Lemma 2.7** *Suppose  $\mathcal{D}$  is a set of classes of modules in  $\sigma[M]$  which are closed under  $M$ -injective envelopes. Then each of the following statements implies the next.*

- (1).  $\bigoplus_{\gamma \in \mathcal{D}} E_\gamma$  is  $M$ -injective whenever  $E_\gamma \in \gamma$  is  $M$ -injective for every  $\gamma \in \mathcal{D}$ .
- (2).  $\bigoplus_{\gamma \in \mathcal{D}} E_\gamma$  is weakly injective in  $\sigma[M]$  whenever  $E_\gamma \in \gamma$  is  $M$ -injective for every  $\gamma \in \mathcal{D}$ .
- (3).  $\bigoplus_{\gamma \in \mathcal{D}} E_\gamma$  is weakly injective in  $\sigma[M]$  whenever  $E_\gamma \in \gamma$  is weakly injective in  $\sigma[M]$  for every  $\gamma \in \mathcal{D}$ .
- (4).  $\bigoplus_{\gamma \in \mathcal{D}} E_\gamma$  is weakly  $R$ -injective in  $\sigma[M]$  whenever  $E_\gamma \in \gamma$  is weakly injective in  $\sigma[M]$  for every  $\gamma \in \mathcal{D}$ .

**Proof.** (1)  $\Rightarrow$  (2) is clear since  $\bigoplus_{\gamma \in \mathcal{D}} E_\gamma$   $M$ -injective implies that  $\bigoplus_{\gamma \in \mathcal{D}} E_\gamma$  is weakly injective in  $\sigma[M]$ .

(2)  $\Rightarrow$  (3). For each  $\gamma \in \mathcal{D}$  suppose  $E_\gamma \in \gamma$  is weakly injective in  $\sigma[M]$ . Let  $F$  be a finitely generated submodule of  $E_M(\bigoplus_{\gamma \in \mathcal{D}} E_\gamma)$ . Since  $E_M(\bigoplus_{\gamma \in \mathcal{D}} E_\gamma) = E_M(\bigoplus_{\gamma \in \mathcal{D}} E_M(E_\gamma))$ , and since by (2)  $\bigoplus_{\gamma \in \mathcal{D}} E_M(E_\gamma)$  is weakly injective in  $\sigma[M]$ , there exists  $L \leq E_M(\bigoplus_{\gamma \in \mathcal{D}} E_\gamma)$  with  $F \leq L \cong \bigoplus_{\gamma \in \mathcal{D}} E_M(E_\gamma)$ . Now for each  $\gamma \in \mathcal{D}$ , let  $K_\gamma$  and  $L_\gamma = E_M(K_\gamma)$  be submodules of  $L$  which are isomorphic to  $E_\gamma$  and  $E_M(E_\gamma)$  respectively under the isomorphism  $L \cong \bigoplus_{\gamma \in \mathcal{D}} E_M(E_\gamma)$ . Since  $F$  is finitely generated, there exists a finite subset  $\mathcal{A} \subseteq \mathcal{D}$  such that  $F \leq \bigoplus_{\gamma \in \mathcal{A}} L_\gamma \cong \bigoplus_{\gamma \in \mathcal{A}} E_M(K_\gamma)$ . Since  $\bigoplus_{\gamma \in \mathcal{A}} K_\gamma$  is weakly injective in  $\sigma[M]$ , there exists  $H \leq E_M(\bigoplus_{\gamma \in \mathcal{A}} K_\gamma)$  such that  $F \leq H \cong \bigoplus_{\gamma \in \mathcal{A}} K_\gamma$ . Since  $H \leq \bigoplus_{\gamma \in \mathcal{A}} L_\gamma$ , we can extend the isomorphism  $H \cong \bigoplus_{\gamma \in \mathcal{A}} K_\gamma$  to get  $F \leq H \oplus (\bigoplus_{\gamma \in \mathcal{D} \setminus \mathcal{A}} L_\gamma) \cong \bigoplus_{\gamma \in \mathcal{D}} K_\gamma \cong \bigoplus_{\gamma \in \mathcal{D}} E_\gamma$  which proves that  $\bigoplus_{\gamma \in \mathcal{D}} E_\gamma$  is weakly injective in  $\sigma[M]$ .

(3)  $\Rightarrow$  (4) is clear since  $\bigoplus_{\gamma \in \mathcal{D}} E_\gamma$  weakly injective in  $\sigma[M]$  implies that  $\bigoplus_{\gamma \in \mathcal{D}} E_\gamma$  is weakly  $R$ -injective in  $\sigma[M]$ .

Recall that a module  $N \in \sigma[M]$  is  *$M$ -singular* if there exist modules  $K \leq L \in \sigma[M]$  such that  $K \leq_e L$  and  $N \cong L/K$ . If  $N$  has no nonzero  $M$ -singular submodules, we say that  $N$  is  *$M$ -nonsingular*. (See [6] for more information about  $M$ -singular and  $M$ -nonsingular modules.) We now characterize for an  $M$ -nonsingular module, when a direct sum of  $M$ -injective modules from the set of disjoint natural classes is again  $M$ -injective.

**Theorem 2.8** *The following statements are equivalent for  $M$   $M$ -nonsingular.*

- (1). *Each  $M$ -nonsingular  $M$ -injective module  $E \in \sigma[M]$  can be written as  $E = E_0 \oplus (\oplus_{\alpha \in \mathcal{C}} E_{(\alpha)})$ .*
- (2).  *$\oplus_{\alpha \in \mathcal{C}} E_{\alpha}$  is  $M$ -injective whenever  $E_{\alpha} \in \alpha$  is  $M$ -injective and  $M$ -nonsingular for every  $\alpha \in \mathcal{C}$ .*
- (3).  *$\oplus_{\alpha \in \mathcal{C}} E_{\alpha}$  is weakly injective in  $\sigma[M]$  whenever  $E_{\alpha} \in \alpha$  is  $M$ -injective and  $M$ -nonsingular for every  $\alpha \in \mathcal{C}$ .*
- (4).  *$\oplus_{\alpha \in \mathcal{C}} E_{\alpha}$  is weakly injective in  $\sigma[M]$  whenever  $E_{\alpha} \in \alpha$  is weakly injective in  $\sigma[M]$  and  $M$ -nonsingular for every  $\alpha \in \mathcal{C}$ .*
- (5).  *$\oplus_{\alpha \in \mathcal{C}} E_{\alpha}$  is weakly  $R$ -injective in  $\sigma[M]$  whenever  $E_{\alpha} \in \alpha$  is weakly injective in  $\sigma[M]$  and  $M$ -nonsingular for every  $\alpha \in \mathcal{C}$ .*
- (6).  *$\mathcal{C}\text{-dim}(L) < \infty$  for every finitely generated  $M$ -nonsingular module  $L \in \sigma[M]$ .*
- (7).  *$\mathcal{C}\text{-dim}(L) < \infty$  for any cyclic submodule  $L \leq M$ .*

**Proof.**

(1)  $\Rightarrow$  (2) Let  $E_{\alpha} \in \alpha$  be  $M$ -injective and  $M$ -nonsingular for all  $\alpha \in \mathcal{C}$ . Let  $E = E_M(\oplus_{\alpha \in \mathcal{C}} E_{\alpha})$ . Then  $E$  is  $M$ -nonsingular. Take  $E_{(\alpha)}$  to contain  $E_{\alpha}$  for  $\alpha \in \mathcal{C}$ . By (1),  $E$  can be written  $E = E_0 \oplus (\oplus_{\alpha \in \mathcal{C}} E_{(\alpha)})$ . First note that  $E_0 = 0$  since  $\oplus_{\alpha \in \mathcal{C}} E_{(\alpha)} \supseteq \oplus_{\alpha \in \mathcal{C}} E_{\alpha}$  is essential in  $E$ . If  $E_{(\beta)}$  properly contains  $E_{\beta}$  for some  $\beta \in \mathcal{C}$ , then, since  $E_{\beta}$  is  $M$ -injective we can write  $E_{(\beta)} = F \oplus E_{\beta}$  for some nonzero  $F$ . Since  $\oplus_{\alpha \in \mathcal{C}} E_{\alpha}$  is essential in  $E$ ,  $F \cap \oplus_{\alpha \in \mathcal{C}} E_{\alpha} \neq 0$ . Write  $f = \sum x_{\alpha} \neq 0$  where  $f \in F$  and  $x_{\alpha} \in E_{\alpha}$ . Then we have  $f - x_{\beta} = \sum_{\alpha \neq \beta} x_{\alpha} \in E_{(\beta)} \cap \sum_{\alpha \neq \beta} E_{(\alpha)}$  which is a contradiction. This shows that  $E_{\alpha} = E_{(\alpha)}$  for all  $\alpha \in \mathcal{C}$  and hence  $E = \oplus_{\alpha \in \mathcal{C}} E_{\alpha}$  which is  $M$ -injective.

(2)  $\Rightarrow$  (3)  $\Rightarrow$  (4)  $\Rightarrow$  (5) is Lemma 2.7.

(5)  $\Rightarrow$  (7) Let  $L$  be a cyclic submodule of  $M$ . Let  $L_0$  be a complement of  $\oplus_{\alpha \in \mathcal{C}} L_{(\alpha)}$  in  $L$ . Further, let  $E_0 = E_M(L_0)$  and  $E_{\alpha} = E_M(L_{(\alpha)})$  for all  $\alpha \in \mathcal{C}$ . Then  $E_M(L) = E_0 \oplus E_M(\oplus_{\alpha \in \mathcal{C}} E_{\alpha})$ . Suppose  $L \leq Y \oplus Z$  where  $Y$  and  $Z$  are cyclic with  $Y \leq E_0$  and  $Z \leq E_M(\oplus_{\alpha \in \mathcal{C}} E_{\alpha})$ . Since  $\oplus_{\alpha \in \mathcal{C}} E_{\alpha}$  is weakly  $R$ -injective in  $\sigma[M]$ , there exists a  $W \leq E_M(\oplus_{\alpha \in \mathcal{C}} E_{\alpha})$  such that  $Z \leq W \cong \oplus_{\alpha \in \mathcal{C}} E_{\alpha}$ . Since  $Z$  is cyclic,  $Z$  embeds in  $\oplus_{\alpha \in \mathcal{A}} E_{\alpha}$  for some finite subset  $\mathcal{A} \subseteq \mathcal{C}$ . It follows that  $\mathcal{C}\text{-supp}(L) \subseteq \mathcal{C}\text{-supp}(Z) \leq \mathcal{A}$  is finite.

(7)  $\Rightarrow$  (2). Let  $E_{\alpha} \in \alpha$  be  $M$ -injective and  $M$ -nonsingular for each  $\alpha \in \mathcal{C}$  and suppose that  $\mathcal{C}\text{-dim}(L) < \infty$  for every cyclic submodule  $L$  of  $M$ . To show  $\oplus_{\alpha \in \mathcal{C}} E_{\alpha}$  is  $M$ -injective, it suffices to show we can extend any map  $f : K \rightarrow \oplus_{\alpha \in \mathcal{C}} E_{\alpha}$  where  $K$  is a submodule of a cyclic submodule  $L$  of  $M$  to a map  $f : L \rightarrow \oplus_{\alpha \in \mathcal{C}} E_{\alpha}$ . For each  $\alpha \in \mathcal{C}$ , let  $K_{\alpha} = (\pi_{\alpha} f)^{-1}(E_{\alpha})$  where  $\pi_{\beta} : \oplus_{\alpha \in \mathcal{C}} E_{\alpha} \rightarrow E_{\beta}$  is a projection onto  $E_{\beta}$ . Let  $f_{\alpha} : K_{\alpha} \rightarrow E_{\alpha}$  be the restriction of  $\pi_{\alpha} f$  to  $K_{\alpha}$ . If  $\ker(f_{\alpha}) \leq_e K_{\alpha}$ , then  $\text{im}(f_{\alpha})$  is  $M$ -singular. This implies that  $f_{\alpha} = 0$ . If  $\ker(f_{\alpha})$  is not an essential submodule of  $L_{\alpha}$ , then there exist a nonzero  $K' \leq K_{\alpha}$  such that  $K' \cap \ker(f_{\alpha}) = 0$ . Therefore  $K' \cong f_{\alpha}(K') \in \alpha$ . Since  $\mathcal{C}\text{-dim}(K) < \infty$ , it follows that  $f_{\alpha} = 0$  for all but a finite number of  $\alpha$ .

Hence  $f(K)$  lies in a finite direct sum  $\bigoplus_{i=1}^m E_{\alpha_i}$ . Since this sum is  $M$ -injective, we can extend  $f$  to a map  $\bar{f} : L \longrightarrow \bigoplus_{i=1}^m E_{\alpha_i} \leq \bigoplus_{\alpha \in \mathcal{C}} E_{\alpha}$  as desired.

(2)  $\Rightarrow$  (1) Given any  $M$ -nonsingular  $M$ -injective module  $E$ ,  $E_{(\alpha)}$  is also  $M$ -nonsingular and  $M$ -injective. By (2),  $\bigoplus_{\alpha \in \mathcal{C}} E_{(\alpha)}$  is  $M$ -injective.  $E$  can then be written  $E_0 \oplus (\bigoplus_{\alpha \in \mathcal{C}} E_{(\alpha)})$ .

(1)  $\Rightarrow$  (6) Let  $L \in \sigma[M]$  be finitely generated. Write  $E = E_M(L)$ . By (1),  $E$  can be written  $E = E_0 \oplus (\bigoplus_{\alpha \in \mathcal{C}} E_{(\alpha)})$ . Since  $L$  is finitely generated,  $L \leq E_0 \oplus (\bigoplus_{i=1}^n E_{(\alpha_i)})$  for some  $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathcal{C}$ . Since  $L$  is essential in  $E$ ,  $E_{\alpha} = 0$  for  $\alpha \notin \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ . Thus  $E_{\alpha} = 0$  for all but finitely many  $\alpha \in \mathcal{C}$  and  $\mathcal{C}\text{-dim}(L) \leq \mathcal{C}\text{-dim}(E) < \infty$ .

(6)  $\Rightarrow$  (7) Obvious.

The following corollary is an immediate consequence of the previous theorem.

**Corollary 2.9** *The following statements are equivalent for  $M$   $M$ -nonsingular and finitely generated.*

- (1).  $\mathcal{C}\text{-dim}(M) < \infty$ .
- (2). Each  $M$ -nonsingular  $M$ -injective module  $E \in \sigma[M]$  can be written as  $E = E_0 \oplus (\bigoplus_{\alpha \in \mathcal{C}} E_{(\alpha)})$ .
- (3).  $\bigoplus_{\alpha \in \mathcal{C}} E_{\alpha}$  is  $M$ -injective whenever  $E_{\alpha} \in \alpha$  is  $M$ -injective and  $M$ -nonsingular for every  $\alpha \in \mathcal{C}$ .
- (4).  $\bigoplus_{\alpha \in \mathcal{C}} E_{\alpha}$  is weakly injective in  $\sigma[M]$  whenever  $E_{\alpha} \in \alpha$  is  $M$ -injective and  $M$ -nonsingular for every  $\alpha \in \mathcal{C}$ .
- (5).  $\bigoplus_{\alpha \in \mathcal{C}} E_{\alpha}$  is weakly injective in  $\sigma[M]$  whenever  $E_{\alpha} \in \alpha$  is weakly injective in  $\sigma[M]$  and  $M$ -nonsingular for every  $\alpha \in \mathcal{C}$ .
- (6).  $\bigoplus_{\alpha \in \mathcal{C}} E_{\alpha}$  is weakly  $R$ -injective in  $\sigma[M]$  whenever  $E_{\alpha} \in \alpha$  is weakly injective in  $\sigma[M]$  and  $M$ -nonsingular for every  $\alpha \in \mathcal{C}$ .

We now want to consider the following subset of  $\mathcal{C}$ ,  $\mathcal{B} = \{\beta \in \mathcal{C} \mid \exists \beta \neq \alpha \in \mathcal{C}, K \leq N \in \alpha \text{ with } N/K \in \beta\}$ .

**Lemma 2.10**  $\mathcal{B} = \{\beta \in \mathcal{C} \mid \text{there exist } M\text{-injective modules } E_{\alpha} \in \alpha \in \mathcal{C} \text{ and } E_{\beta} \in \beta, \beta \neq \alpha \text{ with } \text{Hom}(E_{\alpha}, E_{\beta}) \neq 0\}$ .

**Proof.** Call the right hand side of the desired equality  $\mathcal{B}'$ . Suppose  $\beta \in \mathcal{B}$  with  $K \leq N \in \alpha$  with  $N/K \in \beta$ . Then we can extend the projection map  $N \longrightarrow N/K$  to obtain a nonzero homomorphism  $E_M(N) \longrightarrow E_M(N/K)$  where  $E_M(N) \in \alpha$  and  $E_M(N/K) \in \beta$ . Thus  $\beta \in \mathcal{B}'$ .

Now suppose that  $\beta \in \mathcal{B}'$  and  $0 \neq \phi \in \text{Hom}(E_{\alpha}, E_{\beta})$  where  $E_{\alpha} \in \alpha$  and  $E_{\beta} \in \beta$ . It follows that  $E_{\alpha}/\ker\phi \in \beta$  and  $\beta \in \mathcal{B}$ .

**Lemma 2.11** *Suppose that  $\mathcal{B}$  (as defined above) is a finite set. Given any finite set  $\mathcal{A} \subseteq \mathcal{C}$  and direct sum  $E = \bigoplus_{\alpha \in \mathcal{C}} E_{\alpha}$  of  $M$ -injective modules with  $E_{\alpha} \in \alpha$ , then  $\bigoplus_{\alpha \in \mathcal{A} \cup \mathcal{B}} E_{\alpha}$  is the unique maximal  $\vee(\mathcal{A} \cup \mathcal{B})$ -submodule of  $\bigoplus_{\alpha \in \mathcal{C}} E_{\alpha}$ .*

**Proof.** First we show that  $\bigoplus_{\alpha \in \mathcal{A} \cup \mathcal{B}} E_\alpha$  is a maximal  $\vee(\mathcal{A} \cup \mathcal{B})$ -submodule of  $E$ . Since  $M$ -natural classes are closed under direct sums,  $\bigoplus_{\alpha \in \mathcal{A} \cup \mathcal{B}} E_\alpha$  is a  $\vee(\mathcal{A} \cup \mathcal{B})$ -module. Let  $F$  be a  $\vee(\mathcal{A} \cup \mathcal{B})$ -submodule of  $E$  containing  $\bigoplus_{\alpha \in \mathcal{A} \cup \mathcal{B}} E_\alpha$ . Since  $\bigoplus_{\alpha \in \mathcal{A} \cup \mathcal{B}} E_\alpha$  is a  $M$ -injective, then  $F = F' \oplus (\bigoplus_{\alpha \in \mathcal{A} \cup \mathcal{B}} E_\alpha)$ . We see that

$$F' \cong \frac{F}{\bigoplus_{\alpha \in \mathcal{A} \cup \mathcal{B}} E_\alpha} \subseteq \frac{E}{\bigoplus_{\alpha \in \mathcal{A} \cup \mathcal{B}} E_\alpha} \cong \bigoplus_{\mathcal{C} \setminus (\mathcal{A} \cup \mathcal{B})} E_\alpha$$

Now by the projection argument [2, 1.2, for example]  $F'$  contains a nonzero cyclic submodule, say  $D$ , isomorphic to a submodule of  $E_\gamma$  for some  $\gamma \in \mathcal{C} \setminus (\mathcal{A} \cup \mathcal{B})$ . This implies that  $D \in \gamma$ . It follows that  $F \notin \vee(\mathcal{A} \cup \mathcal{B})$ , since if it were,  $F$  would contain an essential direct sum  $\bigoplus_{\alpha \in \mathcal{A} \cup \mathcal{B}} F_\alpha$  where  $F_\alpha \in \alpha$ . But  $D \cap (\bigoplus_{\alpha \in \mathcal{A} \cup \mathcal{B}} F_\alpha)$  by the projection argument has a nonzero cyclic submodule isomorphic to a submodule of  $F_\alpha$  for some  $\alpha \in \mathcal{A} \cup \mathcal{B}$ . This, however, would contradict the disjointness of the members of  $\mathcal{C}$ .

To show that this is the unique maximal  $\vee(\mathcal{A} \cup \mathcal{B})$ -submodule of  $E$ , we use Daun's criterion [4, 1.6]: That there is a unique  $\vee(\mathcal{A} \cup \mathcal{B})$ -submodule of  $\bigoplus_{\alpha \in \mathcal{C}} E_\alpha$  if there does not exist a nonzero map  $\phi : K \rightarrow W$ , where  $K \in \vee(\mathcal{A} \cup \mathcal{B})$  and  $W \in c(\vee(\mathcal{A} \cup \mathcal{B}))$  are submodules of  $\bigoplus_{\alpha \in \mathcal{C}} E_\alpha$ . (Note that  $c(\alpha) = \{W \in \sigma[M] \mid \forall 0 \neq V \leq W, V \notin \alpha\}$ .) Suppose that such a map exists. Since  $W \leq \bigoplus_{\alpha \in \mathcal{C}} E_\alpha$  and  $W \in c(\vee(\mathcal{A} \cup \mathcal{B}))$ , we may assume that  $W \in \gamma \in \mathcal{C} \setminus (\mathcal{A} \cup \mathcal{B})$ . Now let us extend  $\phi$  to  $\bar{\phi} : E_M(K) \rightarrow E_M(W)$ . Now since  $\mathcal{A} \cup \mathcal{B}$  is finite and  $\bigoplus_{\alpha \in \mathcal{A} \cup \mathcal{B}} K_{(\alpha)} \leq_e K$ , we must have  $E_M(K) = \bigoplus_{\alpha \in \mathcal{A} \cup \mathcal{B}} E_M(K_{(\alpha)})$ . However if  $\bar{\phi}$  restricted to  $E_M(K_{(\alpha)})$  were not zero, we would have either  $\alpha = \gamma$  or  $\gamma \in \mathcal{B}$ . Since these cannot happen,  $\bar{\phi} = 0$  and thus  $\phi = 0$ .

**Theorem 2.12** *Given the following conditions, the implications*

$$(1) \iff (2) \implies (3) \implies (4) \begin{matrix} \implies (5) \\ \implies (6) \end{matrix} \implies (7) \quad \text{hold, and if the set } \mathcal{B} \text{ is finite,}$$

(1)–(7) are equivalent.

- (1). For each  $M$ -injective module  $E \in \sigma[M]$ ,  $E = E_0 \oplus (\bigoplus_{\alpha \in \mathcal{C}} E_{(\alpha)})$ .
- (2).  $\bigoplus_{\alpha \in \mathcal{C}} E_\alpha$  is  $M$ -injective whenever  $E_\alpha \in \alpha$  is  $M$ -injective for every  $\alpha \in \mathcal{C}$ .
- (3).  $\bigoplus_{\alpha \in \mathcal{C}} E_\alpha$  is weakly injective in  $\sigma[M]$  whenever  $E_\alpha \in \alpha$  is  $M$ -injective for every  $\alpha \in \mathcal{C}$ .
- (4).  $\bigoplus_{\alpha \in \mathcal{C}} E_\alpha$  is weakly injective in  $\sigma[M]$  whenever  $E_\alpha \in \alpha$  is weakly injective in  $\sigma[M]$  for every  $\alpha \in \mathcal{C}$ .
- (5).  $\bigoplus_{\alpha \in \mathcal{C}} E_\alpha$  is weakly  $R$ -injective in  $\sigma[M]$  whenever  $E_\alpha \in \alpha$  is weakly injective in  $\sigma[M]$  for every  $\alpha \in \mathcal{C}$ .
- (6).  $\mathcal{C}\text{-dim}(L) < \infty$  for every finitely generated module  $L \in \sigma[M]$ .
- (7).  $\mathcal{C}\text{-dim}(L) < \infty$  for every cyclic submodule of  $M$ .

**Proof.** (1)  $\Rightarrow$  (2) Let  $E_\alpha$  be  $M$ -injective for all  $\alpha \in \mathcal{C}$ . Let  $E = E_M(\bigoplus_{\alpha \in \mathcal{C}} E_\alpha)$ . Take  $E_{(\alpha)}$  to contain  $E_\alpha$  for  $\alpha \in \mathcal{C}$ . By (1),  $E$  can be written  $E = E_0 \oplus (\bigoplus_{\alpha \in \mathcal{C}} E_{(\alpha)})$ . First note that  $E_0 = 0$  since  $\bigoplus_{\alpha \in \mathcal{C}} E_{(\alpha)} \supseteq \bigoplus_{\alpha \in \mathcal{C}} E_\alpha$  is essential in  $E$ . If  $E_{(\beta)}$  properly contains  $E_\beta$  for some  $\beta \in \mathcal{C}$ , then, since  $E_\beta$  is injective we can write  $E_{(\beta)} = F \oplus E_\beta$  for some nonzero  $F$ . Since  $\bigoplus_{\alpha \in \mathcal{C}} E_\alpha$  is essential in  $E$ ,  $F \cap \bigoplus_{\alpha \in \mathcal{C}} E_\alpha \neq 0$ . Write  $f = \sum x_\alpha \neq 0$  where  $f \in F$  and  $x_\alpha \in E_\alpha$ . Then we have  $f - x_\beta = \sum_{\alpha \neq \beta} x_\alpha \in E_{(\beta)} \cap \sum_{\alpha \neq \beta} E_\alpha$  which is a contradiction. This shows that  $E_\alpha = E_{(\alpha)}$  for all  $\alpha \in \mathcal{C}$  and hence  $E = \bigoplus_{\alpha \in \mathcal{C}} E_\alpha$  which is  $M$ -injective.

(2)  $\Rightarrow$  (1) Let  $E$  be an  $M$ -injective module in  $\sigma[M]$ . For each  $\alpha \in \mathcal{C}$ , choose  $E_{(\alpha)} \leq E$ . Since  $E_{(\alpha)}$  is  $M$ -injective, by (2),  $\bigoplus_{\alpha \in \mathcal{C}} E_{(\alpha)}$  is  $M$ -injective and thus there exists an  $E_0$  with  $E = E_0 \oplus (\bigoplus_{\alpha \in \mathcal{C}} E_{(\alpha)})$  as desired.

(2)  $\Rightarrow$  (3)  $\Rightarrow$  (4)  $\Rightarrow$  (5) is Lemma 2.7.

(4)  $\Rightarrow$  (6) Let  $L$  be a finitely generated submodule of  $M$ . Let  $L_0$  be a complement of  $\bigoplus_{\alpha \in \mathcal{C}} L_{(\alpha)}$  in  $L$ . Further, let  $E_0 = E_M(L_0)$  and  $E_\alpha = E_M(L_{(\alpha)})$  for all  $\alpha \in \mathcal{C}$ . Then  $E_M(L) = E_0 \oplus E_M(\bigoplus_{\alpha \in \mathcal{C}} E_\alpha)$ . Suppose  $L \leq Y \oplus Z$  where  $Y$  and  $Z$  are finitely generated with  $Y \leq E_0$  and  $Z \leq E_M(\bigoplus_{\alpha \in \mathcal{C}} E_\alpha)$ . Since  $\bigoplus_{\alpha \in \mathcal{C}} E_\alpha$  is weakly injective in  $\sigma[M]$ , there exists a  $W \leq E_M(\bigoplus_{\alpha \in \mathcal{C}} E_\alpha)$  such that  $Z \leq W \cong \bigoplus_{\alpha \in \mathcal{A}} E_\alpha$ . Since  $Z$  is finitely generated,  $Z$  embeds in  $\bigoplus_{\alpha \in \mathcal{A}} E_\alpha$  for some finite subset  $\mathcal{A} \subseteq \mathcal{C}$ . It follows that  $\mathcal{C}\text{-supp}(L) \subseteq \mathcal{C}\text{-supp}(Z) \leq \mathcal{A}$  is finite.

(5)  $\Rightarrow$  (7) is the same as (5)  $\Rightarrow$  (7) in Theorem 2.8.

(6)  $\Rightarrow$  (7) is clear.

Now if  $\mathcal{B}$  is finite, we show (7)  $\Rightarrow$  (2). Suppose  $\{E_\alpha \mid \alpha \in \mathcal{C}\}$  is a set of  $M$ -injective modules as in (2). Then  $\bigoplus_{\alpha \in \mathcal{C}} E_\alpha$  is  $M$ -injective provided every homomorphism  $f : K \rightarrow \bigoplus_{\alpha \in \mathcal{C}} E_\alpha$  can be extended to  $L$  where  $K$  is a submodule of  $L$  and  $L$  is a cyclic submodule of  $M$ . Since by (7) the  $\mathcal{C}\text{-dim}(L) < \infty$ ,  $\mathcal{C}\text{-dim}(K) < \infty$ . If  $\alpha \in \mathcal{C}\text{-supp}(f(K))$ , either  $\alpha \in \mathcal{C}\text{-supp}(K)$  or  $\alpha \in \mathcal{B}$ . This means that  $f(K)$  is a  $\vee(\mathcal{C}\text{-supp}(K) \cup \mathcal{B})$ -module in  $\bigoplus_{\alpha \in \mathcal{C}} E_\alpha$ . By Lemma 2.11,  $f(K)$  lies in a finite direct sum  $\bigoplus_{\alpha \in \mathcal{C}\text{-supp}(K) \cup \mathcal{B}} E_\alpha$ . Since this is an  $M$ -injective submodule of  $\bigoplus_{\alpha \in \mathcal{C}} E_\alpha$ ,  $f$  may be extended to  $L$  as desired.

**Example 2.13** There exist a ring  $R$  and a set  $\mathcal{C}$  of pairwise disjoint natural classes for which  $\mathcal{C}\text{-dim}(R) < \infty$  (i.e., statement (7) of Theorem 2.12 is true),  $\mathcal{B}$  is infinite and where there exists a direct sum of injective modules which is not injective (i.e., statement (2) of Theorem 2.12 with  $M = R$  is false). This shows that the condition that  $\mathcal{B}$  be finite in Theorem 2.12 is necessary.

Let  $R = F[x_1, x_2, \dots]$  where  $F$  is any field. Let  $I_i = (x_1, x_2, \dots, x_i)$ , and let  $\alpha_i = \mathcal{D}(R/I_i)$ , where  $\mathcal{D}(N) = \{L \mid \text{for every } 0 \neq L' \leq L, \text{ there exists } 0 \neq L'' \leq L' \text{ such that } L'' \text{ can be embedded in } N\}$ . Then  $E(R/I_i) \in \alpha_i$  and  $I_1 \subset I_2 \subset I_3 \subset \dots$ . We claim that  $\bigoplus_{i=1}^{\infty} E(R/I_i)$  is not injective. Suppose it is. Let  $I = \bigcup_{i=1}^{\infty} I_i$  and define  $f : I \rightarrow \bigoplus_{i=1}^{\infty} E(R/I_i)$  by  $\pi_i f(a) = a + I_i$  where  $\pi_i : \bigoplus_{i=1}^{\infty} E(R/I_i) \rightarrow E(R/I_i)$ . Then there exists  $f : R \rightarrow \bigoplus_{i=1}^{\infty} E(R/I_i)$ . Let  $\bar{f}(1) = x$ . Then for any  $a \in I$ ,  $f(a) = \bar{f}(a) = \bar{f}(1)a = xa \in \bigoplus_{i=1}^k E(R/I_i)$  for some  $k$ . If  $a \in I_{j+1}$  where  $j > k$ , then  $\pi_{k+1} f(a) = a + I_{k+1} = 0$  which implies  $a \in I_{k+1}$ . This contradiction proves the claim. We next claim that  $R/I_i \perp R/I_j$  when  $i \neq j$ . Let  $j > i$ . If  $L/I_i \cong K/I_j \neq 0$  for some  $L, K \leq R$ , then  $x_j(L/I_i) = x_j(K/I_j) = 0$ , which can not happen. This shows that  $R/I_i \perp R/I_j$

and consequently that  $\mathcal{C} = \{\alpha_i\}$  forms a pairwise disjoint set of natural classes. It is easy to see that  $\mathcal{C}\text{-dim}(R) = 0$  (and therefore  $\mathcal{C}\text{-dim}(L) = 0$  for every cyclic right ideal of  $R$ ) and  $\mathcal{B}$  is infinite, where  $\mathcal{B} = \{\beta \in \mathcal{C} \mid \text{there exists } \beta \neq \alpha \in \mathcal{C} \text{ with } K \leq L \in \alpha \text{ such that } L/K \in \beta\}$ .

### 3 $\mathcal{C}$ -chains

**Definition 3** Let  $\mathcal{C}$  be a set of pairwise disjoint  $M$ -natural classes. A chain  $N_1 \leq N_2 \leq \dots$  is called an *ascending  $\mathcal{C}$ -chain* (or just a  $\mathcal{C}$ -chain) in  $N$  with  $N_i \leq N$  if  $N_{i+1}/N_i \in \alpha_i \in \mathcal{C}$  and if  $m \neq n$  implies  $\alpha_m \neq \alpha_n$  for all  $i, m, n \geq 1$ . If there exists no infinite ascending  $\mathcal{C}$ -chains in  $N$ , then  $N$  is said to satisfy the *ascending chain condition (ACC) on  $\mathcal{C}$ -chains*.

The following proposition follows easily from the definition.

**Proposition 3.1** *For any  $N \in \sigma[M]$ , we have:*

- (1). *If  $N$  has ACC on  $\mathcal{C}$ -chains, then  $\mathcal{C}\text{-dim}(N) < \infty$ .*
- (2). *If  $0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$  is a short exact sequence, then  $N$  has ACC on  $\mathcal{C}$ -chains if and only if both  $N'$  and  $N''$  do.*
- (3). *If  $N$  has ACC on  $\mathcal{C}$ -chains and  $K$  is finitely subgenerated by  $N$  then  $K$  has ACC on  $\mathcal{C}$ -chains.*

**Theorem 3.2** *Suppose  $M$  is finitely generated. The following statements are equivalent.*

- (1).  *$M$  has ACC on  $\mathcal{C}$ -chains.*
- (2).  $\bigoplus_{\alpha \in \mathcal{C}} E_\alpha$  is  $M$ -injective whenever each  $E_\alpha \in \alpha$  is  $M$ -injective for each  $\alpha \in \mathcal{C}$ .
- (3). *Each  $M$ -injective module,  $E$ , in  $\sigma[M]$  can be written  $E = E_0 \oplus (\bigoplus_{\alpha \in \mathcal{C}} E_{(\alpha)})$ .*

**Proof.**

(1)  $\Rightarrow$  (2). Let  $E = \bigoplus_{\alpha \in \mathcal{C}} E_\alpha$  where each  $E_\alpha$  is  $M$ -injective. Let  $f : N \longrightarrow E$  for some  $N \leq M$ . We wish to show that  $f$  can be extended to  $\bar{f} : M \longrightarrow E$ . If  $f(N)$  is contained in a finite direct sum of  $E_\alpha$ , say  $f(N) \leq \bigoplus_{i=1}^n E_{\alpha_i}$ , then  $\bigoplus_{i=1}^n E_{\alpha_i}$  is  $M$ -injective and  $f$  can be extended as desired. So let us assume that  $f(N)$  is contained in no finite direct sum of  $E_\alpha$ . Let  $n_1 \in N$  be such that  $f(n_1) \neq 0$ . Let  $\mathcal{P}_1 = \{\alpha \in \mathcal{C} \mid \pi_\alpha(n_1) \neq 0\}$ . Note that  $\mathcal{P}_1$  is a nonempty finite set. Let  $\mathcal{P}_1 = \{\alpha_{11}, \alpha_{12}, \dots, \alpha_{1k_1}\}$ . Since  $f(N) \not\leq \bigoplus_{\alpha \in \mathcal{P}_1} E_\alpha$ , there exists  $n_2 \in N$  with  $f(n_2) \notin \bigoplus_{\alpha \in \mathcal{P}_1} E_\alpha$ . Let  $\mathcal{P}_2 = \{\alpha \in \mathcal{C} \setminus \mathcal{P}_1 \mid \pi_\alpha(n_2) \neq 0\}$ . Note that  $\mathcal{P}_2 \cap \mathcal{P}_1 = \emptyset$  and  $\mathcal{P}_2$  is a nonempty finite set. Let  $\mathcal{P}_2 = \{\alpha_{21}, \alpha_{22}, \dots, \alpha_{2k_2}\}$ . For each positive integer  $\ell$ , we can inductively define  $\mathcal{P}_\ell$ : Since  $f(N) \not\leq \bigoplus_{\alpha \in \bigcup_{i=1}^{\ell-1} \mathcal{P}_i} E_\alpha$ , there exists  $n_\ell \in N$  with  $f(n_\ell) \notin \bigoplus_{\alpha \in \bigcup_{i=1}^{\ell-1} \mathcal{P}_i} E_\alpha$ . Let  $\mathcal{P}_\ell = \{\alpha \in \mathcal{C} \setminus \bigcup_{i=1}^{\ell-1} \mathcal{P}_i \mid \pi_\alpha(n_\ell) \neq 0\}$ . Note that  $\mathcal{P}_\ell \cap$

$(\bigcap_{i=1}^{\ell-1} \mathcal{P}_i) = \emptyset$  and  $\mathcal{P}_\ell$  is a nonempty finite set. Let  $\mathcal{P}_\ell = \{\alpha_{\ell 1}, \alpha_{\ell 2}, \dots, \alpha_{\ell k_\ell}\}$ . Now for each  $\alpha_{i j} \in \bigcup_{\ell=1}^{\infty} \mathcal{P}_\ell$ , define

$$N_{i j} = f^{-1} \left( \bigoplus \{E_{\alpha_{h k}} \mid h < i \text{ or both } h = i \text{ and } k \leq j\} \right).$$

We then have an ascending chain

$$N_{11} \leq N_{12} \leq \dots N_{1 k_1} \leq N_{21} \leq N_{22} \leq \dots N_{2 k_2} \leq N_{31} \leq \dots$$

where each subfactor is isomorphic to a submodule of a unique  $E_\alpha$ . This is then a  $\mathcal{C}$ -chain of  $N$ . Since  $n_\ell \in N_{\ell k_\ell}$ , but  $n_\ell \notin N_{\ell-1 k_{\ell-1}}$ , it is an infinitely ascending  $\mathcal{C}$ -chain of  $N$ . Since  $M$  has ACC on  $\mathcal{C}$ -chains and  $N \subseteq M$ , this is a contradiction and  $f(N)$  must lie in a finite direct sum.

(2)  $\Rightarrow$  (1). Suppose there exists an ascending  $\mathcal{C}$ -chain  $N_1 \leq N_2 \leq \dots$ . Let  $N = \bigcup N_k$ , and  $E = \bigoplus_{i=1}^{\infty} E_M(N_{k+1}/N_k)$ . Then  $E$  is  $M$ -injective by (2). Since  $E_M(N_{k+1}/N_k)$  is  $M$ -injective, the inclusion map from  $N_{k+1}/N_k$  to  $E_M(N_{k+1}/N_k)$  can be extended to a map  $f_k : N/N_k \rightarrow E_M(N_{k+1}/N_k)$ . We define a map  $f : N \rightarrow E$  via  $\pi_k(f(a)) = f_k(a + N_k)$ , where  $\pi_k$  is the projection of  $E$  on  $E_M(N_{k+1}/N_k)$ . Using the  $M$ -injectivity of  $E$  we can extend  $f$  to  $\bar{f} : M \rightarrow E$ . Since  $M$  is finitely generated there exists an integer  $j \geq 1$  such that all of the generators of  $M$  are mapped to  $\bigoplus_{i=1}^j E_M(N_{k+1}/N_k)$ . It follows  $\bar{f}(M) \leq \bigoplus_{i=1}^j E_M(N_{k+1}/N_k)$ . Then if  $k > j$  and  $n \in N_{k+1}$ ,  $n + N_k = f_k(n + N_k) = \pi_k(f(n)) = \pi_k(\bar{f}(n)) = 0$ , which implies that  $n \in N_k$ . Hence  $N_{k+1} = N_k$  and the chain stops.

The equivalence of (2) and (3) is by Theorem 2.12.

Recall that a module  $M$  is *locally noetherian* if every finitely generated submodule of  $M$  has the ascending chain condition on submodules.

For a natural class  $\alpha$  and a module  $N \in \sigma[M]$ , Zhou [11] has used finiteness conditions on  $H_\alpha(N) = \{P \leq N \mid N/P \in \alpha\}$  to study  $M$ -injectivity in natural classes. Using Theorem 2.4 of [11], we have the following result.

**Corollary 3.3** *Suppose  $M$  is finitely generated and  $\mathcal{C}$  is a disjoint set of natural classes on  $\sigma[M]$  such that:*

- (a).  $\mathcal{C}$  spans  $\sigma[M]$ ,
- (b).  $M$  has ACC on  $\mathcal{C}$ -chains, and
- (c).  $H_\alpha(L)$  has ACC for each cyclic submodule  $L$  of  $M$  and for each  $\alpha \in \mathcal{C}$ .

*Then  $M$  is locally noetherian.*

**Proof.** To show that  $M$  is locally noetherian it is sufficient to show that every direct sum of  $M$ -injective modules in  $\sigma[M]$  is  $M$ -injective. (This would imply (j) in 27.3 of [10, p. 223], for example.) Assume that for every  $i \in \mathcal{I}$ ,  $E_i \in \sigma[M]$  is  $M$ -injective. Let  $E = \bigoplus_{i \in \mathcal{I}} E_i$ . We must show  $E$  is  $M$ -injective. By Theorem 3.2, each  $E_i$  can be written  $E_{i0} \oplus (\bigoplus_{\alpha \in \mathcal{C}} (E_i)_{(\alpha)})$ . Since  $\mathcal{C}$  spans  $\sigma[M]$ ,  $E_{i0} = 0$ . Then  $E = \bigoplus_{i \in \mathcal{I}} (\bigoplus_{\alpha \in \mathcal{C}} (E_i)_{(\alpha)}) = \bigoplus_{\alpha \in \mathcal{C}} (\bigoplus_{i \in \mathcal{I}} (E_i)_{(\alpha)})$ . Now by Theorem 2.4 of [11], for each  $\alpha \in \mathcal{C}$ ,  $\bigoplus_{i \in \mathcal{I}} (E_i)_{(\alpha)}$  is  $M$ -injective and then by Theorem 3.2 above  $\bigoplus_{\alpha \in \mathcal{C}} (\bigoplus_{i \in \mathcal{I}} (E_i)_{(\alpha)})$  is  $M$ -injective as desired.

**Lemma 3.4** *Let  $N \in \sigma[M]$  and assume that  $M$  is  $M$ -nonsingular. If  $\mathcal{C}\text{-dim}(N) < \infty$  and there exists an ascending  $\mathcal{C}$ -chain  $N_1 \leq N_2 \leq \dots$  in  $N$ , then there exist a positive integer  $m$  such that  $E_M(N_{k+1}/N_k)$  is  $M$ -singular for every  $k \geq m$ .*

**Proof.** Suppose  $N_k$  is not essential in  $N_{k+1}$  for some  $k$ . Then there exist  $N'_k \leq N_{k+1}$  such that  $N'_k \cap N_k = 0$ . Then  $N'_k \cong (N'_k \oplus N_k)/N_k \leq N_{k+1}/N_k \in \alpha_k$ . Since  $\mathcal{C}\text{-dim}(N) < \infty$ , there exist a positive integer  $m$  where  $N_k \leq_e N_{k+1}$  for each  $k \geq m$ . Since  $M$  is  $M$ -nonsingular,  $E_M(N_{k+1}/N_k)$  is singular for every  $k \geq m$  (see [6] page 30).

**Corollary 3.5** *Suppose  $M$  is  $M$ -nonsingular, is finitely generated and has finite  $\mathcal{C}$ -dimension. The following statements are equivalent.*

- (1).  $M$  has ACC on  $\mathcal{C}$ -chains.
- (2).  $\bigoplus_{\alpha \in \mathcal{C}} E_\alpha$  is  $M$ -injective whenever each  $E_\alpha \in \alpha$  is  $M$ -injective.
- (3).  $\bigoplus_{\alpha \in \mathcal{C}} E_\alpha$  is  $M$ -injective whenever each  $E_\alpha \in \alpha$  is  $M$ -singular and  $M$ -injective.
- (4). Any chain  $I_1 \leq I_2 \leq \dots$  in  $M$  such that  $I_{k+1}/I_k \in \alpha_k \in \mathcal{C}$ , where  $E(I_{k+1}/I_k)$  is  $M$ -singular for every  $k$ , stops.

**Proof.** (1)  $\Rightarrow$  (2). By Theorem 3.2.

(2)  $\Rightarrow$  (3). Obvious.

(3)  $\Rightarrow$  (4). Follows easily as in (2)  $\Rightarrow$  (1) of Theorem 3.2.

(4)  $\Rightarrow$  (1). This follows from Lemma 3.4.

**Example 3.6** Let  $T$  be a nonsingular ring and let  $S = \prod_{i=1}^{\infty} T_i$  where  $T_i = T$ . Let  $R$  be the subring of  $S$  generated by  $\bigoplus_{i=1}^{\infty} T_i$  and  $1 \in S$ . For each  $i$  let  $e_i$  denote the idempotent with  $1 \in T$  in the  $i$ -position and 0's elsewhere. Let  $M < R$  be the ideal  $M = \bigoplus_{i=1}^{\infty} T_i$  and consider the natural classes  $\mathcal{C} = \{\alpha_i\}$  where  $N \in \alpha_i$  if and only if  $N(1 - e_i) = 0$ . It is straightforward to show that  $\alpha_i$  is an  $M$ -natural class and that  $\alpha_i \cap \alpha_j = \{0\}$  for  $i \neq j$ .

Now let  $\mathcal{B}$  be the set  $\{\alpha_j \in \mathcal{C} \mid \exists \alpha_j \neq \alpha_i \in \mathcal{C}, K < N \in \alpha_i \text{ with } N/K \in \alpha_j\}$ . Suppose that  $\alpha_j \in \mathcal{B}$ . Then there is an  $i$  with  $K < N \in \alpha_i$  and  $N/K \in \alpha_j$ . But if  $i \neq j$ , then  $e_i(1 - e_j) = e_i$ . Thus  $e_i(1 - e_j)n = e_i n = n$  for each  $n \in N$ . However  $(1 - e_j)N \leq K$  and we must have  $N = K$ . Contradiction. Thus  $\mathcal{B} = \emptyset$ .

Now it is clear that the equivalent conditions of Theorem 2.12 hold; So any direct sum  $\bigoplus_{\alpha \in \mathcal{C}} E_\alpha$  is  $M$ -injective, where  $E_\alpha \in \alpha$  is  $M$ -injective. Thus (2) of Theorem 3.2 and Corollary 3.5 hold. However, since  $T_1 \leq T_1 \oplus T_2 \leq T_1 \oplus T_2 \oplus T_3 \leq \dots$  is an ascending  $\mathcal{C}$ -chain which doesn't stop, (1) in Theorem 3.2 and Corollary 3.5 fail. Here  $M$  is nonsingular, since  $T$  is, but  $M$  is neither finitely generated nor does it have finite  $\mathcal{C}$ -dimension.

## References

- [1] Albu, Toma; Năstăsescu, Constantin. *Relative finiteness in module theory*; Monographs and Textbooks in Pure and Applied Mathematics; Marcel Dekker, Inc.; New York, 1984.
- [2] Dauns, John. Classes of modules. *Forum Math.* **1991**, *3* (4), 327–338.
- [3] Dauns, John. Module types. *Rocky Mountain J. Math.* **1997**, *27* (2), 503–557.
- [4] Dauns, John. Intersections of modules. In *Advances in ring theory*; Jain, S. K., Rizvi, S. Tariq, Eds.; Trends in Mathematics; Birkhäuser Boston: Boston, MA, 1997; 87–103.
- [5] Goodearl, K. R. *Ring theory. Nonsingular rings and modules*; Pure and Applied Mathematics; Marcel Dekker, Inc.; New York–Basel, 1976.
- [6] Nguyen Viet Dung; Dinh Van Huynh; Smith, Patrick F.; Wisbauer, Robert. *Extending modules*; Pitman Research Notes in Mathematics Series; Longman Scientific & Technical, Harlow, England; copublished in the United States with John Wiley & Sons, Inc., New York, 1994.
- [7] Page, Stanley S.; Zhou, Yiqiang. When direct sums of singular injectives are injective. *Ring theory: proceedings of the biennial Ohio State–Denison conference, May 1992*; Jain, S. K., Rizvi, S. Tariq, Eds.; World Scientific Publishing: River Edge, NJ, 1993; 276–286.
- [8] Page, Stanley S.; Zhou, Yiqiang. Direct sums of quasi-injective modules, injective covers, and natural classes. *Comm. Algebra* **1994**, *22* (8), 2911–2923.
- [9] Page, Stanley S.; Zhou, Yiqiang. On direct sums of injective modules and chain conditions. *Canad. J. Math.* **1994**, *46* (3), 634–647.
- [10] Wisbauer, Robert. *Foundations of module and ring theory. A handbook for study and research*, Revised and translated from the 1988 German edition; Algebra, Logic and Applications; Gordon and Breach Science Publishers, Philadelphia, PA, 1991.
- [11] Zhou, Yiqiang. Direct sums of  $M$ -injective modules and module classes. *Comm. Algebra* **1995**, *23* (3), 927–940.
- [12] Zhou, Yiqiang. Decomposing modules into direct sums of submodules with types. *J. Pure Appl. Algebra* **1999**, *138* (1), 83–97.
- [13] Zhou, Yiqiang. Nonsingular rings with finite type dimension. In *Advances in ring theory*; Jain, S. K., Rizvi, S. Tariq, Eds.; Trends in Mathematics; Birkhäuser Boston: Boston, MA, 1997; 323–333.