WHEN DOES A UNIVERSAL COXETER GROUP ACTION HAVE A STRICT FUNDAMENTAL DOMAIN?

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ABSTRACT. We give conditions which guarantee that a geometric action of a free product of cyclic groups of order two on a uniquely geodesic metric space has a strict fundamental domain. Moreover, when this is the case the proof is constructive. The conditions given also turn out to be necessary. This is a simple case of the author's thesis, [O'B07].

1. INTRODUCTION

A universal Coxeter system is a pair (W, S) where W is a free product of cyclic groups of order two and the elements of S are the non-trivial elements of the individual factors. That is (W, S) is a Coxeter system with defining graph consisting only of vertices. Throughout X is a uniquely geodesic metric space and (W, S) a universal Coxeter system which acts geometrically (properly and cocompactly by isometries) on X by an action ρ. We will make reference to this system by listing the quadruple (W, S, X, ρ). See [BH99] for the relevant details about uniquely geodesic metric spaces and geometric actions and [Hum97], [Bou68], or [Dav07] for relevant details about Coxeter groups. The main goal of this article is to present a special case of the author's thesis. As such, while it would not be difficult to formulate a slightly shorter proof this is not done.

Definition 1.1. An element $s \in W$ acts by a *generalized reflection* if for every $x \in X$ every path from x to $s \cdot x$ meets F^s, the fixed point set of s.

Notice that this definition is considerably more general than the usual definition of a reflection which often requires that F^s separates the space into *two* components. Also these components are often assumed or turn out to be convex. However it is stronger than the mere requirement that F^s separates the space. The requirement is clearly equivalent to either of the following:

(R1) s acts freely on the set of components of $X \setminus \mathrm{F}^s$
(R2) every connected s-invariant set meets F^s.

We will pass between the three equivalent definitions without further mention.

Definition 1.2. A component of the complement of a fixed point set will be referred to as a *component*. If we wish to specify exactly which generator a component belongs to we add the appropriate prefix. Thus, if T is a component of $X \setminus \mathrm{F}^s$, we refer to T as an s-component.

Definition 1.3. (W, S, X, ρ) is a *generalized reflection system* if every element of S acts by a generalized reflection. A quadruple that is not necessarily a generalized reflection system will be referred to as simply a *pre-reflection system*.

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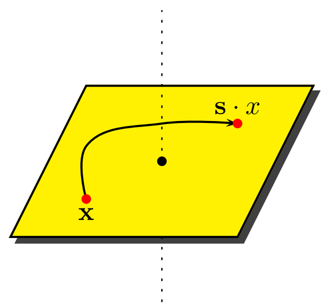
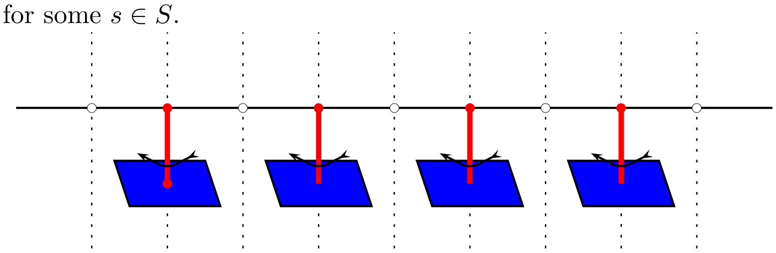
Example 1.4. Let Y be a uniquely geodesic space and let s be an involution acting on Y. We have that any s-invariant geodesic meets the fixed point set of s (a geodesic meets the fixed point set of s in its midpoint). Specifying to the case that Y is a tree, we further have that every connected s-invariant subset of Y contains a geodesic. As such, (R2) holds, and in this special case any pre-reflection system is a generalized reflection system.

Example 1.5. The action of the cyclic group of order two on a square by a 180 degree rotation does not constitute a generalized reflection system.



Moreover, it is not difficult to see that we do not have a generalized reflection system if and only if this $\langle s \rangle$-space equivariantly embeds (in the topological sense) into X for some $s \in S$.



A strict fundamental domain of the action of W on X is a closed, convex set which meets every orbit of W on X exactly once. A strict fundamental domain has the special property that it can be identified with the quotient X/W via a map defined by sending any point in X to the unique point in the strict fundamental domain in its orbit, see [Dav07] or [BH99] for further details. Thus, in particular, the quotient is contractible and as such the group W cannot be torsion free. We are now in a position to state the main theorem.

The Main Theorem. *The action of W on X has a strict fundamental domain if and only if (W, S, X, ρ) is a generalized reflection system.*

Necessity is elementary if we spent additional time discussing how to build the original space X up from its strict fundamental domain and stabilizer information. Again see [BH99] or [Dav07] for details. We prove sufficiency in Theorem 3.6 after some preliminary results.

The notation below does not make sense whenever W contains a central element. When W is a universal Coxeter system this is only possible when S consists of only one element. As such we need to deal with this case separately.

Lemma 1.6. *If $|S| = 1$ the main theorem holds.*

Proof. Let s be the non-trivial element. Let Ψ be the the union of a choice of one component from every pair $T, s{\cdot}T$ and F^s.

Then, almost by definition, Ψ is a strict fundamental domain. Indeed, the orbit of every $x \in X$ consists of either 1 or 2 points. If the orbit size is 1, then $x \in \mathrm{F}^s \subset \Psi$. If the orbit size is 2, then x is in some s-component T. Since $s{\cdot}T \neq T$ by definition of a generalized reflection system, exactly one of the pair $\{x, s{\cdot}x\}$ is in Ψ. In either case, the orbit of x meets Ψ in exactly one point. Lastly, since s-components are open, Ψ is closed, and convexity of Ψ follows from convexity of F^s. □

For every $t \in S$, F^t is convex (hence connected). Moreover for $s \in S$ different from t we have by properness of the action that $\mathrm{F}^s \cap \mathrm{F}^t = \emptyset$. Thus, there exists a unique connected component of $X \setminus \mathrm{F}^s$ which contains F^t. We call this component $\mathrm{T}^y_{st}(W, S, X, \rho)(= \mathrm{T}^y_{st})$ and we define T^n_{st} to be $s{\cdot}\mathrm{T}^y_{st}$. Generically, we call the first type of component a `yes` s-component and the second type of component a `no` s-component [1]. Moreover set

$$\mathrm{T}^y_s := \bigcup_{s \neq t} \mathrm{T}^y_{st}$$

and

$$\mathrm{T}^y := \bigcap_{s \in S} \mathrm{T}^y_s.$$

On a similar note, we define $\widetilde{\mathrm{T}^y_{st}}$ to be $\mathrm{T}^y_{st} \cup \mathrm{F}^s$ and define $\widetilde{\mathrm{T}^n_{st}}$, $\widetilde{\mathrm{T}^y_s}$, and $\widetilde{\mathrm{T}^y}$ in the obvious way. The set $\widetilde{\mathrm{T}^y}$ will often be the strict fundamental domain of the action of W on X.

2. Folding

As pointed out in the introduction, for an element to be a reflection one typically requires that the fixed point set of s separates X into two components which are exchanged by the action of s. Folding can be thought of a way of fixing this problem by replacing the notion of having two components by the notion of having two distinct *kinds* of components, namely `yes` and `no` components. [2] Moreover, in doing so we pick out a set of generators for W that have their corresponding fixed point sets as close together as possible.

Definition 2.1. A pre-reflection system is *folded* if for all $s \in S$ we have $\mathrm{T}^n_s \cap \mathrm{T}^y_s = \emptyset$.

In this section we show that for any generalized reflection system (W, S, X, ρ) there is an automorphism taking the Coxeter system (W, S) to the Coxeter system (W, S') with the new generalized reflection system (W, S', X, ρ) folded. Moreover since the question of the existence of a strict fundamental domain does not depend on S we have:

- (W, S', X, ρ) has a strict fundamental domain if and only if (W, S, X, ρ) does.

[1] The reader should be warned that not every component is a `yes` are a `no` component.

[2] This is not quite accurate. There is a third type of component, but these components turn out to be compact and can essentially be ignored, see Proposition 3.5.

The folding procedure will depend only on the generating set S. Thus we use words to modify S instead of the generalized Coxeter system (W, X, S, ρ) and we reference (W, S, X, ρ) as such.

A *configuration of fixed point sets* is simply an element of $W^{|S|}$. (We can think of the element $(w_s)_{s\in S}$ as corresponding to the collection $(w_s \cdot \mathrm{F}^s)_{s\in S}$.) We assign to every configuration, $\bar{w}$, two numbers $\mathrm{D}(\bar{w})$ and $\mathrm{Y}(\bar{w})$ where

$$\begin{aligned} \mathrm{D}(\bar{w}) &:= \sum_{r,s\in S} d(w_r \cdot \mathrm{F}^r, w_s \cdot \mathrm{F}^s) \\ \mathrm{Y}(\bar{w}) &:= \text{the total number of } \texttt{yes} \text{ components.} \end{aligned}$$

We can now define a lexicographic order on the set of configurations by defining $\bar{w} \prec \bar{v}$ if and only $\mathrm{D}(\bar{w}) < \mathrm{D}(\bar{v})$ or $\mathrm{D}(\bar{w}) = \mathrm{D}(\bar{v})$ and $\mathrm{Y}(\bar{w}) < \mathrm{Y}(\bar{v})$. We use this ordering to prove to the following:

Theorem 2.2. *There exists an automorphism which takes S to a folded S'.*

Proof. If S is not folded, then by definition there exists distinct $s, t, r \in S$ with $\mathrm{T}^n_{st} = \mathrm{T}^y_{sr}$, i.e. $s \cdot \mathrm{T}^y_{sr} = \mathrm{T}^y_{st}$. Define a map $\phi (= \phi_{(s,t,r)})$ by

$$\phi(\widetilde{s}) = \begin{cases} s\,\widetilde{s}\,s & \text{if } \mathrm{F}^{\widetilde{s}} \subset \mathrm{T}^y_{sr}; \\ \widetilde{s} & \text{otherwise.} \end{cases} \tag{1}$$

This clearly induces an isomorphism of the Coxeter system (an automorphism of the group W itself) $(W, S) \to (W, \phi(S))$ (a so called basis conjugating automorphism). Geometrically, we can think of ϕ as folding the fixed points sets corresponding to the elements of S into a configuration where the generators are more likely to be in a single s-component.

Claim. $\phi(S) \prec S$.

To see this let $a, b \in S$. The only case where the distance between the fixed point set of a and the fixed point set b might not be the same before and after the application of ϕ occurs when $\mathrm{T}^y_{sb} \cup \mathrm{T}^y_{sa} = \mathrm{T}^y_{st} \cup \mathrm{T}^y_{sr}$. As such, the geodesic which determines the distance between F^a and F^b meets F^s. On the other hand, by construction of ϕ both $\mathrm{F}^{\phi(a)}$ and $\mathrm{F}^{\phi(b)}$ lie in the same component, namely T^y_{st}. As such, the geodesic which determines the distance between $\mathrm{F}^{\phi(a)}$ and $\mathrm{F}^{\phi(b)}$ cannot be strictly longer than the original geodesic. That is to say that $\mathrm{D}(\phi(S)) \leq \mathrm{D}(S)$ (if we knew that T^y_{sa} we could conclude strict inequality here). What is more is that we are guaranteed that each of the terms in the corresponding summand cannot be strictly greater before the application of ϕ then they were before.

We claim that with the assumption that $\mathrm{D}(\phi(S)) = \mathrm{D}(S)$, we have $\mathrm{Y}(\phi(S)) < \mathrm{Y}(S)$. Indeed, we know for a fact that the number of `yes` s components has decreases, but as of right now we do not know that the total number of `yes` components has gone done (it may have even gone up). However, if for some $c \in S$ we have an additional `yes` component, then one of the generators whose fixed point set lives in T^y_{sr} gets sent by ϕ to a reflection whose fixed point set lives behind F^c. This clearly implies that this fixed point set is strictly closer to F^c after the application of ϕ then before and as such we contradict our assumption that $\mathrm{D}(\phi(S)) = \mathrm{D}(S)$. ■

To finish the proof we note that by properness of the action (actually the locally finite nature of the fixed point sets) that if we continue to apply partial conjugations we must reach a minimal configuration of fixed point sets, and this configuration must be folded. □

3. Proof of the main theorem

In this section we assume that (W, X, S, ρ) is folded.

Lemma 3.1. *For distinct $s, r \in S$ if T is an s component and T' is a r component, then $T \cap T' \neq \emptyset$ implies that either $T = \mathrm{T}^y_{sr}$ or $T' = \mathrm{T}^y_{rs}$.*

Proof. Let x be in the intersection. Assume that $T' \neq \mathrm{T}^y_{rs}$. That means that x and F^s lie in different r-components, i.e., the geodesic from x to some (hence any) point $x_s \in \mathrm{F}^s$ must meet F^r in some point x_r. Since $[\,x_s, x\,]$ and F^s are convex, the geodesic from x_s to x cannot meet F^s again after exiting F^s. In particular, $[\,x_r, x\,]$ does not meet F^s. But this is precisely the condition that $x \in \mathrm{T}^y_{sr}$. □

Remark 3.2. An immediate consequence of this lemma is that if $s \neq r$, $\mathrm{T}^n_s \cap \mathrm{T}^n_r = \emptyset$. Indeed, the folding requirement states that neither or these sets contain a `yes` component.

Since W is a universal Coxeter group, for any $w \in W$ there exists a unique generator $s \in S$ so that the word length of ws, $\ell(ws)$, is less than the word length of w, $\ell(w)$. We denote this generator w^r and $(w^r)^{-1}$ by w^l. These have the interpretation (when reading a word from right to left) that they are the unique letters that start (resp. end) the unique reduced word representing w.

Lemma 3.3. *Let w be a non-trivial element of W. Then $x \in \mathrm{T}^y_{w^r}$ implies that $w{\cdot}x \in \mathrm{T}^n_{w^l}$.*

Proof. Our proof is by induction on $\ell(w)$. If $\ell(w) = 1$, then $w = s$ for some $s \in S$. By definition $s{\cdot}\mathrm{T}^y_s = \mathrm{T}^n_s$. Since $w^l = s = w^r$ the result holds for the $\ell(w) = 1$.

Suppose the result holds for elements of W of word length $n-1$ and let w have length n which we assume is greater than 1. Let $s = w^l$ and x be as in the statement of the lemma. Since $\ell(sw) = n - 1$ and $(sw)^r = w^r$, the induction hypothesis tells us that $sw{\cdot}x \in \mathrm{T}^n_{(sw)^l}$. Since s has order two, $(sw)^l \neq s$ and by Lemma 3.1 (by folding $x \notin \mathrm{T}^y_{(sw)^l}$, so the lemma applies) $sw{\cdot}x \in \mathrm{T}^y_s$ and so $w{\cdot}x = s(sw){\cdot}x \in \mathrm{T}^n_s$ as claimed. □

Lemma 3.4. *If a component is neither a `yes` or a `no` component, then it is compact.*

Proof. Let T be such a component and let $x \in T$. Cocompactness and convexity allow us to prove that T is compact by showing that $x's$ orbit in X meets T in only one point, namely itself. Let $w \neq 1$ be an element of W. We show that $w{\cdot}x \in T \cup s{\cdot}T$ implies that $w = s$. By definition of a generalized reflection system, $s{\cdot}T \neq T$ and hence the result. Since multiplication by s does not change this claim we assume $s \neq w^r, w^l$. Let $r = w^l$ and $t = w^r$. Since $x \notin \mathrm{T}^y_s$, $x \in \mathrm{T}^y_t$. Lemma 3.3 tells us that $w{\cdot}x \in \mathrm{T}^n_r \subset \mathrm{T}^y_s$ where the last inclusion is due to Lemma 3.1. But since T is not a `yes`-component $w{\cdot}x \notin T$, as desired. □

Proposition 3.5. *For every element of $x \in X$ exactly one of the following is true*

- $x \in \widetilde{\mathrm{T}^y}$
- $x \in \mathrm{T}^n_s$ *for a unique $s \in S$ or*
- *x lies in some compact component of $X \setminus \mathrm{F}^s$ for a unique $s \in S$.*

Proof. This is an immediate consequence of the above lemmas. $\square$

Now for $s \in S$, the compact components discussed in the lemma are obviously invariant under the action of s. To the set $\widetilde{\mathrm{T}^y}$ we add in exactly one component for every pair $T, s{\cdot}T$ of compact components. We call this set $\Psi (= \Psi(W, S, X, \rho))$.

Theorem 3.6. $\Psi(W, S, X, \rho)$ *is a strict fundamental domain.*

Proof. It is clear that Ψ is convex and closed It remains to show that Ψ meets every orbit exactly once. We break up the proof into two parts, each part having its own cases.

- Every orbit meets Ψ at *most* once.
  By construction of Ψ there are two possibilities for an $x \in \Psi$:
  (1) $x \in \widetilde{\mathrm{T}^y}$
  In this case if $x \in \mathrm{F}^s$ for some $s \in S$, we can assume that $w^r \neq s$ since this does not change the number of times that $W \cdot x$ meets Ψ. Thus we have that $x \in \mathrm{T}^y_{w^r}$ and Lemma 3.3 applies to show that $w{\cdot}x \in \mathrm{T}^n_t$ where $t = w^l$. Since T^n_t is not a `yes` component or a compact component, we have that $w{\cdot}x \notin \Psi$.
  (2) x is in some s-compact component.
  This case is the same if $w^r \neq s$; for then $x \in \mathrm{T}^y_{w^r}$ which, as in the first case, implies that $\mathrm{T}^n_{w^l}$ which does not meet Ψ. On the other hand, if $s = w^r$, then by construction of Ψ, $s{\cdot}x \notin \Psi$ and $s{\cdot}x$ lies in a `yes` component of every generator except for s's. Now since $s = w^r$, $s \neq (ws)^r$. Moreover, since $s{\cdot}T$ is not a `yes` component (it is compact) Lemma 3.1 implies $s{\cdot}x \in \mathrm{T}^y_{(ws)^r}$ which implies that $w{\cdot}x = wss{\cdot}x \in \mathrm{T}^n_{w^l}$ in the case that $\ell(w) \neq 1$ and $w{\cdot}x \in s{\cdot}T$ in the case that $\ell(w) = 1$. Since neither of these components meet Ψ we are done with the first part of the proposition.

- Every orbit meets Ψ *at least* once.
  By Proposition 3.5, there are three cases to consider:
  (1) By definition $\widetilde{\mathrm{T}^y} \subset \Psi$. Hence, if $x \in \widetilde{\mathrm{T}^y}$ we are done.
  (2) If x is in some compact s-component and $x \notin \Psi$, then by the way that compact components were added to Ψ, $s{\cdot}x \in \Psi$.
  (3) If x is in T^n_{st}, then we must have that $d(s{\cdot}x, \mathrm{F}^r) \leq d(x, \mathrm{F}^r)$ for all $r \in S$ with strict inequality whenever $\mathrm{F}^r \subset \mathrm{T}^y_{st}$ and $s{\cdot}x \in \mathrm{T}^n_{rs}$. If $s{\cdot}x \notin \widetilde{\mathrm{T}^y}$, then we must have that this strict inequality holds at least once. Thus we see that if we select a point of $x's$ orbit with

$$\sum_{r \in S} d(-, \mathrm{F}^r)$$

  minimized (such a minimum exists by properness) we are not in a `no` component of any generator. As such we are in one of the first two cases. $\square$

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