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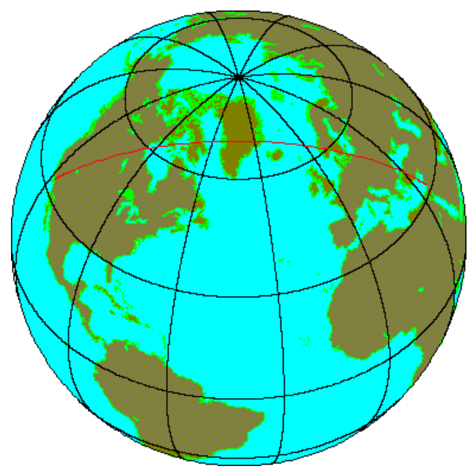
Math 234

Being Relatively General about General Relativity

Nearly every science fiction movie tries to wield general (and special) relativity as a mysterious tool that makes it possible to reach the far reaches of our universe. The general theory of relativity is the view of gravitation developed by Albert Einstein. It uses a complicated mathematical language known as differential geometry (a mix between geometry and calculus) to describe non-flat space as the culprit of the gravitational effect. In many instances, this idea better describes the phenomena of gravity as compared to Sir Isaac Newton's 16th century push and pull ideas (and in instances where it does not better describe gravity, it describes it just as well). In this theory, acceleration is caused when an object follows the least distance path of a curved space; similar to how a kid gains speed while rolling down a steep (non-flat) hill (see image on left for an example of a least distance path on the curved surface of the Earth). Furthermore, Einstein's theory explained that mass is what causes the space to curve in the first place. To fully understand and appreciate what the above implies, one must be able to open his or her mind and eliminate any preconceived notions of the universe. In order to successfully accomplish this, we must invoke a tool of the human mind known as imagination. Although Einstein himself claimed this was the most important aspect of what it took to establish today's nearly world-wide accepted idea, he had to be able to use the ideas of many great scientists and mathematicians that came before him.



Galileo Galilei was the first of three physicists who helped build the groundwork for classical physics that will be discussed. Galileo was born February 15, 1564 in what is today Pisa, Italy and was able to overcome the incorrect classical ideas passed down from the ancient Roman and Greek philosophical approach to physics (Geroch, 37).

Instead of thinking purely of what was beautiful, Galileo was able to come to grasp what was true through experiment. Galileo disproved many supposed laws of the universe that were passed down for over a millennium by the Greeks and Romans with simple household experiments. Of most importance to the development of the theory of relativity is his, now seemingly commonsensical, ideas for adding velocities referred to as the *Galilean transformations*. These transformations can be illustrated through the following example. If a person is driving a car at 20 miles per hour, and he or she throws a ball out the front of the window at 5 miles per hour, how fast would an observer see the ball move? Everyday experiences tells us relative to a person on the ground the ball would be moving at 20 miles per hour plus 5 miles per hour or 25 miles per hour and relative to the person in the car the ball would be moving at just 5 miles per hour. This is a *Galilean transform*. A second consequence is the even more obvious fact that if the person in the car sees the ball in the air for five seconds, then the person on the ground sees it also in the air for only five seconds. These transformations have two important consequences: the first being that the distance between two points does not depend on how fast you are traveling, and that the time it takes for an event to occur also does not depend on how fast you are moving. These two facts are known as *space-time invariance.* The truly remarkable thing about relativity is that it disproves these simple, common sense ideas.

After Galileo, the laws of motion put forth by Sir Isaac Newton further embellished the theories of classical physics. In 1687, Newton's book, *Principia,* became the backbone of classical physics (Bergmann, 8). In this book, Newton described his three laws of motion that are now the very basis of classical mechanics. One particular law (the second law) defines the most general form of what we now colloquially refer to as *force* as the rate at which momentum changes. In grade school, students are taught that force is given by the mathematical relationship, force equals mass multiplied by acceleration ($F = ma$). However, this definition is not the most general case of what Newton meant by force. We will later see how changes in direction also can cause an apparent force.

A common *myth* tells of an apple falling on Newton's head, from this he was able to convince himself that what made the apple fall to the ground was the same thing that kept the moon in its orbit around the earth. In either case, the earth was pulling on the object in question (i.e. exerting a force), thus making it fall. However, in the latter case, the object is literally missing the earth and forever destined to spend its entire existence falling. Newton based his law on what is referred to in physics as an inverse square relationship. An inverse square relationship for a force decreases in strength by the distance squared. An example of this would be the effect of an everyday butter gun. Imagine a butter gun held from a plate so that it is just far enough away to cover one slice of toast with its buttery spray. Now if the gun were pulled away from the toast so it is now twice as far as it was originally, it would now cover four pieces of toast (two squared is four). Similarly, if the gun were three times as far away, it would now cover nine pieces of toast (three squared is nine). In each successive case, each individual slice of toast would have less of a buttery coating applied to it.

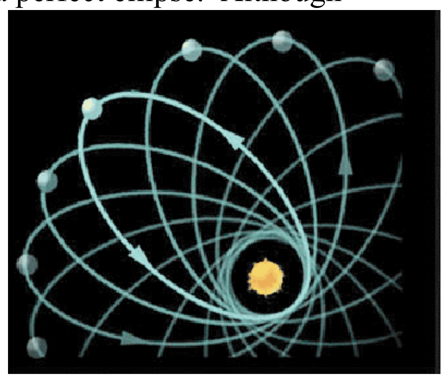
Newton's law of gravitation is highly successful. Through the use of elementary calculus and this law, one can predict the shape of a planet's orbit around the sun, determine how fast a rocket needs to go to leave the earth's gravitational field (as well as how strong the field is), determine how long any object takes to go around any other large object due to gravitational attraction between the objects (i.e. the earth going around the sun), and obtain any calculation commonly thought of as being associated with gravitation. Newton's theory of gravity is so precise and simple that it is still the most used theory of its kind. When men were sent to the moon, for example, general relativity was not used (even though it had been established for forty years at this time), but Newton's gravity was used to predict the paths that would need to be taken in order to guide eighteen Americans a quarter of a million miles to the moon.[*]

During the two centuries following Newton's pioneering work, numerous powerful and dramatic confirmations of his law of gravitation were made. For example, in 1781, William Herschel discovered the planet Uranus. After careful observations of the planet, Newton's laws of gravitation predicted the presence of an additional planet. The very night that a German observatory got word of the detailed calculation using

[*] You may be wondering what the point of using Newton's theory if it was proven wrong. It turns out that general relativity and Newton's gravitation theory nearly describe identical events in everyday occurrences. The difference is so slight in normal circumstances that the difference in precision is far outweighed by the difference in ease in which calculations are actually done using Newton's gravity.

Newton's law of gravitation, an observation of the eighth planet was made that we now call Neptune. Newton's law of gravitation was so powerful and so universal that it could be used to predict the existence of undiscovered planets (Bergmann, 112).

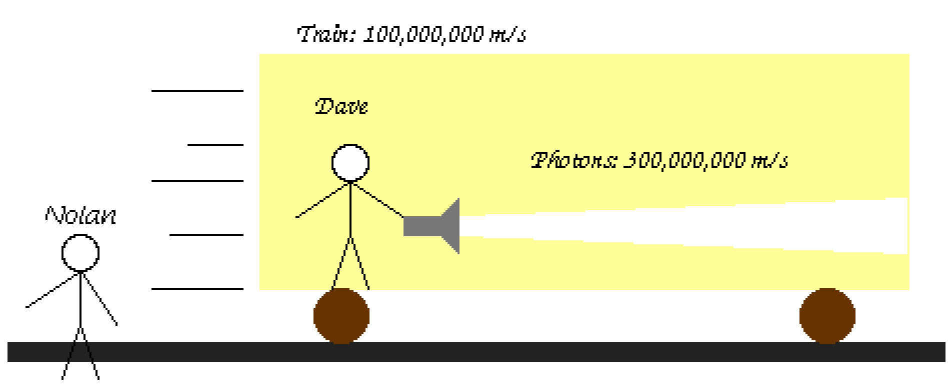
Classical physics was going strong, but, of course, there had to be some faults to it. By the end of the nineteenth century, astronomers began to notice a peculiar motion of the inner most planet of our solar system, Mercury. The planet's closest point to the sun in its elliptical orbit appeared to be ever so slowly moving around every year (as shown in on left) in a rosette shaped pattern. Newton's theory predicts that a planet without any attraction from any outside object would be moving in a perfect ellipse. Although Mercury's attraction to the other planets caused much of the planet's bizarre motion, there was still some motion left unexplained. Once again, astronomers began looking for an unseen planet. This time, the planet was not believed to be further out than any other planet, but was further in. They dubbed the hypothetical planet *Vulcan*. It does not exist. So, astronomers began to wonder if it were possible to simply tweak the inverse square form of Newton's law. However, any modification to Newton's theory would render the updated theory incapable of describing the motions of the outermost planets (Bergmann, 116) .



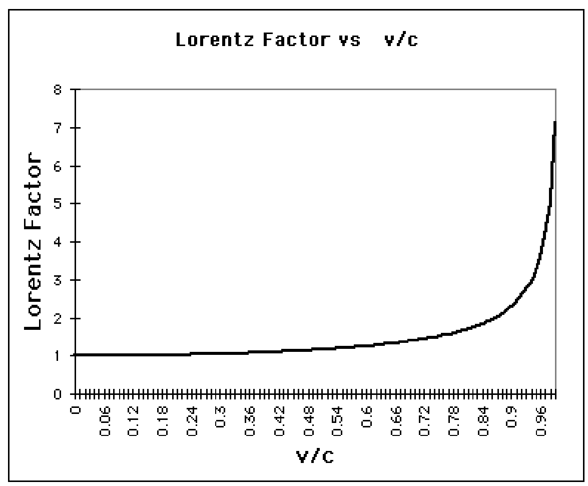
Another large failing of classical physics seemed less directly attached to gravitation. During the middle part of the nineteenth century, a physicist by the name of James Clark Maxwell was able to combine the forces of electricity and magnetism into one. The combined force, referred to as electromagnetism, describes the force that makes your clothes stick together in the dryer and the force that causes magnets to stick to the fridge as the interaction of particles, known as photons, jumping between the two objects. In a sense, the two objects are playing catch with the photons. Photons are also the particles that make up light. Combining Maxwell's equations shows that the speed of light is about 186,000 miles per second (this is fast enough to go around the earth about 8 times per second!). Due to the fact that sound waves travel through air and water waves travel on water, scientists were led to the conclusion that light waves had to be traveling in something as well. Scientists dubbed this medium the "ether."

Now we can begin to tie a couple of ideas together. Since light can be seen from distant stars, ether (since it's the medium in which light travels) has to fill the space between the star and us. Therefore, one would be able to measure his or her speed relative to the speed of the ether by measuring differences in the speed of light at different times in the year (similar to how you would notice a difference in the speed of a bullet as it flew by you coming from a man moving forwards or backwards). Once again, the experimenters of physics set out and (as in the case of Mercury) they came up empty handed. No matter how the scientists attempted to measure the speed of light, the speed came to be measured as 186,000 miles per second. This lead to only one possible conclusion: the measured speed of light is independent of the velocity of the measuring party. This fact goes against deep-rooted common sense ideas illustrated earlier by Galileo's transforms. To illustrate this fact, let us return from our car example but now with an extra twist. Consider the same person in the car that threw the ball has now decided to turn their headlights on. If the person in the car could somehow measure the speed of light, he or she would record the speed of the light photon as 186,000 miles per second. Amazingly enough, a person on the ground would also measure the exact same speed for the light as it went by him! This directly contradicts the additive nature of velocities that we have so ingrained in our minds from everyday experience.

Around the time of these two discoveries, a young man working in a Swiss patent office had a solution to the latter, and later the former, problem in classical physics. The young man's name was Albert Einstein, and he came up with two laws to describe nature based on the observations made in the previous paragraph. The first was that the laws of physics were exactly the same no matter how fast the object or the observer is moving. The second is that the speed of light is a law of physics. All that this second statement implies is that it has the same value no matter how fast you are moving (yes, one should be in disbelief of this very fact initially). The implications of this would spark a revolution in the very foundations of physics.



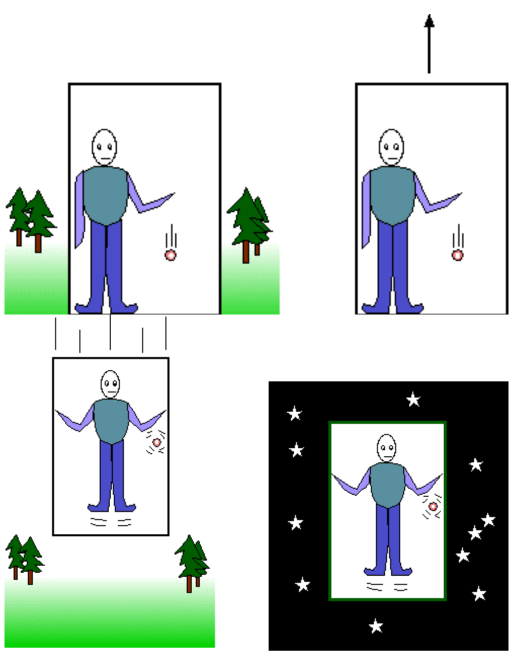
To illustrate these implications, consider the following example commonly referred to as Einstein's train. There are two observers, call them Nolan and Dave. Dave is on a train going at one-third the speed of light, and Nolan is an observer stationed on the ground. Dave turns on a flashlight and observes the light moving at 186,000 miles per second (300,000 meters per second). How fast does Nolan see the light? As Einstein's postulate states, Nolan is also going to view the photons of light as having a speed of 186,000 miles a second. Therefore, after one second, (by the definition of velocity) the light photon initially observed by Nolan and Dave is 186,000 miles ahead of both of them! How could this be when Dave is moving at 1/3 the speed of light? This leads to one of only two conclusions: either time is going more slowly for Dave, or the distance between Dave and the photon is shorter. Both of these answers turn out to be correct and equivalent; it really does not matter which way you look at it, because they describe the same thing. Furthermore, it turns out in the case explained above, Dave's time would be passing only 94% as fast as Nolan's. Thus, if Nolan and Dave were twins, after a years trip Nolan would now be twenty-two days older than Dave simply because Dave had been moving at such high velocities for so long!



Anyone reading this should hopefully be asking him or herself how they do not notice these dramatic effects in everyday life. The reason is we do not travel at high enough speeds where we would notice such things. The graph, in the figure directly to the left, depicts how much time is slowed (or length is contracted) as speed is increased. The Lorentz factor on the left side of the graph tells how many times more slowly time is passing as compared to the fraction of the speed of light on the bottom (v/c). Apparently, time slows considerably as the speed of light is approached. It turns out if somebody would reach the speed of light, he or she would not experience any time at all. This is just part of what makes it completely impossible to reach the speed of light. This is opposed to what we typically see on many science fiction movies such as *Star Trek* when Captain Kirk jumps into the *Enterprise* and breaks the speed of light as if it were not even there. If the speed of light were to be broken, a complete revision of the laws of physics would have to be made. We currently can bring objects arbitrarily close to the speed of light, but we cannot break it (some places take particles and accelerate them beyond 99.99% the speed of light!).

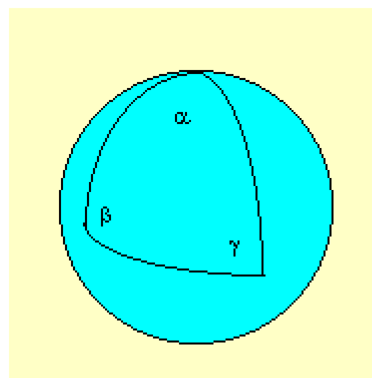
The special theory of relativity, described above, has tremendous experimental backing. The world's most accurate clocks have been put on airplanes, and flown around the world, while an identical clock was left positioned on the earth. The results were just as expected from what was outline above. The clock on the ground had more seconds tick away than the clock in flight. Heavy particles that decay in a certain amount of time apparently decay in a longer amount of time when moving at fast speeds. Einstein's famous formula, Energy equals mass times the speed of light squared, ($E = mc^2$) is a direct consequence of the variance of time as velocity increases. If special relativity was not correct, the atomic bomb would never have gone off, and the sun would not shine. As Einstein himself stated, "Who would imagine that this simple law [constancy of the velocity of light] has plunged the conscientiously thoughtful physicist into the greatest intellectual difficulties?, (Misner, Thorne, and Wielder 43).

To develop the special theory of relativity only about a year was needed, but to develop the general theory of relativity ten years were needed. The question that naturally arises is, to what happens when there is acceleration present? This question is handled within the confines of special relativity. However, to explain a uniform gravitational field that causes a uniform acceleration, another amazing mental jump was made by Albert Einstein.



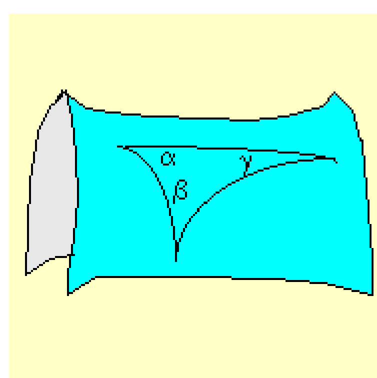
Things falling freely in a gravity field all accelerate by the same amount, so they move the same way as if they were in a region of zero gravity — ``weightlessness''!

Consider the following: Jim is in an elevator that is completely closed in to the rest of the universe. There is no way for him to know of anything that is happening around him at any time while he is in the elevator, and he can only make observations within the confines of this elevator. Jim is curious about the nature of his confines, and he decides to drop a rock, and notices the rock fall to the floor. He then concludes that he is on the surface of the earth. However, Einstein noted that it is quite possible that Jim may be experiencing an upward acceleration, similar to that experienced by an astronaut as he or she accelerates away from our planet, equal to that caused by the earth (32 feet per second per second). This would give exactly the same feeling, as Jim would feel if he were on the earth's surface. Next consider that Jim drops an object and it does not fall to the ground. He now concludes that he is in outer space where there is no gravity. But, once again, the presumptuous Jim could be wrong. Einstein pointed out that some maniacal physicist could have cut the line that was supporting the elevator, and it began falling freely towards the ground. This gave Einstein, what he felt was "the happiest thought of his life". The equivalence between the gravitational field and the usual phenomena of acceleration is what he called the equivalence principle, and is the fundamental basis of the general theory of relativity.



After the theory of equivalence in 1907, Einstein published nothing further on the subject until 1911. At that time, he was able to verify the equivalence principle for light, by noting that light, when under the influence of the gravitational field, was curved. In 1912, he realized that the space and time transformations outlined about would not hold in a simple manner. It is then that he determined that "If all accelerated systems are equivalent, then Euclidean geometry couldn't hold in all of them". It is taught in grade school that the angles in a triangle add up to 180 degrees. This is what makes a space Euclidean. In curved (i.e. non Euclidean) space, this will not be the case.

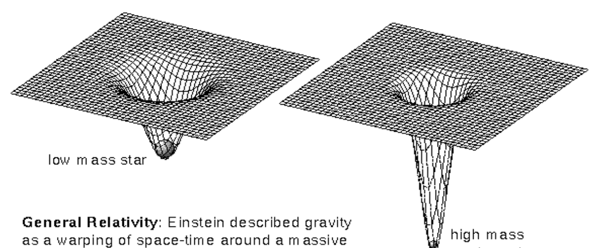
Consider the following two examples where the angles of a triangle in fact do not add up to 180 degrees. First consider the surface of a sphere as depicted on the left. Here, as you may be able to visualize, the angles do not add up to 180 degrees, but they in fact add up to more than the magic number for a space to be Euclidean. Thus, intuition pays off (for once in this paper) and the surface of a sphere is curved (I bet if it were not, you would throw this paper down right now in disgust). Furthermore, since the angles add up to *more*



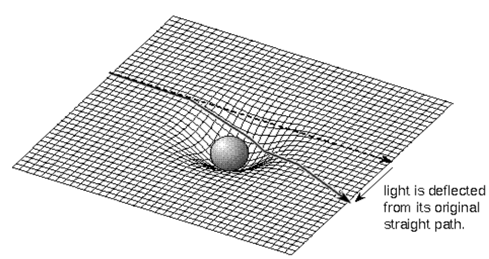
than 180 degrees, the curvature is said to be positive. Next, consider a triangle placed on an object similar to a horse's saddle. Here, the angles add up to less than 180 degrees, because of this, the space is said to have negative curvature.

What exactly does curvature imply? Picture a person on a merry-go-round. The faster this playground object would spin, the more the person feels as if they were being pushed off the side. Even though there is no acceleration here (the merry-go-round is moving at constant speed), there is still an apparent force, commonly referred to as centripetal force. The force the person would feel is due to the curved (circular) path that he or she follows while being whipped around the central axis of the merry-go-round (hence a force due to changes in direction as mentioned previously). In general, the velocity squared times the curvature of the path gives this so called centripetal force. Curvature for a circle (like the merry-go-round) is one divided by the radius of the particular circle, so the smaller the circle, the more the tendency for the person to get whipped off the side. Different curves have various degrees of curvature, and a intuitively one thinks of a straight line as having no curvature, which is why there is no force felt when walking a straight line (unless the person had a little too much to drink, then there is nearly an infinite amount of force pushing the person off the straight line).

The curvature of space and time in the presence of a gravitational field is what Einstein began to work on in order to replace Newton's push-pull ideas of gravity. The task was far from easy. The mathematics needed to explain such a theory was one developed fifty years earlier by mathematicians Georg F. B. Riemann, Elvin Bruno Christoffel, and Gregorio Ricci in order to describe curved spaces of any number of dimensions (Wald, 24). At that time, a new branch of mathematics known as *differential geometry* was constructed that made heavy use of something called *tensor analysis*. Previous to Einstein, primarily mathematicians used these complicated mathematical techniques. One particular tensor, known as the Riemann-Ricci tensor, describes everything that can be known about how a particular space is curved. This is exactly the mathematical tool that Einstein needed. Setting this tensor equal to zero tells how space is curved where there is no mass present (i.e. in orbit around the Earth). From knowing how much the space was curved, one could tell how much of an effect gravity would have on the surrounding area. Inside of a mass, Einstein used a different tensor, which now bears his name (the Riemann-Ricci tensor mentioned above is a special case of the Einstein tensor mentioned here). In an amazing insight, and many years of labor, Einstein set this tensor that describes the geometry of the region equal to another tensor that describes the amount of mass in the local area (in case you are interested what I am saying is that $\mathbf{G} = 8\pi\mathbf{T}$ or $\mathbf{R} - 1/2\mathbf{g}K = 8\pi\mathbf{T}$ inside of a mass in juxtapose with $\mathbf{R} = 0$ outside of a mass, where bold letters are these so-called tensors; **G** is the Einstein tensor and **R** is the Riemann-Ricci tensor). This formulation can be used when talking about objects in the Earth's interior. These are the Einstein field equations and are the very basis for all of general relativity (Dirac, 30). For the first time in the history of physics, a completely geometrical object (the Einstein tensor) was set equal to a completely physical object (the so called stress energy tensor, **T**). The result of these ideas is beautiful: matter tells space how to curve; curvature tells matter how to move. The more matter, the more curvature, the more acceleration – this is gravitational effect.



General Relativity: Einstein described gravity as a warping of space-time around a massive object. The stronger the gravity, the more space-time is warped.



General Relativity: Light travels along the curved space taking the shortest path between two points. Therefore, light is deflected toward a massive object! The stronger the local gravity is, the greater the light path is bent.

To help visualize the above, imagine a large rubber sheet with a grid on it (see images to the right). Clearly, the rubber sheet is only two dimensional, while space/time is four dimensional (the three spatial directions and the one time "direction"), so this is just an illustration of the theory. Now suppose a massive object like a bowling ball is added to the sheet. What happens to the grid on the sheet? It gets stretched and distorted. This is exactly what happens to space and time when a large object is in space. The object warps the space and time around it. And, like the grid (how we measure space and time) the distance between time and space is longer. Near a massive body the amount of time measured by a clock would be greater than the amount of time that would be measured by a clock away from the massive body. Now try to imagine an object, let's say a marble, shot across the rubber sheet. If the object does not approach the bowling ball, its path will not be diverted. However, as the object nears the bowling ball its path is some what deflected.

These insights clearly did not come to Einstein all at once. In 1912 Einstein wrote, "in all my life I have not labored nearly so hard, and I have become imbued with great respect for mathematics, the subtler part of which I had in my simple-mindedness regarded as pure luxury until now" (O'Conner and Robertson). In 1913, Einstein and another physicist by the name of Marcel Grossmann, who had originally given Einstein the idea to use tensor calculus and differential geometry in the first place, published a paper where the tensor calculus of Ricci and Riemann was used to make further advances. However, at this time the theory was not quite right. When his friend and long-time mentor, Max Planck came to visit Einstein in that same year, he told Einstein, "As an older friend, I must advise you against it for in the first place you will not succeed, and even if you succeed no one will believe you" (O'Conner and Robertson).

In 1915, Levi-Civita pointed out some technical errors in Einstein's use of the tensor calculus. From there, in June of 1915, Einstein spent a week at Gottingen where he lectured for six two-hour sessions on his (at the time, still incorrect) October 1914 version of general relativity. At these lectures, mathematicians David Hilbert and Felix Klein "were convinced *completely" (*quote from Einstein, O'Conner and Robertson*).* Both Hilbert and Einstein took the final steps when they realized Einstein's 1914 flaw, and they both published the final form of the gravitational field equations within a few days of each other.

The measured success of general relativity is naturally much more difficult to obtain than that of the special theory. However, one of the largest successes of the theory is that it almost perfectly predicts the strange motion of Mercury about the sun. The implications of general relativity are what many science fiction movies are based on. Such implications include objects known as black holes. These objects have such intense gravitational fields that they literally warp space-time such that time cannot proceed at all (in the rubber sheet example, imagine an object so massive that it stretches the distance between the grid lines out infinitely long).

The main goal of many physicists is to be able to describe the laws of physics in as mathematical of terms possible, similar to the way gravity is described by space and time curvature. This is what is meant when a theoretical physicist speaks of beauty. The General Theory of Relativity is indeed very beautiful, and now lies at the base of these future goals in the same way that Newton, Galileo and Maxwell's laws inspired Einstein to ask questions such as "what is gravity"? Now we are taunted by questions such as "what causes mass to make space curve in the first place". This is just the beginning of what it takes for a physicist to use every bit of their most important tool, imagination.