GEOMETRIC INSIGHT INTO RIGIDITY AND THE AUTOMORPHISM GROUP OF A RIGHT-ANGLED COXETER GROUP.

MARK O'BRIEN

Abstract. We give a short geometric proof of the splitting of the automorphism group and rigidity (first proven in [Tit88] and [Rad01] respectively) for right-angled Coxeter groups by showing how to transform the region between fixed point sets for a given generating set into a strict fundamental domain.

1. Introduction

Throughout this document (W, S) is a right-angled Coxeter system, $\Sigma(W, S)$ is the associated Davis complex (see [Dav07]), $\Gamma(W, S)$ its defining graph, and X is an arbitrary CAT(0) space (see [BH99]). When discussing an action of W on X the action is always assumed to be geometric, i.e. proper, cocompact, and by isometries. With exception of the second section we attempt to minimize the use of specifics about the inner structure of the Davis complex.

First our protagonist.

Definition 1.1. A closed set is a *strict* set if it meets every orbit at *most* once, and a *fundamental domain* if it meets every orbit at *least* once. It is a strict fundamental domain if it is both a strict set and a fundamental domain.

It follows from general results proven in [FR40] that there are exactly four types of automorphisms of a right-angled Coxeter group. The table below defines these.

Name	**Definition**	**Condition for Existence**
Inner Automorphism	$s \mapsto wsw^{-1}$	$w \in W$
Graph Symmetry	$s_i \mapsto s_{\sigma(i)}$	$\sigma \in \mathrm{Aut}(\Gamma(W, S))$.
Transvection, $\mathrm{Tv}_{(s,t)}$	$s \mapsto st, r \mapsto r$ for $r \neq s$	$\mathrm{C}(s) \subset \mathrm{C}(t)$
Partial Conjugation, $\mathrm{Pc}_{(s,T)}$	$t \mapsto sts$ for $t \in T$	$T \subset \pi_0(\Gamma(W, S) \setminus \mathrm{St}(s))$

Table 1: In the table $T \subset S$, $\mathrm{C}(s)$ is the centralizer of s in W, and $\mathrm{St}(s)$ is the star of s in $\Gamma(W, S)$. Also, maps on generators are understood to induce the obvious map on W.

Remark 1.2. Inner automorphisms are really products of partial conjugations. Indeed if $\mathrm{St}(s)$ does not separate $\Gamma(W, S)$ then a partial conjugation by s *is* an inner automorphism, i.e. conjugation by s. However I separate the two in the table above because they will play very different roles in what is to follow.

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Remark 1.3. While we introduce this list now, we will see how these arise quite naturally when studying configurations of fixed point sets. Thus, it is as a naturalist that we point out the various animals that will be seen before we start our journey. Indeed, we will see how each has a specific role in changing a configuration of fixed point sets into a strict fundamental domain.

It is our main goal to simultaneously reprove the following results:

The Main Theorem.
1. *Right-angled Coxeter groups are rigid,* [Rad01].
2. *The automorphism of a right-angled Coxeter group is generated by the elements of Table 1,* [FR40].
3. *The full automorphism group splits as a semi-direct product of automorphism of the first and fourth type by automorphism of the second and third type,* [Tit88].

2. Lemmas about the Davis complex

This section we prove several results that no doubt appear elsewhere in the literature. From our viewpoint, these can be taken as properties that hold in the Davis complex that we wish to make use of so we provide proofs for completeness.. The arguments in this section hold for any Coxeter group W.

Lemma 2.1. *Let Δ be a strict fundamental domain for the action of $W \curvearrowright \Sigma(W,S)$. For any strict fundamental domain, Δ, $\Delta = \overline{\Delta^\circ}$ (the closure of the interior) and a point in Δ is in $\partial\Delta$ if and only if it is in the wall of some reflection.*

Proof. We first show that the boundary of Δ (considered as a subset of Σ), $\partial\Delta$, is contained in the union of walls. Let U be a neighborhood of Δ which meets the orbit of $x \in \partial\Delta$ in a unique point. Such a neighborhood exists since the action of $W \curvearrowright \Sigma(W,S)$ is proper. Any neighborhood of x in Σ meets a point y outside of Σ. Translate that point into Σ via an element $w \in W$. Such a w exists since Δ is a fundamental domain. The assumption on U guarantees that $w \cdot x = x$.

Now suppose that $x \in \Delta^\circ$. If $w \in W$ fixes x, then w must fix any neighborhood of x contained entirely inside of Δ. However in the Davis complex fixed point sets are nowhere dense. This proves the first claim and finishes the proof of the second. □

Proposition 2.2. *If Δ and Δ' are strict fundamental domains for the action of W on Σ, then there exists a unique $w \in W$ with $w \cdot \Delta = \Delta'$.*

Proof. If such a w exists, then it is clearly unique. Let Δ be a strict fundamental domain and $K = K(W,S)$ be the standard strict fundamental domain which is used to define the Davis complex (see [Dav07] for more details about K). Baire's category theorem guarantees that Δ has an interior point x. Translate x into $K(W,S)$ via an element $w \in W$. We claim that $w \cdot \Delta = K(W,S)$. The above lemma implies that Δ° is an intersection of (open) half spaces and since (open) half spaces are convex in the Davis complex Δ° is convex. As such for any $y \in \Delta^\circ$, the path $w \cdot [\,x, y\,]$ does not meet a wall and is thus contained in K. Thus w causes the interiors to line up and taking closures gives the result. □

Remark 2.3. The above proposition does not necessarily hold if the space were not the Davis complex, for instance see [**?**] for some simple examples.

3. Turning

As usual (W, S) is a right-angled Coxeter group and $\Sigma = \Sigma(W, S)$ is the corresponding Davis complex. Now suppose that (W, S') is another Coxeter system with the same Coxeter group W. At this point, we do not assume rigidity of right-angled Coxeter groups, that is we do not assume that there is an automorphism which takes S to S'. Independent of whether we assume rigidity or not, the elements of S' do not necessarily act by reflections, i.e. the fixed point sets need not separate Σ. For instance if we look at the usual action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ acting by reflections on the square, it is possible to take one generator as one of the two usual reflections but have the other generator act by a 180 rotation. In this section we show that transvections are precisely what is needed in order to fix this problem.

The following is well known and is proved in many places, see for instance [Rad01] or [Dav07].

Lemma 3.1. *Suppose G is a finite subgroup of W. Then there exists $T \subset S$ generating a finite subgroup of W and $w \in W$ with $G \subset wW_Tw^{-1}$.* □

The lemma is proved by taking the corresponding fixed point set of G, which exists since Σ is CAT(0), and translating it back to the strict fundamental domain of Σ. It is a wonderful example of something that is easily proven by use of the Davis complex.

As a direct consequence of this lemma and properties of proper actions on CAT(0) spaces (actually involutions acting on a uniquely geodesic space) we get the following lemma. It will play an implicit role in almost everything that will follow.

Lemma 3.2. *Any two finite order elements of W commute if and only if their fixed point sets meet.* □

Proposition 3.3. *These exists transvections whose product* Tv *has the property that* $\mathrm{Tv}(S')$ *consists of reflections of* Σ*.*

Proof. In the Davis complex a fixed point set is the wall of a reflection if and only if it is not contained in any other fixed point set. Thus our plan is to show that if Σ^s is not the wall of a reflection that there exists a transvection that makes the fixed point set larger without shrinking the fixed point set of any other element of S'.

If Σ^s does not separate then, its fixed point set is contained in the fixed point set of some conjugate of an S generator whose fixed point set does separate. Since this element of S has finite order Lemma 3.1 applies to show that it is of the form $ws_1 \cdots s_n w^{-1}$ $(s_1, \ldots, s_n \in S')$ when written in terms of the S' generators. Since $\Sigma^s \subset \Sigma^{ws_1 \cdots s_n w^{-1}} = w \cdot \Sigma^{s_1 \cdots s_n}$ we have that $\Sigma^s \subset \Sigma^{s_1 \cdots s_n}$. This implies that will verify later

(1) $$\mathrm{C}(s) \subset \mathrm{C}(s_1) \cap \cdots \cap \mathrm{C}(s_n).$$

If s appears in the $s_1, \ldots, s_n$ then define $\varphi(s) = s_1 \cdots s_n$ and leave all other elements of S' alone. This is a product of transvections by (1). If s does not appear then we define $\varphi(s) = ss_1 \cdots s_n$ and again leave all other elements of S' alone. This is also a product of transvections by (1). Since Σ^s is confined to $\Sigma^{s_1 \cdots s_n}$ we have the claim that the fixed point of $\phi(s)$ is strictly larger than the fixed point set of s. Since we leave alone all other elements of S' alone their fixed point sets are unaffected and the result follows. □

Definition 3.4. S' is *turned* if for any flag $s_1 \cdots s_n$ we have

$$\Sigma^{s_1 \cdots s_n} = \Sigma^{s_1} \cap \cdots \cap \Sigma^{s_n}$$

The following proposition is clear. In the statement and for the rest of the document an s-component is simply a connected component of the complement of the fixed point set of s in Σ. This terminology makes sense and will be used even if Σ^s does not separate Σ.

Proposition 3.5. *The following are equivalent*

1. *S' acts by reflections.*
2. *For every $s \in S', t \in \mathrm{C}(s)$, and every s-component T we have that $t \cdot T \neq T$ if and only $t = s$.* □

4. Folding

Based on the previous section we assume S' acts by reflections.

Definition 4.1. For any generator $s \in S'$ and t not commuting with s we have that the fixed point set of t does not meet the fixed point set of s. As such there is a unique component of s that contains Σ^t. We define that component to be T^y_{st}. We call such a component a `yes` component. We denote the other s-component by T^n_{st} and call such a component a `no` component. Moreover we let T^y_s (resp T^n_s) be the union of the `yes` (resp. `no`) components. If s happens to be central in W, then we select either s component to be T^y_s and call the other T^n_s. Lastly we let

$$\widetilde{\mathrm{T}^y}(S') := \bigcap_{s \in S'} \mathrm{T}^y_s \cup \Sigma^s.$$

Remark 4.2.

- For our original generating set S (that we have neglected for some time), up to manipulation by central elements $\widetilde{\mathrm{T}^y}(S)$ is the standard strict fundamental domain for the Davis complex $\Sigma(W, S)$.
- At this point there is no reason to expect that a particular s-component is not both a `yes` and `no` component at the same time.

Our goal is to "make" $\widetilde{\mathrm{T}^y}(S')$ into a strict fundamental domain. We will eventually see with hindsight that at this moment there is exactly one reason why $\widetilde{\mathrm{T}^y}(S')$ may not be a strict fundamental domain. Notice that there is no reason to expect that for a given s and t, r not commuting with s that $\mathrm{T}^y_{st} = \mathrm{T}^y_{sr}$.

Definition 4.3. S' is folded if for every s, t and r as above that $\mathrm{T}^y_{st} = \mathrm{T}^y_{sr}$.

Proposition 4.4. *These exists partial conjugations whose product* Pc *has the property that* $\mathrm{Pc}(S')$ *is folded.*

Proof. If S' is not folded we have an s, r, t with $\mathrm{T}^y_{sr} \neq \mathrm{T}^y_{st}$. This implies that in the graph $\Gamma(W, S')$ that $\mathrm{St}(s)$ separates r from t. This is precisely the condition for there to be a partial conjugation

$$\phi(\widetilde{s}) = \begin{cases} s\,\widetilde{s}\,s & \text{if } \Sigma^{\widetilde{s}} \subset \mathrm{T}^y_{sr}; \\ \widetilde{s} & \text{otherwise.} \end{cases} \tag{2}$$

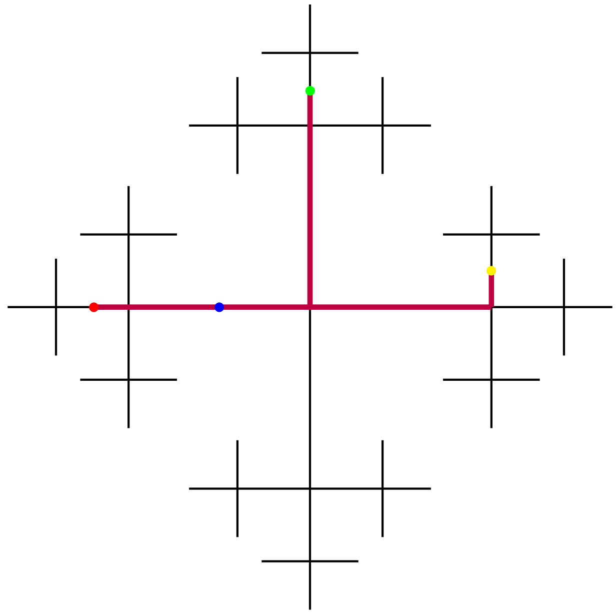

Figure 1: An action of the free product of four copies of $\mathbb{Z}_2$ on the regular tree of valence four. Indicated are the fixed point sets of the four generators which act in the obvious way. Also indicated is a specific subset of $\widetilde{\mathrm{T}^y}$. Note that the actual $\widetilde{\mathrm{T}^y}$ is not compact.

The fixed point sets corresponding to elements of $\phi(S')$ are closer together than the fixed point sets of S' in the sense that

$$\sum_{a,b\in S'} d\Big(\Sigma^{\phi(a)}, \Sigma^{\phi(b)}\Big) < \sum_{a,b\in S'} d\big(\Sigma^a, \Sigma^b\big). \tag{3}$$

To see this let $a, b \in S$. The only case where the distance between the fixed point set of a and the fixed point set b is not the same before and after the application of ϕ occurs when Σ^a and Σ^b appear in opposite s components. As such, the geodesic which determines the distance between Σ^a and Σ^b meets Σ^s. On the other hand, by construction of ϕ both $\Sigma^{\phi(a)}$ and $\Sigma^{\phi(b)}$ lie in the same component, namely T^y_{st}. As such, the geodesic which determines the distance between $\Sigma^{\phi(a)}$ and $\Sigma^{\phi(b)}$ does not meet Σ^s. Moreover the length of the first geodesic the length of a path from $\Sigma^{\phi(a)}$ and $\Sigma^{\phi(b)}$ which meets Σ^s. Hence, for a and b in this second case, the distance between their fixed point sets is strictly less after the application of ϕ. Since there exists a pair that fits into this case, namely r and t, (3) holds.

Since at every step we can assume that some element $s \in S'$ is fixed we use the locally finite nature of fixed point sets to guarantee that this process cannot continue indefinitely. When the process ends we have a folded generating set. □

5. Verifying Strict Fundamental Domain Properties

We now assume that S' acts by reflections and is folded.

Lemma 5.1. *$\mathrm{T}^n_s \cap \mathrm{T}^n_t \neq \emptyset$ if and only if s and t commute.*

Proof. If s and t commute, then turning implies that t fixes T^n_s. Since T^n_s and the midpoint of $[\,x, t{\cdot}x\,]$ is fixed by t we must have that Σ^t meets T^y_s. Now since that action of W on Σ is faithful we must have that T^y_s is not contained in Σ^t. Thus we must have that both T^y_t and T^n_t meets T^y_s.

Conversely if s and t do not commute, then select a point in $\mathrm{T}^n_s \cap \mathrm{T}^n_t$ and fix points $x_s \in \Sigma^s$ and $x_t \in \Sigma^t$. If the geodesic from x to x_s (resp. x_t) meets Σ^t (resp Σ^s) then x must be in T^y_s (resp. T^y_t). Since we are folded `yes` and `no` components do not meet and we must have that the interior of the path $[\,x_s, x\,]$ does not meet the fixed point set of t and so x lies in T^y_t. □

The following proof completely parallels the classical proof that one finds in many guises in the literature about Coxeter groups. In a book such as [Hum97] one sees it stated in terms of root systems and in a book such as [**?**] one sees it in terms of components. We give the proof for two reasons: it shows exactly why we turned and folded and secondly it is considerably easier in the right-angled case.

Lemma 5.2. *Let $x \in \widetilde{\mathrm{T}^y}(S')$. Suppose that for every $s \in S'$ with $\ell(ws) = \ell(w) - 1$ we have that $x \notin \Sigma^s$. Then $w{\cdot}x \in \mathrm{T}^n_t$ (for $t \in S'$) if and only if $\ell(sw) = \ell(w) - 1$.*

Proof. For the case that $\ell(w) = 1$, i.e. $w = s$ we must have that $x \notin \Sigma^s$ by assumption. That is $x \in \mathrm{T}^y_s$ and so $s{\cdot}x \in \mathrm{T}^n_s$ which by folding does not meet $\widetilde{\mathrm{T}^y}$. On the other hand if $t \neq s$ there are two cases to consider based on whether or not s and t commute. If t and s do not commute then Lemma 5.1 says that $s{\cdot}x \in \mathrm{T}^y_t$. On the other hand if s and t do commute then t by turning s preserves $\mathrm{T}^y_t \cup \Sigma^t$. Thus since $x \in \mathrm{T}^y_t \cup \Sigma^t$ we also have that $s{\cdot}x$ is in this same set and can not be in T^n_t.

Now suppose the result holds for all w with $\ell(w) = n - 1$ which we assume to be greater than 1. Since $\ell(w) \neq 1$ there exists an $s \in S$ with $\ell(sw) = \ell(w) - 1$. If $w{\cdot}x \notin \mathrm{T}^n_s$, then $sw{\cdot}x \in \mathrm{T}^y_s$ contradicting the induction hypothesis. Thus we can assume that left multiplication $t \in S'$ adds to the length of w and $w{\cdot}s \in \mathrm{T}^n_{s'}$. Then for the same s as above if s commutes with s', then as before in the $n = 1$ case $s{\cdot}w \in \mathrm{T}^n_{s'}$ as well. Lastly if s does not commute with s', then again by the above lemma we must have that $w{\cdot}x \in \mathrm{T}^y_s$ or $sw{\cdot}x \in \mathrm{T}^n_s$ again contradicting the induction hypothesis (since $\ell(s(sw)) = \ell(w) = \ell(sw) + 1$). □

Notice that from a formal standpoint the $n = 1$ case is in some sense the same as the inductive step.

Theorem 5.3. *$\widetilde{\mathrm{T}^y}(S')$ is a strict fundamental domain.*

Proof. Let $x \in \widetilde{\mathrm{T}^y}$. The above lemma implies that $\widetilde{\mathrm{T}^y}$ is strict for any element $w \in W'$ either fixes x or moves it into a `no` component and thus by folding does not meet $\widetilde{\mathrm{T}^y}$ again.

To see that $\widetilde{\mathrm{T}^y}$ is a fundamental domain we now let x be anywhere Σ. If $x \notin \widetilde{\mathrm{T}^y}$ then x is in the `no` component of some generator s. Convexity of T^y_s and the fact that the fixed point sets of maximal finite subgroups lie completely in T^y_s tells us that $s{\cdot}x$ is strictly closer to the fixed point sets of all maximal finite subgroups (which are points since they are points in the original generating set S). As such we must eventually have that the orbit of x meets $\widetilde{\mathrm{T}^y}$ (alternatively one can note that the closure of any union of components of the complement of the union of all fixed point sets is a fundamental domain for W and $\widetilde{\mathrm{T}^y}(W, S')$ clearly has this property). $\square$

6. Main Claims

What we have shown so far is that we can start with any initial generating set S' and by applying transvections and partial conjugations arrive at a new generating set S'' with the property that $\widetilde{\mathrm{T}^y}(S'')$ is a strict fundamental domain. Moreover by Proposition 2.2 we know that $\widetilde{\mathrm{T}^y}(S'')$ and $\widetilde{\mathrm{T}^y}(S)$ differ by translation.

Corollary 6.1. *In the situation above there exists an inner automorphism which takes S'' to S.*

Proof. By 2.2 there exists w with $w \cdot \widetilde{\mathrm{T}^y}(S) = \widetilde{\mathrm{T}^y}(S'')$. Now elements of S'' are characterized by the following properties

(1) For every $t \in S''$, $t \cdot \widetilde{\mathrm{T}^y}(S'') \cap \widetilde{\mathrm{T}^y}(S'') \neq \emptyset$, and
(2) if $w \in W$ also satisfies $w \cdot \widetilde{\mathrm{T}^y}(S'') \cap \widetilde{\mathrm{T}^y}(S'') \neq \emptyset$, then $\Sigma^w \subset \Sigma^s$ for some $s \in S''$.

The result of the corollary follows from the fact that wSw^{-1} satisfies the conditions above. $\square$

Combining all of these results gives the main theorem.

Lemma 6.2. $<\mathrm{PartialConjugations}> \cap <\mathrm{GraphSymmetries}, \mathrm{Transvections}>= \mathrm{Id}_W$

Proof. Show that for a $\phi = \mathrm{Tv}$ or σ, that if $\phi(s)$ is conjugate to s then $\phi(s) = s$. Since for a product transvections Tv we have that $\mathrm{C}(\mathrm{Tv}(s)) = \mathrm{C}(s)$ and $\mathrm{C}(wsw^{-1}) = \mathrm{C}(s)$ implies that $w \in \mathrm{C}(w)$. Moreover we have if s is conjugate to r that $s = r$. $\square$

Corollary 6.3. *Aut splits as a semidirect product.*

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Department of Mathematics, Tufts University, Medford, Massachusetts, 02155
E-mail address: mark.o_brien@tufts.edu